The left task of the right job

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Abstract

A manager has to assign "left" or "right" employees to "left" or "right" positions of a job. Both positions involve a left task and a right task, with complementarities. The left task is more productive in the left position, and *vice versa*. A left employee is more able in the left task, and *vice versa*. Employees are not aware of their type, but the manager is. The manager strategically assigns employees to positions that maximize their productivity. We exhibit an informative symmetrical equilibrium that matches employees to positions according to their type. We exhibit two other asymmetrical equilibria upon which employees are assigned against their type. These equilibria are shown to be stable, in contrast to the symmetrical equilibrium. Furthermore, these equilibria are shown to be efficient if tasks are complementary enough.

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"But the contingent perceptions are important not because they are definitive guides to individual interests and well-being (this they are not), but because the perceptions (including illusions) have influence – often a major impact – on actual states and outcomes"

Amartya Sen, *Gender and Cooperative Conflicts*

1 Introduction

Social experiences have crucial impacts on self-perceptions. This is the so-called *looking-glass self* effect, introduced by Cooley (1902).[1] It describes how individuals retrieve information on who they are through how they are perceived by the others.

For instance, when a worker is assigned to a specific position, or when a pupils is assigned to a specific school, he would infer he has specific attributes that have driven the choice of the organization. This inference depends on the other positions he could have obtained, and which have not been chosen. A manager who has to take care of an expiring division rather than an emerging division, would infer his firmness is more valuable than his flexibility. A poorly able researcher will have to teach more, and through this decision the university informs him about its evaluation of abilities.

Reciprocally, self-perceptions have crucial impact on social experiences. Decision makers make their decisions on the basis of their anticipation of the actions their decisions induce. A firm would have low interest to assign an employee to a position he would not be motivated for.

However, employees' abilities and attributes of positions lie in a multi-dimensional space. A position assignment could have a positive effect on the employee's motivation with regard to one dimension, but a negative effect with respect to the other. From the firm's perspective, the best matching between employees and positions gets more complex when

[1] Cited by Bénabou and Tirole (2003).

taking into account the multiplicity of dimensions. Moreover, as argued by Holmstrom and Milgrom (1991), complementarities between tasks might exclude to consider the redesign of the job as a solution.[2]

At first glance, upon increasing returns of abilities, the firm has incentives to match employees who have high specific ability to a position where this ability is the most productive. In that case, individuals are correctly informed about their type, *i.e.* their comparative advantage, by observing the position they obtain.

In this paper, we will show that such a matching is not so obviously the one preferred by the firm when the different tasks involve complementarities.

We consider a manager who has to assign an employee to one of two positions. There are two types of employees, say "left" and "right" employees, and two types of positions as well. Each position involves a left task and a right task. The two positions differ only in the relative productivity of the tasks. In the left position, the left task is more productive, and *vice-versa*. The employee has two abilities, each associated with one task. A left employee is more able in the left task, and *vice versa*. When assigned to a position, the employee performs in each task. Contributions lead to an aggregate production with complementarity between the tasks. The employee derives utility from his total contribution, and disutility from effort. However, the employee lacks information on the tasks to be done. He has a uniform prior concerning his abilities relative to the tasks. He infers his abilities from the assignment. Then, he chooses his levels of efforts accordingly. In contrast, the manager is informed on the abilities needed for the tasks, and has observed the corresponding levels of abilities on the employee. She anticipates the employee's efforts conditional on the assignment. Then she compares the aggregate production conditional on the assignment, and proceeds to the assignment that maximizes the aggregate production.

This model exhibits a trade-off the firm has to solve. The trade-off results from the cost and benefit of assigning the employee on the basis of the observed abilities, *i.e.* of being

[2]And teachers do not consider science and humanities as substitutes factors of knowledge.

informative. As we will see, the solution crucially relies on the level of complementarity between the tasks.

We derive three equilibria from the model. In the first equilibrium, the manager assigns the employee according to his observed type. This equilibrium matches the employee's highest ability to the most productive task of the position. This equilibrium is symmetrical: left employees are assigned to left positions, and *vice versa*. This equilibrium is informative: an employee derives correctly his type from the assignment. Upon this equilibrium, the employee correctly infers that he is "good" or "bad" with respect to the abilities involved. Then he derives higher utility from his contribution in the most productive task of the assignment, and lower utility from his contribution in the least productive task. Then he exerts higher effort in the most productive task, where he is more able, and lower effort in the least productive task, where he is less able.

In the two other equilibria, the assignment does not necessarily match the employee's highest ability to the highest productive task of the position. These equilibria are asymmetrical. They derive from asymmetrical beliefs of the employee with respect to the assignments.

To get the intuition, suppose an employee would be more likely assigned to the left position, unless his left ability is substantially below his right ability. Such an employee believes it is sufficient to be "not so bad" in the left task for being assigned to the left position. Equivalently he believes he would be assigned to the left position unless he is "very bad" in the left task. Upon such a belief, in case he observes the "very bad" assignment, he would infer a very low left ability, relative to his right ability. Then he would exert a very low effort in the left task and an high effort in the right task in that position. If on the contrary he observes the expected "not so bad" assignment, he would exert medium efforts in both tasks. Therefore his couple of efforts conditional on the assignment are asymmetrical. Such an asymmetry of efforts is anticipated by the manager. Because of complementarities, the similar efforts upon the "not so bad" message are more productive than the divergent efforts upon the "very bad" message. Therefore the manager is more likely to deliver the "not so bad" message, *i.e.* to assign the employee to the left position, as expected.

Note that this might occur against the observed employee's type, i.e. even if the employee has higher right ability. Furthermore, the more complementary the tasks are, the more severe the impact of the divergent efforts is. This results in a more asymmetrical assignment rule.

Now consider the efficiency, *i.e.* the preferred equilibrium of the manager, previous to her observation of abilities. If "good" and "bad" messages are delivered, employees exerts high and low effort according to their type, and according to the most productive task in their position. But complementarities increase the cost of the low effort, relative to the benefit of the matching of types. Thus complementarity decreases the efficiency of the symmetrical equilibrium. In contrast, if "not so bad" and "very bad" messages are delivered, employees exert similar medium efforts in both tasks if the "not so bad" message is observed, and divergent efforts if the "very bad" message is observed. The later case is very detrimental with complementarity, but it occurs for few employees, because it is delivered conditional on a narrow range of abilities. Furthermore, the more complementarity there is, the less it occurs. Then complementarity decreases the cost of misallocation, relative to the benefit of the similar medium efforts. Thus complementarity increases the efficiency of the asymmetrical equilibrium.

Then with more complementarities, the cost and benefit of being informative, derived from the symmetrical equilibrium, turns out to be higher than the cost of misallocation and the benefit of withholding information, derived from the asymmetrical equilibria. The firm would then prefer to commit to the asymmetrical equilibria if it could.

A key assumption of the model is that the manager has private information concerning the employee's abilities. This is due, for instance, to her superior information concerning the tasks to be done. Upon this assumption, Bénabou and Tirole (2003) investigate the informative effect of incentives on the employee. The authors show that giving the employee high incentives might convey bad news about his ability, in contrast with the usual agency theory. In the same line, Bénabou and Tirole (2006) model the psychological "overjustification effect" of rewards or punishments, that may crowd out the self-image motivation to contribute.

In our model, the manager's decision convey positive and negative informations simultaneously. The assignment has a positive effect on the motivation in the task primarily associated with the position obtained, but it has also a negative effect on the motivation in the other task. This result derives from the implicit comparison brought by the assignment (Chakraborty and Harbaugh, 2007).

In equilibrium, the manager responds to symmetrical or asymmetrical beliefs of the employee, and the employee responds consequently to a symmetrical or asymmetrical assignment rule. Then a crucial question of our model is the selection of equilibrium. Efficiency points to the asymmetrical equilibria. We furthermore show that the symmetrical equilibrium is unstable, while the asymmetrical ones are stable. One may furthermore interpret that the asymmetrical assignment conforms to some common prejudice. For instance, men or women may have different self-perceptions of their role in society, to which they associate specific attributes of jobs.[3] According to Bertrand (2011), occupational segregation by gender have been shown to explain much of the gender gap in earnings. The author notes: "First and foremost is the possibility that there are important differences in psychological attributes and preferences between men and women that may make some occupations more attractive to women and others more attractive to men".

In this line, an employee's motivation concerning a position would be asymmetrical with respect to the primarily task associated with that position. Such an employee would

[3]For instance, Mellström and Johannesson (2008) experimentally investigate the effect of monetary incentives on blood donors. The results differ between men and women, with a significant crowding out effect for women only. This suggests that women would be more pro-social than men.

exert asymmetrical efforts with respect to the different positions. Our main result is that upon complementarities, the firm has interest to conform to such wrong beliefs. Individuals would get trapped themselves in an equilibrium that goes against their comparative advantage. Furthermore, in contrast with statistical discrimination (Arrow, 1973; Phelps, 1972), this would occur although the interaction takes place with a perfectly informed principal.

The favoritism we consider does not result from any subjective evaluation that would be counter-productive (Prendergast and Topel, 1993; Bentley W., 2003), nor it is sustained by inefficient practices ((Becker, 1957), see also Mechtenberg (2009) where biased teachers inefficiently misallocate students). In contrast, the model gives a rational reason to the principal to chooses the asymmetrical equilibrium, if he could, in line with the firm's interest.

Our work is closely related to the recent literature on confidence management (Crutzen et al., 2013; Kamphorst and Swank, 2016). In Crutzen et al. (2013), the authors study the impact on efforts of providing employees with their rank concerning their abilities. The authors show that in general, the firm has interest to withhold such an information. However, the authors focus on symmetrical equilibria, and exclude the asymmetrical ones from their scope. In the same line, Kamphorst and Swank (2016) investigate the promotion decision of a manager concerning two employees, with regard to their abilities. As we do, the authors derive asymmetrical equilibria from wrong beliefs of the employees. However, Kamphorst and Swank (2016) note the inefficiency of the asymmetrical equilibria they exhibit.

The paper is organized as follows. We present the model setup in Section 2, and optimal strategies in Section 3. Results are established in Section 4. Existence of the asymmetrical equilibria for each level of complementarity between the tasks is derived in Section 4.1. Then, in Section 4.2, we associate the complementarity level to the degree of asymmetry occurring at those equilibria. In Section 4.3, we give argument concerning the robustness of the effect of complementarity. In Section 4.4, we investigate selection criteria. The asymmetrical equilibria are shown to be stable, and efficient with enough complementarity. Finally, in Section 4.5, we study the impact of the differentiation of the tasks within each position. In Section 5, we conclude. Proofs are given in Appendix.

2 Benchmark Model

We consider an informed manager that must assign an employee to one of two positions. We denote by m, $m \in \{1, 2\}$, the position to which the employee is assigned. The activities in each position involve two types of task, $i = 1$ and $i = 2$. Both positions involve the same types of tasks.

The employee has two abilities a_1 and a_2 concerning the tasks. We assume that abilities are independently drawn by Nature from the uniform distribution on the interval $[0, 1]$. The distribution of abilities is assumed to be known by the employee.

The employee exerts effort e_i on task i. The employee's contribution in each task relies on his abilities a_i and his efforts e_i. We assume that effort and ability are complementary factors of his contribution. In position m, on task i, the employee contributes to a level of

$$y_i(m) = \eta_i(m) a_i e_i, \tag{1}$$

where

$$\eta_i(m) = \begin{cases} 1 \text{ if } i = m, \\ \eta \geq 1 \text{ if } i \neq m, \end{cases}$$

represents the productivity of task i (resp. $\neg i$) relative to the other task $\neg i$ (resp. i) in position m.

We assume that the output derived from the employee's contribution in each task of position m is given by the CES function

$$Y(m) = \left(\frac{y_1^r(m)^r + y_2^r(m)}{2} \right)^{\frac{1}{r}} \tag{2}$$

where $r \in (0,1]$ measures the complementarity between the tasks[4]. The manager makes her decision m^* in order to maximize the aggregate output (2).

The employee has no expertize on the tasks to be done, in contrast to the manager. In particular, he is not aware of his abilities, nor of the level of complementarity r. The employee derives utility from his contribution in each task, and disutility from effort in each task. In task i, the employee's utility is given by

$$y_i(m) - \frac{1}{2}e_i^2,$$

where $\frac{1}{2}e_i^2$ represents his cost of effort. Then the utility function of the employee is given by

$$U(e_1, e_2) = \sum_{i=1,2} \left(y_i(m) - \frac{1}{2}e_i^2 \right). \tag{3}$$

The employee is not aware of his abilities. Conditional on the decision m, the employee chooses his efforts' levels $e^*(m) = (e_1^*(m), e_2^*(m))$ to maximizes his expected utility

$$E[U|m].$$

Note that our assumptions involve few symmetries. In particular, the relative productivity of the important task of each position, represented by η, is constant across the positions. Also, the degree of complementarity of the tasks is constant across the positions and within each position. Although these assumptions could seem artificial, it is a feature of our model. Precisely, they permit us to derive the asymmetrical mechanism of favoritism endogenously, *i.e.* without any assumed asymmetry.

The timing of the game is as follows:

[4]Recall that the elasticity of substitution $\rho_{1/2}(y_{1,m}, y_{2,m})$ of $Y(m)$ is then $\rho = \frac{1}{1-r}$. As r tends to 1, ρ goes to $+\infty$: substitution between factors tends to be perfect. As r goes to 0, ρ converges to 1: the factors are more and more complements. We exclude the case $\rho < 1 \iff r < 0$ of strict complementarity between the tasks because in that case, one can easily show that the assignment to position m would imply a lower expected ability in the most productive task, reverting the incentive role of the assignment. Then the players would enter in an infinite loop of lies.

1. Nature draws abilities a_1 and a_2, and reveals them to the manager, but not to the employee (he has a uniform prior);

2. the manager assigns the position m to the employee;

3. the employee observes the manager's decision and update his beliefs about his abilities;

4. the employee chooses his level of efforts $e_i(m)$ according to his posterior beliefs;

5. payoffs are realized.

We look for perfect Bayesian equilibria of this game, that is: (i) employee's efforts strategy is optimal, given his beliefs about his abilities; (ii) the manager's decision strategy is optimal, given the employee's efforts strategies and beliefs; (iii) beliefs are updated according to Bayes' rule.

3 Strategies

Given the decision m, the employee exerts effort to maximize his expected utility

$$(e_1^*(m), e_2^*(m)) = \arg\max_{(e_1,e_2)} \; E(U(e_1, e_2)|m).$$

Reciprocally, given the effort strategy $m \mapsto (e_1(m), e_2(m))$ of the employee, the manager decides in order to maximize the aggregate output $Y(m)$. For instance, the manager decides $m = 1$ if

$$\left(\frac{y_1^r(1) + y_2^r(1)}{2}\right)^{\frac{1}{r}} > \left(\frac{y_1^r(2) + y_2^r(2)}{2}\right)^{\frac{1}{r}}. \tag{4}$$

Note that given the effort strategy of the employee, the set of abilities (a_1, a_2) such that the manager decides $m = 1$ (resp. $m = 2$) is defined up to the set of abilities (a_1, a_2) such that $\left(\frac{y_1^r(1)+y_2^r(1)}{2}\right)^{\frac{1}{r}} = \left(\frac{y_1^r(2)+y_2^r(2)}{2}\right)^{\frac{1}{r}}$. In particular if $e_1(1) = e_1(2)$ and $e_2(1) = e_2(2)$ then $y_1(1) = y_1(2)$ and $y_2(1) = y_2(2)$. Then the manager has no incentive to decide either

$m = 1$ or $m = 2$. Reciprocally, in case the decision does not rely on the employee's abilities, the efforts in each task are equal across the decisions. This conditions define a *babbling equilibrium*, upon which the manager randomizes its decisions.

Henceforth, let us assume $e_1(1) \neq e_1(2)$ or $e_2(1) \neq e_2(2)$.

In case $r > 0$, (4) gives $m = 1$ if

$$\begin{aligned} & y_1^r(1) + y_2^r(1) > y_1^r(2) + y_2^r(2) \\ \iff & a_1^r \left(\eta^r e_1^r(1) - e_1(2)^r\right) > a_2^r \left(\eta^r e_2^r(2) - e_2^r(1)\right) \end{aligned}$$

If $e_1^r(1) > e_1^r(2)$, and $e_2^r(2) \geq e_2^r(2)$, that is if $e_1(1) > e_2(1)$ and $e_2(2) \geq e_1(2)$ since $r > 0$, then we obtain $m = 1$ if

$$\begin{aligned} & a_1 > a_2 \left(\frac{\eta^r e_2^r(2) - e_2(1)^r}{\eta^r e_1^r(1) - e_1(2)^r}\right)^{\frac{1}{r}} \\ \iff & a_1 > t \times a_2, \end{aligned} \tag{5}$$

where

$$t = \left(\frac{\eta^r e_2^r(2) - e_2(1)^r}{\eta^r e_1^r(1) - e_1(2)^r}\right)^{\frac{1}{r}}. \tag{6}$$

Similarly, upon the same conditions, we have $Y(2) > Y(1)$ if $a_1 < t \times a_2$, and $Y(2) = Y(1)$ if $a_1 = t \times a_2$. Since the later case occurs with a zero probability, let us gather it w.l.o.g. with the case $Y(1) > Y(2)$. The following lemma permits us to focus on this specific strategy for the manager.

Lemma 1. *If $r > 0$, then up to a relabeling of the decisions, upon an influential equilibrium it is necessary that $e_1(1) > e_2(1)$ and $e_2(2) > e_1(2)$, and the manager conforms necessarily to the following* decision-making rule*:*

$$\begin{cases} m^* = 1, \text{ if } a_1 \geq t a_2, \\ m^* = 2, \text{ if } a_1 < t a_2, \end{cases} \tag{7}$$

where t is given by (6)*.*

There is no equilibrium if $r < 0$.

The proof is given in Appendix. Let us now derive the effort strategy of the employee, conditional on the decision-making rule (7) of the manager. Then for instance

$$\begin{aligned}(e_1^*(1), e_2^*(1)) =& \arg\max_{(e_1,e_2)}\ E(U(e_1,e_2)|m=1)\\ =& \arg\max_{(e_1,e_2)} \iint\limits_{a_1 \geq ta_2} \left(\sum_{i=1,2} \left(y_i(1) - \frac{1}{2} e_i^2 \right) \right) \mathrm{d}a_1 \, \mathrm{d}a_2\end{aligned}$$

and since U is separable across the tasks, we obtain

$$\begin{aligned}e_i^*(1) =& \arg\max_{e_i} \iint\limits_{a_1 \geq ta_2} \left(\eta_i(1) a_1 e_1 - \frac{1}{2} e_1^2 \right) \mathrm{d}a_1 \, \mathrm{d}a_2\\ =& \eta_i(1) E[a_1 | a_1 \geq ta_2],\end{aligned}$$

where $E[a_1|a_1 \geq ta_2]$ represents the expected ability of the employee conditional on the decision. Then we obtain

$$\begin{aligned}e_1^*(1) = \eta E[a_1|a_1 \geq ta_2]; \quad e_2^*(1) = E[a_2|a_1 \geq ta_2];\\ e_1^*(2) = E[a_1|a_1 < ta_2]; \quad e_2^*(2) = \eta E[a_1|a_1 < ta_2].\end{aligned} \tag{8}$$

Now given the uniform prior of abilities $(a_1, a_2) \in [0,1]^2$ for the employee, we compute the optimal efforts levels and find, if $t \leq 1$:

$$\begin{aligned}e_1^*(1) &= \eta \tfrac{1}{3} \tfrac{3-t^2}{2-t}, \quad & e_2^*(1) &= \tfrac{1}{3} \tfrac{3-2t}{2-t},\\ e_1^*(2) &= \tfrac{t}{3}, & e_2^*(2) &= \eta \tfrac{2}{3};\end{aligned} \tag{9}$$

and similarly, if $t \geq 1$, at:

$$e_1^*(1) = \eta \frac{2}{3}, \quad e_2^*(1) = \frac{t}{3}, \quad e_1^*(2(t) = \frac{1}{3} \frac{3-2t}{2-t}, \quad \eta e_2^*(2) = \eta \frac{1}{3} \frac{3-t^2}{2-t}.$$

Figure 1 represents a decision-making rule (7) with $t \leq 1$, and the corresponding effort levels $e_i^*(m)$ with $\eta = 1$.

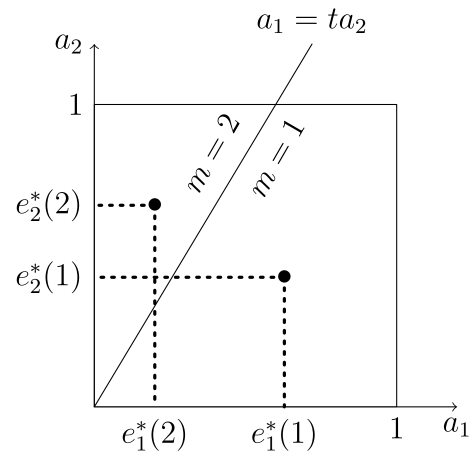

Figure 1: The decision-making rule, and the according beliefs of employees

From (6) and (9), an equilibrium occurs at any $t \in \mathbb{R}^+$ which solves

$$t = \left(\frac{(\eta e_2^*(2))^r - e_2^*(1)^r}{(\eta e_1^*(1))^r - e_1^*(2)^r} \right)^{\frac{1}{r}}.$$

Then if $t \leq 1$, an equilibrium occurs at $t \in (0, 1]$ which solves

$$t = \left(\frac{\left(\eta^2 \frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\eta^2 \frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r} \right)^{\frac{1}{r}}.$$

Setting for any $r \in (0, 1]$,

$$f_r(t) = \left(\frac{\left(\eta^2 \frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\eta^2 \frac{1}{3}\frac{3-t^2}{2-t})\right)^r - \left(\frac{t}{3}\right)^r} \right)^{\frac{1}{r}},$$

an equilibrium occurs at any $t \in (0, 1]$ which solves

$$t = f_r(t). \tag{10}$$

If $t \geq 1$, an equilibrium occurs *iff* $t \in [1, +\infty)$ is solution of $t = \frac{1}{f_r(t)}$, which amounts to say that t is the inverse of a solution of (10). The case $t \geq 1$ mirrors the case $t \leq 1$ with respect to a permutation of the tasks' labels.

Note that from (7), if $t < 1$, then the manager may assign the employee to position $m = 1$ even if the employee's comparative advantage is not his ability in tasks of type 1.

The employee's ability is misallocated relative to the most productive task of that position. Symmetrically, if $t > 1$, employees might be assigned to position 2 despite an higher ability in task 1. The parameter t in the decision-making rule of the manager represents thus some degree of "favoritism" of the decision with respect to some position against the other. Note that in equilibrium this relies on the beliefs of the employee. Henceforth, decision-making rules with t far from 1 (t near to 0 or much greater than 1) are said to be of high *degree of favoritism*, and decision-making rules with t close to 1 are said to be of low degree of favoritism. There is no favoritism when $t = 1$. We call t the *level of favoritism* of the decision-making rule (so that if $t \leq 1$, $1 - t$ represents the degree of favoritism).

4 Results

We first study the case of equal productivity of the tasks within positions, that is we assume $\eta = 1$. Then, the negative impact of an eventual misallocation derives from the lack of matching of high ability with high effort, which are complementary. Note that in the case $\eta = 1$, the assignment to the position may be interpreted as a *cheap talk* from the manager. The manager makes one of two recommendations to the employee, on the behalf of her private information. Then the existence of an influential equilibrium, concerning the employee's efforts, requires *credible* recommendations.

In particular, this setting permits us to isolate the informative aspect of the assignment. In Section 4.5, we will consider the impact of $\eta > 1$ on the equilibrium.

4.1 Equilibria

Let us first remark that an equilibrium occurs at $t = 1$. Indeed, $t = 1$ is a solution of (10). At $t = 1$, the manager's decision-making rule is

$$\begin{cases} m^* = 1, \text{ if } a_1 \geq a_2, \\ m^* = 2, \text{ if not,} \end{cases}$$

and the employee's optimal efforts are

$$e_i^*(m) = \begin{cases} \frac{2}{3}, \text{ if } m = i, \\ \frac{1}{3}, \text{ if } m \neq i. \end{cases}$$

Hence the manager assigns the employee to the position that matches his highest ability with the most important task of the position, and the employee exerts higher effort where he is more able, and more productive. In particular, this rule does not favor the assignment on one position or the other.

The next proposition asserts that besides the above symmetrical equilibrium, there are *asymmetrical equilibria.*

Proposition 1. *For each* $r \in (0, 1]$*, there exists a unique equilibrium with level of favoritism* $\underline{t}(r)$*, with* $0 < \underline{t}(r) \leq \frac{1}{2}$*, and a unique equilibrium with level of favoritism* $\overline{t}(r)$*, with* $\overline{t}(r) \geq 2$.

We relegate the proof of Proposition 1 to the Appendix. Proposition 1 ensures that on both sides of the symmetrical equilibrium occurring at $t = 1$, there are two other equilibria with level of favoritism $\underline{t}(r) \leq \frac{1}{2}$ and $\overline{t}(r) \geq 2$. At these equilibria, the manager assigns an employee to the position that does not necessarily match the highest ability of the employee to the most productive task, and the employee does not necessarily exert his highest effort on the task he his more able.

In order to isolate the effect of complementarity, let us first examine the case of perfect substitution. Under perfect substitution, the manager decides upon the comparison of

$a_1e_1^*(1)+a_2e_2^*(1)$ and $a_1e_1^*(2)+a_2e_2^*(2)$. In particular, in equilibrium the manager compares the sum of abilities weighted by the effort's level in each task.

Let us derive the levels of effort obtained from an asymmetrical assignment rule, and show how they confirm the rule's asymmetry. Assume w.l.o.g. that Position 1 is favored, so that the manager decides $m = 1$ as soon as $a_1 \geq ta_2$, with $t < 1$.

Then note first that the two possible assignments distinguish one another by their informativeness concerning the abilities. Indeed, since Position 1 is favored, the employee expects to be assigned to Position 1. Decision $m = 1$ is poorly informative. In contrast, Decision $m = 2$ is unexpected. It is observed as the result of contrasted abilities. Then Decision $m = 2$ is more informative than $m = 1$.

Second, the asymmetry of informativeness leads to asymmetrical levels of effort. Indeed, because $m = 1$ is poorly informative, the derived effort are close to the prior $E[a]$, and then close one another. In contrast, efforts are far from one another when $m = 2$. Precisely, when $m = 2$ the effort in task 1 is strongly lowered relative to the effort in task 2.

Finally, the asymmetry of effort's levels leads to a comparison of aggregate outputs that confirms the rule's asymmetry. Indeed, given the loss of effort in task 1 in case $m = 2$, Decision $m = 2$ is made when the contribution in task 2 compensates the poor contribution in task 1, relative to the contributions in case $m = 1$. The rule's asymmetry is confirmed because in case $m = 2$ the gain in effort in task 2 does not compensates the poor contribution in task 1, so that the ability in task 2 must do it. Then Decision $m = 2$ is made only if a_2 is much higher than a_1.

Figure 2 illustrate the various optimal level of effort $e_i^*(m)$, with respect to the level of favoritism $t = 1$, conditional on the uniform distribution of abilities we have assumed.

At $t = 1$, $e_2^*(2) - e_2^*(1) = e_1^*(1) - e_1^*(2)$ so that the gain in effort in task 2 does compensates the loss of effort in task 1. This confirms the symmetrical rule. While t goes from 1 to 0, $e_2^*(2) - e_2^*(1)$ decreases and $e_1^*(1) - e_1^*(2)$ increases. Then the gain in effort in task 2 does not compensates the loss of effort in task 1, confirming the asymmetrical rule. In equilibrium, at $t = \frac{1}{2}$ the lack of compensation precisely identifies with the level of favoritism.

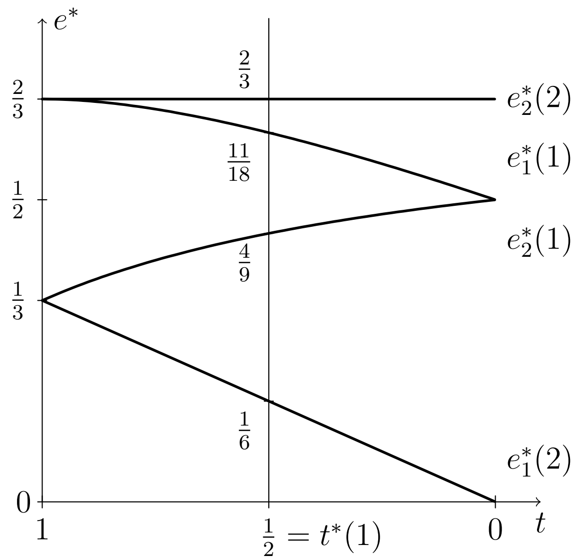

Figure 2: Optimal efforts as a function of the level of favoritism t

4.2 Complementarity exacerbates favoritism

Proposition 1 is an existence result. As such, nothing is said about the degree of favoritism in equilibrium. The following proposition associates the complementary level of the tasks to the degree of favoritism in equilibrium.

Proposition 2. *a) Favoritism is exacerbated with complementarity : the more complementarity the tasks are (r decreasing), the more discriminatory the decision-making rule of the manager is ($\underline{t}(r)$ decreases, and $\overline{t}(r)$ increases).*

b) Moreover, when complementarity equals substitutability between the tasks ($r \to 0$), favoritism reaches its maximum degree ($\underline{t}(r) \to 0$, $\overline{t}(r) \to \infty$), i.e., the employee obtains the favored position whatever his abilities.

We relegate the proof of Proposition 2 to Appendix. Figure 3 illustrates the degree of favoritism occurring at $t = 1$ (the symmetrical equilibrium) and at $\underline{t}(r)$ (the asymmetrical equilibrium) with respect to the complementary level r.

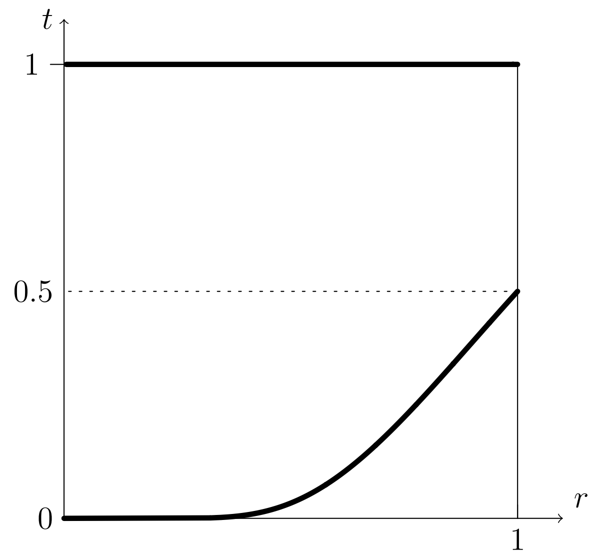

Figure 3: Discrimination degree occurring at the equilibria

At $r = 1$ (perfectly substitutable case), we have $\underline{t}(1) = \frac{1}{2}$, and as r decreases, the asymmetrical equilibrium occurring at $\underline{t}(r)$ is more and more discriminatory. It goes to a maximum degree of favoritism ($t = 0$) when complementarity equals substitutability between the tasks ($r \to 0$).

Let us now provide the details of the effect of complementarity on the asymmetrical equilibria.

Assume w.l.o.g. that Position 1 is favored. Then the manager decides upon the comparison of $a_1 \left(e_1^r(1) - e_1^r(2)\right)^{\frac{1}{r}}$ and $a_2 \left(e_2^r(2) - e_2^r(1)\right)^{\frac{1}{r}}$ with $e_1^r(1) - e_1^r(2) > e_2^r(2) - e_2^r(1)^r$, that is with

$$e_1^r(1)^r + e_2^r(1) > e_2^r(2) + e_1^r(2). \tag{11}$$

Recall that when Position 1 is favored, Decision $m = 1$ is poorly informative and thus $e_1(1)$ and $e_2(1)$ get close to the prior $E[a]$. In contrast, Decision $m = 2$ is highly informative, and $e_1(2)$ and $e_2(2)$ represent respectively low and high efforts. With complementarities, divergent efforts are less productive than convergent efforts. Precisely, the more concave

$(x, y) \mapsto x^r + y^r$ is, the more likely (11) holds. And the more likely (11) holds, the more discriminatory the assignment is.

Therefore, favoritism is exacerbated because (i) efforts conditional on the expected decision are similar, while efforts conditional on the unexpected decision are divergent, and (ii) the impact of divergent efforts on the aggregate production is more detrimental with complementarity.

4.3 Robustness to the prior

The asymmetrical equilibria derive from the comparison of the anticipated levels of efforts across the decisions. In case of perfect substitutability between tasks, the favoritism comes from the lack of compensation of effort in one task relative to the fall down of effort in the other task conditional on the unexpected decision. This crucially relies on the prior distribution of abilities. In contrast, with complementarities, the asymmetrical equilibria derive from the similarity of efforts conditional on the expected decision and the divergence of effort conditional on the unexpected decision. As we will see, this is robust to change of prior. To illustrate this robustness, we investigate a specific prior distribution of abilities which precludes the asymmetrical equilibria in case of perfect substitution. The prior is chosen so that the cause of the asymmetrical decisions is ruled out in case of perfect substitution.

Specifically, we consider a uniform prior on the set of abilities (a_1, a_2) such that

$$a_1 + a_2 \leq 1,$$

$$a_1 \geq 0,\ a_2 \geq 0,$$

as illustrated in Figure 4. Upon such a prior, an employee who is revealed to be good in task 1 infers he is bad in task 2 and reciprocally. It might represent a situation where the tasks require incompatible abilities. Say for instance that task 1 requires assertiveness, while task 2 requires passivity.

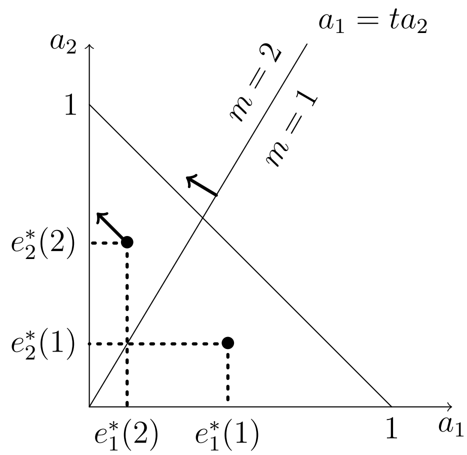

Figure 4: The decision-making rule with dependent prior abilities

With such a prior distribution of abilities, we obtain, if $t \leq 1$,

$$e_1^*(1) = \frac{1}{3}\frac{2t+1}{1+t}, \quad e_2^*(1) = \frac{1}{3}\frac{1}{1+t}, \quad e_1^*(2) = \frac{1}{3}\frac{t}{1+t}, \quad e_2^*(2) = \frac{1}{3}\frac{t+2}{1+t},$$

so that for any $t \leq 1$,

$$e_1^*(1) - e_1^*(2) = e_2^*(2) - e_2^*(1).$$

Under perfect substitution between the tasks, we obtain a unique symmetrical equilibrium:

$$t = \frac{e_2^*(2) - e_2^*(1)}{e_1^*(1) - e_1^*(2)} \iff t = 1,$$

so that this setting precludes the asymmetrical equilibria.

However, note that (11) holds with enough complementarity (r sufficiently low), even if $e_1^*(1) - e_1^*(2) = e_2^*(2) - e_2^*(1)$.

Figure 5 illustrates the degree of favoritism as a function of the complementarity level in this setting. The symmetrical equilibrium is the only equilibrium with low complementarity, but increasing the complementarity level (as r decreases) reintroduces the asymmetrical equilibria.

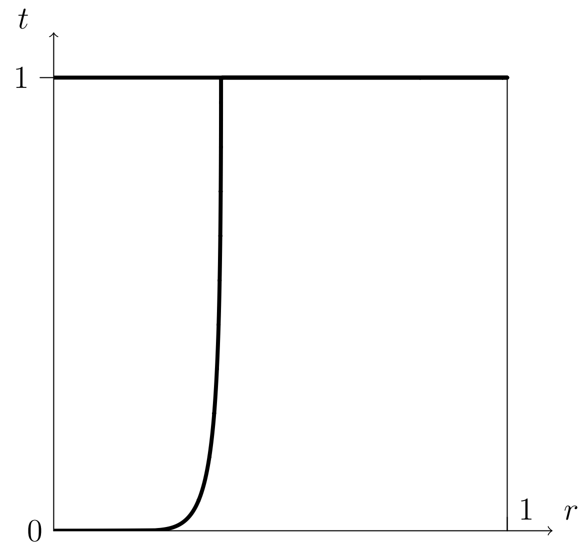

Figure 5: Discriminatory equilibrium with complementarity in case of dependent abilities

4.4 Selection

We investigate two arguments to select between the equilibria. Stability and efficiency both point to the asymmetrical equilibria.

4.4.1 Stability

We say that an equilibrium is stable if, starting from beliefs not too far from the equilibrium's beliefs, then the resulting best responses of the agents (employee and manager) lead to beliefs closer to the equilibrium's beliefs. It is said unstable if not.

Proposition 3. *For each complementary level, the symmetrical equilibrium is unstable and the asymmetrical equilibria are stable. Furthermore, the more complementary the tasks are, the more important is the amplification of an anticipated small degree of favoritism from the agents.*

The symmetrical equilibrium is unstable because of the confirmation and amplification, by the manager, of any anticipated level of favoritism near to 1 from the employee. When

$\hat{t} < 1$ with $\hat{t} \cong 1$, the employee anticipates the decision-making rule of the manager to be in favor of Position 1, but slightly discriminatory. Up to a certain extent, the response of the manager confirms and amplifies this favoritism when responding $f_r(\hat{t}) < \hat{t}$. Symmetrically, the manager's response $f_r(\hat{t})$ is more discriminatory than the anticipated $\hat{t}$ when $\hat{t} > \underline{t}(r)$. There is a limit to this discriminatory effect because the comparison of abilities does operate as a force into the decision-making rule of the manager. Indeed, effort and ability being complementary in the output, when the ability is too low, effort matters less. The manager's best response $f_r(\hat{t})$ will be less discriminatory than the anticipated $\hat{t}$ of the employees when $\hat{t} < \underline{t}(r)$.

Let us note that the more complementarity there is, the more stable the asymmetrical equilibria are. For instance, at $r = \frac{3}{4}$ we compute at the symmetrical equilibrium $f'_{\frac{3}{4}}(1) \cong 3$, and at asymmetrical equilibrium $t^*(r) \cong \frac{1}{4}$ we compute $f'_{\frac{3}{4}}(\frac{1}{4}) \cong 0.35$. This implies: if $\hat{t}$ is the level of favoritism anticipated by the employee, at a distance of $\left|\hat{t} - \underline{t}(r)\right|$ of $\underline{t}(r)$, then the responding level of the manager $f_r(\hat{t})$ satisfies $\left|f_r(\hat{t}) - \underline{t}(r)\right| \leq 0.35 \left|\hat{t} - \underline{t}(r)\right|$, hence almost three times closer to the equilibrium level.

4.4.2 Efficiency

Let us now investigate the efficiency of the equilibria. From the firm's perspective, it is measured as the expected value $E(Y_t)$ of the aggregate production, with regard to abilities. It represents the expected payoff of the manager before he observes the abilities.

Proposition 4. *From the firm's perspective, with complementarity, favoritism becomes more and more efficient. Favoritism is more efficient than no favoritism when the maximum degree of favoritism is reached, i.e., when complementarity equalizes substitutability between tasks (as $r \to 0$).*

The intuition of Proposition 4 relies on the interaction between informativeness of the equilibria and complementarity between the tasks. The symmetrical equilibrium is symmetrically informative concerning the abilities. Whatever is the decision, it leads to divergent efforts. The informativeness of an asymmetrical equilibrium depends on the decision. With respect to the favored position, it is highly informative when the assignment is unexpected, and not so informative when the assignment is expected. However, the more favoritism there is, the more likely the expected assignment is. Therefore the informativeness of an asymmetrical equilibrium decreases with the degree of discrimination.

With a low level of complementarity ($r \cong 1$), effort being complementary to ability, and aggregate output being linear in the individual outputs, it is likely to be higher when the highest ability of the employee is associated with his highest effort. Revealing information is thus efficient. In contrast, with complementarity, as we have seen, revealing information tends to decrease the manager's payoff.

With this in consideration, let us compute the optimal level of favoritism the manager would commit to if she could. At any $t \leq 1$, and level $r = \frac{1}{n}$ with $n \in \mathbb{N}^*$, we find

$$\begin{aligned}
E(Y_t) = &\Pr(m=1) \iint\limits_{a_1 \geq t a_2} \left(\frac{(a_1 e_1(1))^{\frac{1}{n}} + (a_1 e_2(1))^{\frac{1}{n}}}{2} \right)^n \mathrm{d}a_1 \, \mathrm{d}a_2 \\
&+ \Pr(m=2) \iint\limits_{a_1 < t a_2} \left(\frac{(a_1 e_1(2))^{\frac{1}{n}} + (a_1 e_2(2))^{\frac{1}{n}}}{2} \right)^n \mathrm{d}a_1 \, \mathrm{d}a_2 \\
= &\frac{1}{2^n} \sum_{k=0}^{k=n} \left(\frac{n!}{k!(n-k)!} \frac{1}{3(1+\frac{k}{n})} \times \right. \\
&\left. \left(e_1(1)^{\frac{k}{n}} e_2(1)^{1-\frac{k}{n}} \frac{3 - t^{1+\frac{k}{n}}(2-\frac{k}{n})}{2-\frac{k}{n}} + e_1(2)^{\frac{k}{n}} e_2(2)^{1-\frac{k}{n}} t^{1+\frac{k}{n}} \right) \right)
\end{aligned}$$

As seen in Figure 6, the optimal t for the manager to commit to is non discriminatory only in case of no complementarity between the tasks ($r = 1$). With complementarity, the manager prefers, if she could, commit to an out of equilibrium level of favoritism. Considering the equilibria, there is a threshold $\tilde{r}$ for the complementarity level ($\tilde{r} \cong \frac{1}{3}$) below which the manager prefers to commit to an asymmetrical equilibrium as soon as $r \leq \tilde{r}$.

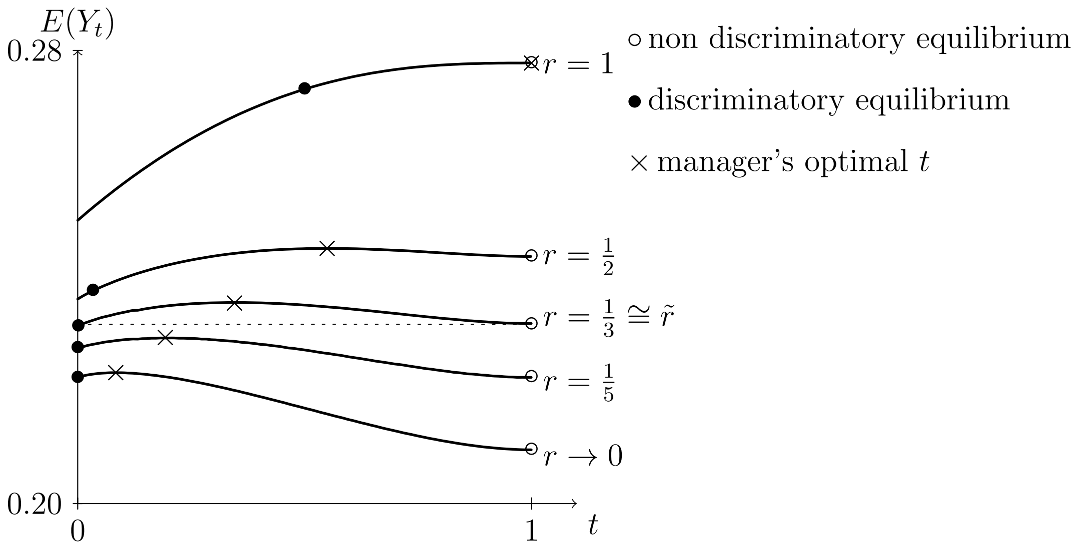

Figure 6: Efficiency with regard to the degree of favoritism t

4.5 Design of the tasks

Let us now rule out the purely informative aspect of the manager's decision, and consider differentiated task relative to their productivity. Recall that $\eta \geq 1$ measures the differentiation between the tasks in the output.

Clearly, the more important the difference of the tasks is, the more it should prevent the manager to misallocate the abilities. Our aim is to investigate the impact of complementarity on that effect.

The differentiation of the tasks put an increasing importance on the ability and effort in the most important task of a position. Therefore the force that leads to favoritism vanishes, and the manager's incentive to favor a position decreases. However, complementarity severely mitigates this effect. High complementarity needs an extreme differentiation of the tasks for the favoritism to cease.

Proposition 5. *For any level of differentiation of the tasks η, the asymmetrical equilibria occurring at $\underline{t}(r)$ and $\overline{t}(r)$ still occur provided enough complementarity between the tasks.*

Figure 7 illustrates where the discriminatory equilibria occur for different values of η.

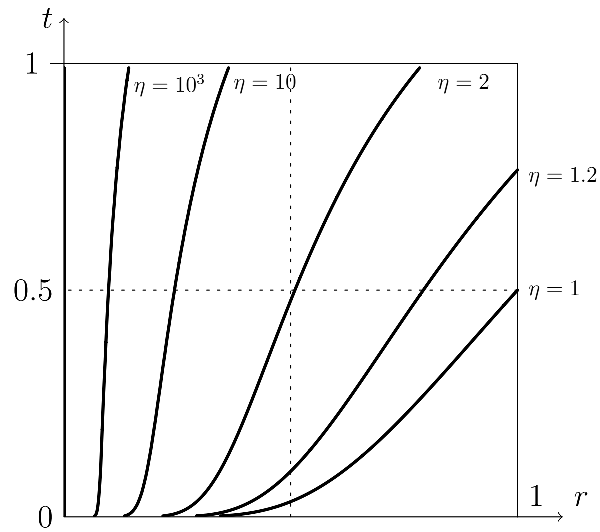

Figure 7: Equilibrium $\underline{t}(r)$ with respect to r and η

The more the task are differentiated, the less favoritism occurs. It provides a way out to favoritism by conveniently designing positions with respect to the tasks. However, complementarity requires an higher level of differentiation to avoid favoritism to occur. Consider for instance $\eta = 10^3$, so that the important task of a Position is a thousand times more productive than the least important task. Then reading Figure 7, the asymmetrical equilibria still occur with enough complementarity (r sufficiently small). Furthermore, with enough complementarity again, the degree of favoritism might still be very high ($\underline{t} \to 0$).

5 Conclusion

We have shown how asymmetrical treatments might endogenously arise when assigning employees to positions that involve multiple complementary tasks, and how more complementarity might exacerbate this asymmetry.

The mechanism we have exhibited concerns the processing of information derived from the assignment. A key assumption we have made is that employees perceive their abilities from the assignment, and exert their optimal efforts accordingly.

Positional assignments leads to divergent efforts when the assignment is based on the employees' abilities. This effect is detrimental to the production when positions involve complementary tasks. Asymmetrical assignment, derived from wrong beliefs of employees, mitigates the informative effect of the assignment. It thereby reduces the difference between the employee's contributions. This effect is profitable to the firm.

Such asymmetrical treatment arises without any taste for discrimination (Becker, 1957), nor exogenous asymmetry of information with respect to some type of employee (Arrow, 1973; Phelps, 1972). On the contrary, we have assumed perfect information from the manager concerning the abilities of the employee. Furthermore, the biased assignment does not derive from subjective evaluation that would be detrimental to the organization (Prendergast and Topel, 1993; Bentley W., 2003). On the contrary, it is in the interest of the firm to commit to such assignments when tasks are complementary enough. Conversely, such assignments are detrimental from the employees' perspective. They might be misinformed concerning their comparative advantage.

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A Proof of Lemma 1

Proof. The other cases with $r > 0$ are derived similarly to the case given in the lemma and proved in the text. The possible manager's strategies are given in Table 1.

Table 1: Designer's strategies in case $r > 0$

Strategy	Condition on efforts	Rule
(S_0)	$e_1(1) = e_1(2),\ e_2(1) = e_2(2)$	Random decision
(S_1)	$e_1(1) > e_1(2),\ e_2(1) \geq e_2(2)$	$m = 1$ *iff* $a_1 \geq ta_2$.
(S_2)	$e_1(1) < e_1(2),\ e_2(1) \leq e_2(2)$	$m = 1$ *iff* $a_1 \leq ta_2$.
(S_3)	$e_1(1) \geq e_1(2),\ e_2(2) < e_2(1)$	$m = 1$ *iff* $a_1 \geq 0$.
(S_4)	$e_1(1) \geq e_1(2),\ e_2(2) < e_2(1)$	$m = 2$ *iff* $a_2 \geq 0$.

In case $r < 0$, the inequalities in the Rule column of Table 1 must be reversed.

Let us note that in Case (S_3) and (S_4) the decisions are uninformative with respect to the abilities of the employees. Therefore $e_i(1) = e_i(2)$ for $i \in \{1, 2\}$ and the corresponding conditions on the level of efforts are not fulfilled. Therefore strategies (S_3) and (S_4) do not occur as equilibrium strategies.

In case (S_1), Decision $m = 1$ increases effort in the task $i = 1$ and decreases effort in task $i = 2$, and decision $m = 2$ produces the opposite effects. The case (S_2) mirrors the case (S_1) by switching the labels of the messages. Therefore w.l.o.g. we can focus on (S_1), *i.e.* (7) in case $r > 0$.

In case $r < 0$, conditional $e_1(1) > e_2(1)$ and $e_2(2) > e_1(2)$, the corresponding rule we can focus on is given by:

$$\begin{cases} m^* = 1, \text{ if } a_1 \leq ta_2, \\ m^* = 2, \text{ if } a_1 > ta_2. \end{cases} \tag{7'}$$

Now according to the optimal level of efforts of the employee (8), since $\eta \geq 1$, and

$$E[a_1|a_1 \geq ta_2] \geq E[a_1|a_1 < ta_2],\ E[a_2|a_1 \geq ta_2] \leq E[a_2|a_1 < ta_2],$$

the conditions on efforts given in Table 1 are fulfilled in case (S_1). In case $r < 0$, when the manager conforms to the rule (7'), the employee's efforts across the tasks does not fulfill the corresponding conditions. For instance, while it is necessary that $e_1^*(1) \geq e_1^*(2)$, the decision making rule (7') implies inferences that gives $e_1^*(1) < e_1^*(2)$. □

B Proof of Proposition 1

Proof. We will first prove the existence of each equilibrium, and next their uniqueness.

Existence. Let $r \in (0, 1]$. We will first prove that there exists (at least) a t_r^* such that

$$t_r^* = f_r(t_r^*)$$

with $0 < t_r^* \leq \frac{1}{2}$. By reversing the roles of Employee 1 and Employee 2, we obtain that there exists also (at least) an equilibrium in case $t \geq 1$ at a $t_r^* \geq 2$.

An equilibrium $0 < t_r^* \leq \frac{1}{2}$ exists if and only if

$$t = f_r(t)$$

has a solution $t = t_r^* \in (0, \frac{1}{2}]$. Since $t \mapsto f_r(t)$ is continuous, and $[0, \frac{1}{2}]$ is compact and convex, we get the result from Brouwer's fixed point theorem, if we prove that the range $[0, \frac{1}{2}]$ is stable under $t \mapsto f_r(t)$ (and ensure that $t = 0$ is not a solution of $t = f_r(t)$, but $f_r(0) \neq 0$ is straightforward). We will at the end furthermore prove that $f_r(t) < \frac{1}{2}$ if $r < 1$. Clearly, for any $r \in (0, 1]$ and any $t \in [0, \frac{1}{2}]$, $f_r(t) > 0$. So we need

$$f_r(t) \leq \frac{1}{2}, \tag{12}$$

with equality only at $r = 1$.

The proof of (12) is in two steps; first we will show

Step A.

$$\text{for any } r \in (0, 1], t \mapsto f_r(t) \text{ is increasing with } t \in \left[0, \frac{1}{2}\right], \tag{13}$$

so that for any $t \in [0, \frac{1}{2}]$, $f_r(t) \leq f_r(\frac{1}{2})$, and next we will show

Step B.

$$\text{for any } r \in (0,1], f_r\left(\frac{1}{2}\right) \leq \frac{1}{2}, \tag{14}$$

equality occurring only when $r = 1$.

Proof of Step A. For (13), from composition with $x \mapsto x^{\frac{1}{r}}$, which is increasing, it is enough to show that $t \mapsto \frac{\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r}$ increases. To this goal, we will state that:

$$t \mapsto \left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r \text{ increases with } t, \tag{13a}$$

while

$$t \mapsto \left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r \text{ decreases with } t. \tag{13b}$$

Proof of (13a). The derivative of $t \mapsto \left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r$ is $t \mapsto r\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r \frac{1}{(3-2t)(2-t)}$, which is positive for $t \in (0, \frac{1}{2}]$.

Proof of (13b). The derivative of $t \mapsto \left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r$ is $t \mapsto r\left(\frac{1}{3}\right)^r \left(\left(\frac{3-t^2}{2-t}\right)^r \frac{(1-t)(3-t)}{(3-t^2)(2-t)} - \frac{1}{t^{1-r}}\right)$. It is negative if and only if

$$\left(\frac{3-t^2}{2-t}\right)^r \frac{(1-t)(3-t)}{(3-t^2)(2-t)} \leq \frac{1}{t^{1-r}} \iff \left(\frac{3-t^2}{t(2-t)}\right)^r \leq \frac{(3-t^2)(2-t)}{t(1-t)(3-t)}. \tag{15}$$

For $t \in (0, \frac{1}{2}]$, we have $\frac{3-t^2}{t(2-t)} > 1$, hence $\left(\frac{3-t^2}{t(2-t)}\right)^r \leq \frac{3-t^2}{t(2-t)}$, and $\frac{3-t^2}{t(2-t)} \leq \frac{(3-t^2)(2-t)}{t(1-t)(3-t)}$, hence (15). This completes the proof of (13).

Thus for each $r \in (0,1]$, each $t \in [0, \frac{1}{2}]$,

$$f_r(t) \leq f_r\left(\frac{1}{2}\right) = \left(\frac{\left(\frac{2}{3}\right)^r - \left(\frac{4}{9}\right)^r}{\left(\frac{11}{18}\right)^r - \left(\frac{1}{6}\right)^r}\right)^{\frac{1}{r}}.$$

Proof of Step B. Our goal is (14), that is for all $r \in (0,1]$,

$$\left(\frac{\left(\frac{2}{3}\right)^r - \left(\frac{4}{9}\right)^r}{\left(\frac{11}{18}\right)^r - \left(\frac{1}{6}\right)^r}\right)^{\frac{1}{r}} \leq \frac{1}{2} \iff \left(\frac{2}{3}\right)^r - \left(\frac{4}{9}\right)^r \leq \left(\frac{11}{36}\right)^r - \left(\frac{1}{12}\right)^r. \tag{16}$$

The mean value theorem yields

$$\left(\frac{2}{3}\right)^r - \left(\frac{4}{9}\right)^r = \frac{r}{a^{1-r}}\left(\frac{2}{3} - \frac{4}{9}\right) = \frac{2r}{9}\frac{1}{a^{1-r}}$$

for a $a \in [\frac{4}{9}, \frac{2}{3}]$, and

$$\left(\frac{11}{36}\right)^r - \left(\frac{1}{12}\right)^r = \frac{r}{b^{1-r}}\left(\frac{11}{36} - \frac{1}{12}\right) = \frac{2r}{9}\frac{1}{b^{1-r}}$$

with $b \in [\frac{1}{12}, \frac{11}{36}]$. Since $\frac{4}{9} > \frac{11}{36}$, we have $a > b$, and thus $\frac{2r}{9}\frac{1}{a^{1-r}} \leq \frac{2r}{9}\frac{1}{b^{1-r}}$, which gives (16). Let's remark finally that for $r \in (0,1]$, the equality $\frac{2r}{9}\frac{1}{a^{1-r}} = \frac{2r}{9}\frac{1}{b^{1-r}}$ occurs only at $r = 1$.

Uniqueness. The proof of the uniqueness of an equilibrium $t_r^* = \underline{t}(r)$ for each $r \in (0,1]$ relies on the following fact:

$$\text{for each } r \in (0,1], \text{ if } t_r^* \text{ is an equilibrium, then } f_r'(t_r^*) < 1. \tag{17}$$

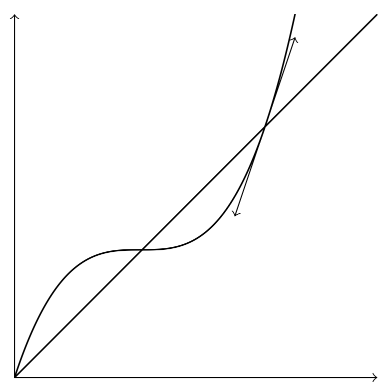

Figure 8: If there are multiple equilibria t_r^*, then at one of them $f_r'(t_r^*) > 1$

Indeed, $t \mapsto f_r(t)$ is C^∞ on $]0, \frac{1}{2}]$, and its set of fixed points is not dense in any interval $(a,b) \subset [0, \frac{1}{2}]$, with $a \neq b$ (because if not, by continuity, $f_r(t)$ would be equal to t on (a,b), and thus $f_r''(t)$ would be equal to 0 for $t \in (a,b)$, but trivially it is not). Hence, given $r \in (0,1]$, if f_r possesses at least two equilibria, we may suppose that given an equilibrium $t_r^{*_1}$, there exists another equilibrium $t_r^{*_2}$ such that there is no other equilibria between them. Let $t_r^{*_1}$ and $t_r^{*_2}$ be two such equilibria, and assume w.l.o.g. $t_r^{*_1} < t_r^{*_2}$. Then $t \mapsto f_r(t) - t$ vanishes at $t_r^{*_1}$ and $t_r^{*_2}$ and does not on $(t_r^{*_1}, t_r^{*_2})$. Thus there is some $\alpha \in (t_r^{*_1}, t_r^{*_2})$ such that either $t \mapsto f_r(t) - t$ increases on $(t_r^{*_1}, \alpha)$ and decreases on $(\alpha, t_r^{*_2})$, or $t \mapsto f_r(t) - t$

decreases on $(t_r^{*_1}, \alpha)$ and increases on $(\alpha, t_r^{*_2})$. By continuity, in the first case $f_r'(t_r^{*_1}) \geq 1$, while in the second case $f_r'(t_r^{*_2}) \geq 1$. This contradicts (17) in both cases.

We relegate the proof of (17) to a standalone proof below. □

Proof of (17). We place ourselves at an equilibrium t_r^*, where we want to state $f_r'(t_r^*) < 1$. For easier reading, in the further we will simply write t instead of t_r^*. The relation $t = f_r(t)$ will be used only at Step 3B below.

We plan the proof as follows : first (*Step 1*), we compute and set $f_r'(t) \leq A_r(t) + B_r(t)$. Next (*Step 2*), we get upper bound of $A_r(t)$ (*Step 2A*) and $B_r(t)$ (*Step 2B*) independent of t, with functions $A(r)$ and $B(r)$ respectively. Finally (*Step 3*), we get upper bounds A and B of $A(r)$ (*Step 3A*) and $B(r)$ (*Step 3B*) such that $A + B < 1$.

Step 1. The derivative of $t \mapsto f_r(t)$ is

$$f_r'(t) = \left(\frac{\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r} \right)^{\frac{1}{r}} \left(\frac{\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r \frac{1}{(3-2t)(2-t)}}{\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r} - \frac{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r \frac{(1-t)(3-t)}{(3-t^2)(2-t)} - \frac{1}{3}\left(\frac{t}{3}\right)^{r-1}}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r} \right)$$

$$= f_r(t) \left(\frac{\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r \frac{1}{(3-2t)(2-t)}}{\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r} - \frac{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r \frac{(1-t)(3-t)}{(3-t^2)(2-t)}}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r} + \frac{\frac{1}{t}\left(\frac{t}{3}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r} \right)$$

and since

$$\frac{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r \frac{(1-t)(3-t)}{(3-t^2)(2-t)}}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r} \geq 0,$$

we have

$$f_r'(t) \leq f_r(t) \left(\frac{\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r \frac{1}{(3-2t)(2-t)}}{\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r} + \frac{\frac{1}{t}\left(\frac{t}{3}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r} \right),$$

that is

$$f_r'(t) \leq A_r(t) + B_r(t), \tag{18}$$

with

$$A_r(t) = \frac{\left(\frac{\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r} \right)^{\frac{1}{r}}}{\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r} \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r \frac{1}{(3-2t)(2-t)},$$

and

$$B_r(t) = \frac{\left(\frac{(\frac{2}{3})^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - (\frac{t}{3})^r}\right)^{\frac{1}{r}}}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - (\frac{t}{3})^r} \frac{1}{t}\left(\frac{t}{3}\right)^r.$$

Step 2. We will now get upper bounds of $A_r(t)$ and $B_r(t)$ with functions $A(r)$ and $B(r)$ that rely on r only.

Step 2A. For $A_r(s)$, we deduce from (13a) and (13b) that $t \mapsto \frac{\left(\frac{(\frac{2}{3})^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - (\frac{t}{3})^r}\right)^{\frac{1}{r}}}{(\frac{2}{3})^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r} = \frac{\left((\frac{2}{3})^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r\right)^{\frac{1}{r}-1}}{\left(\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - (\frac{t}{3})^r\right)^{\frac{1}{r}}}$ increases. Hence it is less than it's value at $t = \frac{1}{2}$, which is $\frac{\left(\left(\frac{2}{3}\right)^r - \left(\frac{4}{9}\right)^r\right)^{\frac{1}{r}-1}}{\left(\left(\frac{11}{18}\right)^r - \left(\frac{1}{6}\right)^r\right)^{\frac{1}{r}}}$. Also, since $t \mapsto \frac{1}{3}\frac{3-2t}{2-t}$ and $t \mapsto \frac{1}{(3-2t)(2-t)}$ decrease, we have $\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r \frac{1}{(3-2t)(2-t)} \leq \frac{1}{3}\left(\frac{1}{2}\right)^r$. Hence for each $t \in (0, \frac{1}{2}]$,

$$A_r(t) \leq A(r) \tag{19}$$

with

$$A(r) = \frac{1}{3}\left(\frac{1}{2}\right)^r \frac{\left(\left(\frac{2}{3}\right)^r - \left(\frac{4}{9}\right)^r\right)^{\frac{1}{r}-1}}{\left(\left(\frac{11}{18}\right)^r - \left(\frac{1}{6}\right)^r\right)^{\frac{1}{r}}}.$$

Step 2B. We want to get an upper bound of $B_r(t)$. We now use the fact that we are at the equilibrium, that is t satisfies

$$\left(\frac{(\frac{2}{3})^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - (\frac{t}{3})^r}\right)^{\frac{1}{r}} = t.$$

In particular, we may write

$$B_r(t) = \frac{\left(\frac{(\frac{2}{3})^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - (\frac{t}{3})^r}\right)^{\frac{1}{r}}}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - (\frac{t}{3})^r} \frac{1}{t}\left(\frac{t}{3}\right)^r = \frac{\left(\frac{t}{3}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - (\frac{t}{3})^r} = \frac{\left(\frac{1}{3}\left(\frac{(\frac{2}{3})^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - (\frac{t}{3})^r}\right)^{\frac{1}{r}}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{1}{3}\left(\frac{(\frac{2}{3})^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - (\frac{t}{3})^r}\right)^{\frac{1}{r}}\right)^r}$$

$$= \frac{1}{\left(\frac{3-t^2}{2-t}\right)^r \frac{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - (\frac{t}{3})^r}{(\frac{2}{3})^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r} - 1} \tag{20}$$

Our next goal is then to get a lower bound of $\left(\frac{3-t^2}{2-t}\right)^r \frac{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - (\frac{t}{3})^r}{(\frac{2}{3})^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}$ independent of t. For that, let's prove that it's decreasing with t. Its derivative with respect to t is equal to

$$\frac{r\left(\frac{3-t^2}{2-t}\right)^r}{\left((\frac{2}{3})^r-\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r\right)^2}\Bigg(\frac{(1-t)(3-t)}{(3-t^2)(2-t)}\left(\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r-(\frac{t}{3})^r\right)\left((\frac{2}{3})^r-\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r\right)$$
$$+\left(\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r\frac{(1-t)(3-t)}{(3-t^2)(2-t)}-\left(\frac{t}{3}\right)^r\frac{1}{t}\right)\left(\left(\frac{2}{3}\right)^r-\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r\right)$$
$$-\left(\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r-\left(\frac{t}{3}\right)^r\right)\left(\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r\frac{3-2t}{2-t}\right)\Bigg)$$

which sign is that of

$$\frac{(1-t)(3-t)}{(3-t^2)(2-t)}\left(\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r-(\frac{t}{3})^r\right)\left((\frac{2}{3})^r-\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r\right)$$
$$+\left(\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r\frac{(1-t)(3-t)}{(3-t^2)(2-t)}-\left(\frac{t}{3}\right)^r\frac{1}{t}\right)\left(\left(\frac{2}{3}\right)^r-\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r\right)$$
$$-\left(\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r-\left(\frac{t}{3}\right)^r\right)\left(\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r\frac{3-2t}{2-t}\right)$$

The sign of the second term $\left(\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r\frac{(1-t)(3-t)}{(3-t^2)(2-t)}-\left(\frac{t}{3}\right)^r\frac{1}{t}\right)\left(\left(\frac{2}{3}\right)^r-\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r\right)$ is that of $\left(\left(\frac{3-t^2}{2-t}\right)^r\frac{(1-t)(3-t)}{(3-t^2)(2-t)}-(t)^r\frac{1}{t}\right)$; it is negative since $\left(\frac{3-t^2}{t(2-t)}\right)^r\leq\frac{3-t^2}{t(2-t)}$ (because $\frac{3-t^2}{t(2-t)}>1$), $t^r\geq t$ (because $t<1$), and also $\frac{(1-t)(3-t)}{(2-t)^2}\leq 1$.

The sign of the sum of the first and third terms is negative *iff*

$$\left(\frac{2}{3}\right)^r-\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r-\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r\frac{(3-t^2)(3-2t)}{(1-t)(3-t)}\leq 0$$

Now $r\mapsto(\frac{2}{3})^r-\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r$ increases, and $r\mapsto-\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r$ also. At $r=1$ the sum is equal to $-\frac{2}{3}\frac{2t^3-2t^2-5t+6}{(1-t)(3-t)}$; it is negative (since $2t^3-2t^2-5t+6>0$ if $t\geq 0$). This completes the proof that $t\mapsto\left(\frac{3-t^2}{2-t}\right)^r\frac{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r-(\frac{t}{3})^r}{(\frac{2}{3})^r-\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}$ decreases; it is thus greater than its value at $t=\frac{1}{2}$, that is

$$\left(\frac{3-t^2}{2-t}\right)^r\frac{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r-(\frac{t}{3})^r}{(\frac{2}{3})^r-\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}\geq\left(\frac{11}{6}\right)^r\frac{\left(\frac{11}{18}\right)^r-\left(\frac{1}{6}\right)^r}{\left(\frac{2}{3}\right)^r-\left(\frac{4}{9}\right)^r}=\frac{\left(\frac{121}{108}\right)^r-\left(\frac{11}{36}\right)^r}{\left(\frac{2}{3}\right)^r-\left(\frac{4}{9}\right)^r}. \tag{21}$$

Hence

$$B_r(t)\leq B(r)$$

with $B(r) = \frac{1}{\frac{\left(\frac{121}{108}\right)^r - \left(\frac{11}{36}\right)^r}{\left(\frac{2}{3}\right)^r - \left(\frac{4}{9}\right)^r} - 1}$.

Step 3. So we proved

$$f_r'(t) \leq A_r(t) + B_r(t) \leq A(r) + B(r).$$

with

$$A(r) = \frac{1}{3}\left(\frac{1}{2}\right)^r \frac{\left(\left(\frac{2}{3}\right)^r - \left(\frac{4}{9}\right)^r\right)^{\frac{1}{r}-1}}{\left(\left(\frac{11}{18}\right)^r - \left(\frac{1}{6}\right)^r\right)^{\frac{1}{r}}}$$

and

$$B(r) = \frac{1}{\frac{\left(\frac{121}{108}\right)^r - \left(\frac{11}{36}\right)^r}{\left(\frac{2}{3}\right)^r - \left(\frac{4}{9}\right)^r} - 1}.$$

In order to get upper bounds of $A(r)$ and $B(r)$, we need accurate upper bound and lower bound of expressions of the form $b^r - a^r$, with $b > a > 0$. We have

$$b^r - a^r = \int_a^b \frac{1}{r} x^{r-1} \mathrm{d}x.$$

As seen in Figure 9, from convexity of $x \mapsto \frac{1}{r}x^{r-1}$, the integral is limited by two trapezes' area, so that

$$r(b-a)\left(\frac{a+b}{2}\right)^{r-1} \leq b^r - a^r \leq r(b-a)\frac{a^{r-1} + b^{r-1}}{2} \qquad (22)$$

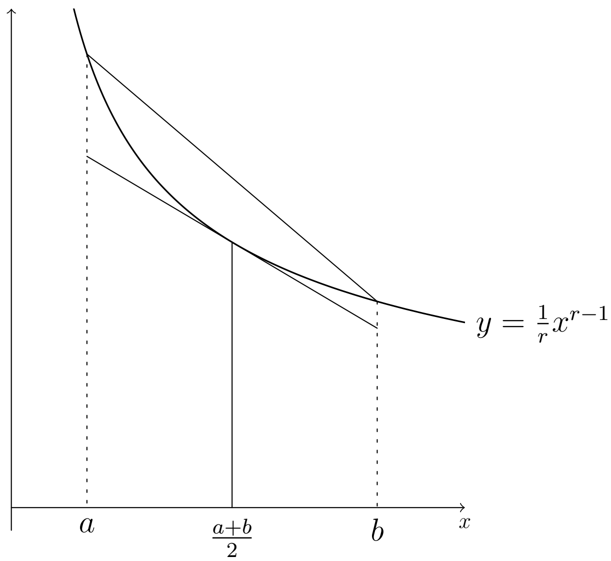

Figure 9: Trapezoidal minoration and majoration of $b^r - a^r$

Step 3A. We want to get an upper bound for $A(r)$. From (22),

$$\left(\frac{2}{3}\right)^r - \left(\frac{4}{9}\right)^r = \left(\frac{2}{3}\right)^r \left(1^r - \left(\frac{2}{3}\right)^r\right) \leq \left(\frac{2}{3}\right)^r r \left(1 - \frac{2}{3}\right) \frac{1 + \left(\frac{2}{3}\right)^{r-1}}{2}$$

with $\left(\frac{2}{3}\right)^{r-1} \leq \frac{3}{2}$ so that

$$\left(\frac{2}{3}\right)^r - \left(\frac{4}{9}\right)^r \leq \left(\frac{2}{3}\right)^r \frac{5r}{12}.$$

Also, from (22) again,

$$\left(\frac{11}{18}\right)^r - \left(\frac{1}{6}\right)^r \geq \frac{8r}{7} \left(\frac{7}{18}\right)^r.$$

We obtain

$$A(r) \leq \frac{1}{r} \frac{48}{35} \left(\frac{3}{4}\right)^r \left(\frac{35}{96}\right)^{\frac{1}{r}}$$

This last expression derivates to

$$\frac{48}{35} \left(\frac{3}{4}\right)^r \left(\frac{35}{96}\right)^{\frac{1}{r}} \frac{1}{r} \left(\frac{-1}{r} + \ln\left(\frac{3}{4}\right) + \frac{-1}{r^2} \ln\left(\frac{35}{96}\right)\right)$$

which sign is that of $\frac{-1}{r}+\ln\left(\frac{3}{4}\right)+\frac{-1}{r^2}\ln\left(\frac{35}{96}\right)=\frac{1}{r^2}\left(r^2\ln\left(\frac{3}{4}\right)-r-\ln\left(\frac{35}{96}\right)\right)$ which is positive if $0<r<\alpha$ with $\alpha=\frac{1-\sqrt{1+4\ln\left(\frac{3}{4}\right)\ln\left(\frac{35}{96}\right)}}{2\ln\left(\frac{3}{4}\right)}$ and negative if $\alpha<r\leq 1$. Hence finally

$$A(r)\leq\frac{1}{\alpha}\frac{48}{35}\left(\frac{3}{4}\right)^{\alpha}\left(\frac{35}{96}\right)^{\frac{1}{\alpha}}<0.39. \tag{23}$$

Step 3B. In order to get an upper bound of $B(r)=\frac{1}{\frac{\left(\frac{121}{108}\right)^r-\left(\frac{11}{36}\right)^r}{\left(\frac{2}{3}\right)^r-\left(\frac{4}{9}\right)^r}-1}$, we need to get a lower bound for $\frac{\left(\frac{121}{108}\right)^r-\left(\frac{11}{36}\right)^r}{\left(\frac{2}{3}\right)^r-\left(\frac{4}{9}\right)^r}$. From (22),

$$\left(\frac{121}{108}\right)^r-\left(\frac{11}{36}\right)^r\geq r\frac{22}{27}\left(\frac{77}{108}\right)^{r-1},$$

and

$$\left(\frac{2}{3}\right)^r-\left(\frac{4}{9}\right)^r\leq r\frac{1}{9}\left(\left(\frac{2}{3}\right)^{r-1}+\left(\frac{4}{9}\right)^{r-1}\right),$$

so that

$$\frac{\left(\frac{121}{108}\right)^r-\left(\frac{11}{36}\right)^r}{\left(\frac{2}{3}\right)^r-\left(\frac{4}{9}\right)^r}\geq\frac{22}{3}\frac{1}{\left(\frac{72}{77}\right)^{r-1}+\left(\frac{48}{77}\right)^{r-1}}\geq\frac{22}{3}\frac{1}{\frac{77}{72}+\frac{77}{48}}=\frac{96}{35}$$

(since $\frac{x^r}{x}\leq\frac{1}{x}$ if $0<x\leq 1$). This gives

$$B(r)\leq\frac{1}{\frac{96}{35}-1}=\frac{35}{61}. \tag{24}$$

Combined with (23), we obtain for each $r\in(0,1]$, and each $t\in[0,\frac{1}{2}]$ such that $t=f_r(t)$, the upper bound

$$f_r'(t)\leq 0.39+\frac{35}{61}<1$$

as desired. □

C Proof of Proposition 2

Proof. a) We want to show that the function $r\mapsto\underline{t}(r)$ increases. It does *iff* for each $r,r'\in(0,1]$

$$\text{if } \underline{t}(r')<\underline{t}(r),\ \text{then } r'<r. \tag{25}$$

For this purpose, let's first state that for any $r \in (0,1]$ we have

$$t < f_r(t) \text{ for any } t \in [0, \underline{t}(r)) \tag{26}$$

(we have also $t > f_r(t)$ for any $t \in (\underline{t}(r), \frac{1}{2}]$).

Indeed: $t \mapsto f_r(t) - t$ does not vanishes on $[0, \underline{t}(r))$, since $\underline{t}(r)$ is the unique equilibrium, and $f_r(0) - 0 = f_r(0) > 0$.

Now because of (26), if for $r, r' \in (0,1]$, we have $\underline{t}(r') \in [0, \underline{t}(r))$, that is $\underline{t}(r') < \underline{t}(r)$, then

$$f_{r'}(\underline{t}(r')) = \underline{t}(r') < f_r(\underline{t}(r')). \tag{27}$$

It is therefore sufficient to prove that $f_{r'}(\underline{t}(r')) < f_r(\underline{t}(r'))$ implies $r' < r$ in order to get (25). This is a particular case of the implication

$$\text{at any } t \in [0, \frac{1}{2}], \text{ if } r' \geq r, \text{ then } f_{r'}(t) \geq f_r(t),$$

which amounts to say that at any $t \in [0, \frac{1}{2}]$, $r \mapsto f_r(t)$ increases. Let's prove it.

Since for any $0 < x < 1$, the function $r \mapsto x^{\frac{1}{r}}$ is increasing, it is enough to prove that

$$r \mapsto \frac{\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t})\right)^r - \left(\frac{t}{3}\right)^r} \tag{28}$$

is increasing for any $t \in [0, \frac{1}{2}]$.

Let $t \in (0, \frac{1}{2}]$. For any $r \in (0,1]$, the sign of the derivative of (28) is that of

$$\begin{aligned} g_t(r) = & \left(\left(\frac{2}{3}\right)^r \ln\left(\frac{2}{3}\right) - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r \ln\left(\frac{1}{3}\frac{3-2t}{2-t}\right)\right)\left(\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r\right) \\ & - \left(\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r\right)\left(\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r \ln\left(\frac{1}{3}\frac{3-t^2}{2-t}\right) - \left(\frac{t}{3}\right)^r \ln(t)\right). \end{aligned}$$

Now if we write

$$\begin{aligned} g_t(r) = & \frac{1}{r}\left(\left(\frac{2}{3}\right)^r \ln\left(\left(\frac{2}{3}\right)^r\right) - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r \ln\left(\left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r\right)\right)\left(\left(\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r\right) \\ & - \frac{1}{r}\left(\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r\right)\left(\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r \ln\left(\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r\right) - \left(\frac{t}{3}\right)^r \ln\left(\left(\frac{t}{3}\right)^r\right)\right) \end{aligned}$$

then setting $h(x) = x\ln(x)$, $a = \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r$, $b = \left(\frac{2}{3}\right)^r$, $c = \left(\frac{t}{3}\right)^r$ and $d = \left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r$ we obtain

$$g_t(r) = \frac{1}{r}(b-a)(c-d)\left(\frac{h(b)-h(a)}{b-a} - \frac{h(d)-h(c)}{d-c}\right)$$

which sign is that of $\left(\frac{h(b)-h(a)}{b-a} - \frac{h(d)-h(c)}{d-c}\right)$. But this is the difference of the slope of (AB) and the slope of (CD), where A, B, C and D are the points of on the graph of h with abscissa a, b, c and d respectively (see Figure 10). Since h is convex and since

$$\frac{t}{3} < \frac{1}{3}\frac{3-2t}{2-t} < \frac{1}{3}\frac{3-t^2}{2-t} < \frac{2}{3},$$

so that

$$c < a < d < b,$$

we get the result from the Three Chords Lemma[5] applied

first to the triangle CAB, where it gives that the slope of (AD) is higher than the slope of (CD), and

second to the triangle ADB, where it gives that the slope of (AB) is higher than the slope of (AD); so that

the slope of (AB) is higher than the slope of (CD): this amounts to say $g_t(r) > 0$.

[5] The Three Chords Lemma states that if X, Z, Y are three points on the graph of a convex function, with abscissa $x_X < x_Z < x_Y$, then the slope of (XZ) is lower than the slope of (XY), which is lower than the slope of (YZ).

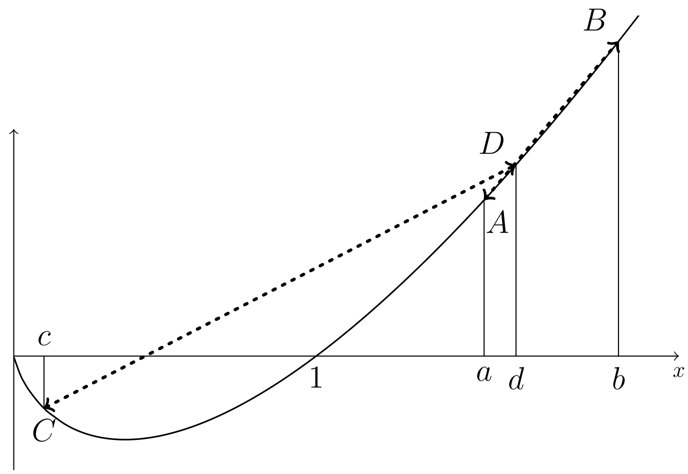

Figure 10: The slopes of (AB) and (CD) with A, B, C, D on $y = x\ln(x)$

b) The result relies on $\lim_{r\to 0} \underline{t}(r) = 0$, that we will prove. We have to prove precisely that, as $r \to 0$, $t = 0$ is solution of

$$t = \left(\frac{\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r} \right)^{\frac{1}{r}}.$$

From l'Hôpital's Rule

$$\lim_{r\to 0} \frac{\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\frac{1}{3}\frac{3-t^2}{2-t}\right)^r - \left(\frac{t}{3}\right)^r} = \frac{\ln\left(\frac{2}{3}\right) - \ln\left(\frac{1}{3}\frac{3-2t}{2-t}\right)}{\ln\left(\frac{1}{3}\frac{3-t^2}{2-t}\right) - \ln\left(\frac{t}{3}\right)},$$

and since $0 < \frac{\ln\left(\frac{2}{3}\right) - \ln\left(\frac{1}{3}\frac{3-2t}{2-t}\right)}{\ln\left(\frac{1}{3}\frac{3-t^2}{2-t}\right) - \ln\left(\frac{t}{3}\right)} < 1$ for any $t \in (0, \frac{1}{2}]$, we have

$$\lim_{r\to 0} \left(\frac{\ln\left(\frac{2}{3}\right) - \ln\left(\frac{1}{3}\frac{3-2t}{2-t}\right)}{\ln\left(\frac{1}{3}\frac{3-t^2}{2-t}\right) - \ln\left(\frac{t}{3}\right)} \right)^{\frac{1}{r}} = 0.$$

□

D Proof of Proposition 3

Proof. Assume the employee anticipates a level $\hat{t}$ of favoritism. To $\hat{t}$ correspond efforts $e_i(m)[\hat{t}]$ and the response of the manager is a decision-making rule based on a level of favoritism equal to $\left(\frac{e_2^r(2)[\hat{t}]-e_2^r(1)[\hat{t}]}{e_1^r(1)[\hat{t}]-e_1^r(2)[\hat{t}]}\right)^{\frac{1}{r}} = f_r(\hat{t})$. For an equilibrium occurring at t^*, it is stable if $|f_r(\hat{t}) - t^*|$ is less than $|\hat{t} - t^*|$. Therefore stability is measured by the derivative $f_r'(t^*)$: if $f_r'(t^*) < 1$, then the equilibrium is stable, and if $f_r'(t^*) > 1$, it is unstable.

Consider the symmetrical equilibrium. We have $f_r'(t^*) = f_r'(1) = \frac{\left(\frac{1}{3}\right)^r}{\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\right)^r} = \frac{2}{2^r-1} > 2$ since $2^r - 1 < 2 - 1 = 1$. Moreover, $f_r'(1)$ increases with r, and $\lim_{r\to 0} f_r'(1) = +\infty$: the more complementary the tasks are, the lower r is, and the more the symmetrical equilibrium is unstable.

For an asymmetrical equilibrium, we state $f_r(\underline{t}(r)) < 1$ in the Appendix, Equation (17). The proof follows the statement. □

E Proof of Proposition 4

Proof. We need to compare the mean value $E(Y_t)$ of the aggregate production between the non discriminatory equilibrium and the discriminatory equilibria. Setting t^* the discriminatory degree of the decision-making rule ($t^* = 1$ or $t^* = \underline{t}(r)$), we have

$$\begin{aligned} E(Y_{t^*}) &= P(m=1)E(Y_{t^*}(1)) + P(m=2)E(Y_{t^*}(2)) \\ &= \int_{a_1 \geq t^* a_2} Y_{t^*}(1)\,\mathrm{d}a_1\,\mathrm{d}a_2 + \int_{a_1 < t^* a_2} Y_{t^*}(2)\,\mathrm{d}a_1\,\mathrm{d}a_2 \\ &= \int_{a_2=0}^{1}\int_{a_1=t^* a_2}^{1} \left(\frac{(A_{11}(t^*)a_1)^r + (A_{21}(t^*)a_2)^r}{2}\right)^{\frac{1}{r}} \mathrm{d}a_1\,\mathrm{d}a_2 \\ &\quad + \int_{a_2=0}^{1}\int_{a_1=0}^{t^* a_2} \left(\frac{(A_{12}(t^*)a_1)^r + (A_{22}(t^*)a_2)^r}{2}\right)^{\frac{1}{r}} \mathrm{d}a_1\,\mathrm{d}a_2 \end{aligned} \tag{29}$$

with A_{ij}'s given by (9).

At $r = 1$, we have at the non discriminatory equilibrium $E(Y_{t^*=1}) = \frac{5}{18} \cong 0.278$, and at the discriminatory equilibrium, occurring at $t^* = \underline{t}(1) = \frac{1}{2}$, we have $E(Y_{t^*=\frac{1}{2}}) = \frac{59}{216} \cong 0.273$. Therefore discrimination is not efficient relative to no discrimination.

As r tends to 0, $\underline{t}(r)$ tends to 0 and at $t^* = 0$, we have $A_{11}(0) = \frac{1}{2}$, $A_{21}(0) = \frac{1}{2}$, $A_{12}(0) = 0$, and $A_{22}(0) = \frac{2}{3}$. Moreover, $Y = \left(\frac{(y_1)^r + (y_2)^r}{2}\right)^{\frac{1}{r}}$ tends to $\sqrt{y_1 y_2}$.

With a maximum degree of discrimination ($r \to 0$, $\underline{t}(r) \to 0$), we obtain at the non discriminatory equilibrium

$$\begin{aligned}E(Y_{t^*=1}) &= \int_{a_1 \geq a_2} \sqrt{\frac{2}{3}a_1 \frac{1}{3}a_2}\,\mathrm{d}a_1\,\mathrm{d}a_2 + \int_{a_1 < a_2} \sqrt{\frac{1}{3}a_1 \frac{2}{3}a_2}\,\mathrm{d}a_1\,\mathrm{d}a_2\\ &= \frac{\sqrt{2}}{3}\int_{a_1,a_2} \sqrt{a_1 a_2}\,\mathrm{d}a_1\,\mathrm{d}a_2 = \frac{2\sqrt{2}}{27} \cong 0.21,\end{aligned}$$

and at the discriminatory equilibria

$$E(Y_{t^*\to 0}) = \int_{a_1,a_2} \sqrt{\frac{1}{2}a_1 \frac{1}{2}a_2}\,\mathrm{d}a_1\,\mathrm{d}a_2 = \frac{1}{2}\int_{a_1,a_2} \sqrt{a_1 a_2}\,\mathrm{d}a_1\,\mathrm{d}a_2 = \frac{2}{9} \cong 0.22.$$

Therefore discrimination is efficient relative to no discrimination. □

F Proof of Proposition 5

Proof. We will prove that at $t = 0$, $f_r(0) > 0$, while for each $t \in (0, 1]$, there exists r sufficiently small such that $f_r(t) < t$. The result is thus a consequence of Bolzano's Theorem.

For each $\eta \geq 1$, for each $r \in (0, 1]$,

$$f_r(0) = \left(\left(\frac{4}{3}\right)^r - \left(\frac{1}{\eta^2}\right)^r\right)^{\frac{1}{r}} > 0.$$

Now fix $t \in (0, 1]$. Then

$$\begin{aligned}\lim_{r\to 0}\left(\frac{f_r(t)}{t}\right)^r &= \lim_{r\to 0}\frac{\left(\eta^2\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{t^r\left(\left(\eta^2\frac{1}{3}\frac{3-t^2}{2-t})\right)^r - \left(\frac{t}{3}\right)^r\right)} = \lim_{r\to 0}\frac{\left(\eta^2\frac{2}{3}\right)^r - \left(\frac{1}{3}\frac{3-2t}{2-t}\right)^r}{\left(\eta^2\frac{1}{3}\frac{t(3-t^2)}{2-t}\right)^r - \left(\frac{t^2}{3}\right)^r}\\ &= \frac{\ln\left(\eta^2\frac{2}{3}\right) - \ln\left(\frac{1}{3}\frac{3-2t}{2-t}\right)}{\ln\left(\eta^2\frac{1}{3}\frac{3-t^2}{2-t}\right) - \ln\left(\frac{t}{3}\right)} = \frac{\ln\left(\eta^2\frac{2(2-t)}{3-2t}\right)}{\ln\left(\eta^2\frac{3-t^2}{t(2-t)}\right)}.\end{aligned}$$

and since for all $t \in (0, 1]$, $\frac{2(2-t)}{3-2t} < \frac{3-t^2}{t(2-t)}$ we obtain $\lim_{r\to 0}\left(\frac{f_r(t)}{t}\right)^r < 1$. So for r sufficiently small, $f_r(t) < t$. □