Foundations of Discounted Utility with Satiation

Manel Baucells Lin Zhao

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Abstract

We provide preferences foundations for discounted satiation, a utility model that relaxes the time separability axiom of discounted utility. Our goal is to account for diminishing marginal utility in continuous-time, whereby recent consumption decreases the marginal utility of current consumption. Relaxing one of our preference axioms—conditional stationarity–yields an axiomatization of general Hindy-Huang-Kreps preferences, of which the satiation and the durability model are special cases. We clarify the relationships between the satiation, durability and discounted utility models, and illustrate their differences by comparing the optimal plans in different situations.

Keywords: Temporal Separability, Inter-temporal substitution, Satiation, Discounted Utility

1 Introduction

In 1960, Koopmans provided preference foundations for the discounted utility model. After introducing time separability—consumption in any one period has no effect on the preference between alternative sequences of consumption in the remaining future, and conversely—Koopmans (1960, p. 292) admits: “One cannot claim a high degree of realism for such a postulate, because there is no clear reason why complementarity of goods could not extend over more than one time period. It may be surmised, however, that weaker forms of this postulate would still allow similar results to be reached." The separability axiom was soon criticized by Hicks (1965, p. 261) and many others. Our goal is to provide an axiomatic foundation for temporal preferences over consumption that relaxes time separability.

We also want to go beyond a discrete-time setup, where preferences implicitly depend on what is meant by 'a period'. Mas-Colell, Whinston, and Green (1995, p. 18) note that "commodities consumed at different times should be viewed rigorously as distinct commodities [...] Thus, one commodity might be 'bread consumed in the month of February,' even though, in principle, bread consumed at each instant in February should be distinguished." Discrete-time models put aside the potentially important detail of how consumption ought to be distributed inside the period.

Kopylov (2010) is the first to axiomatize a continuous-time version of discounted utility. Separability now requires that consumption instants before time t does not affect preferences over consumption occurring at t or after. At face value, this condition runs against any sensible notion of diminishing marginal returns. At least for short time horizons, it seems that marginal utility should decrease as a function of recent past consumption. Otherwise, how can the illustrious second scoop of ice cream be less enjoyable than the first scoop consumed a moment ago?

Our focus is on the effect that past consumption has on current preferences. Becker (1992) argues that both satiation and habit formation are relevant factors influencing preferences; and Read, Loewenstein, and Rabin (1999, p. 183) state that "The most important taste change effects are habit formation and satiation." While habit formation has received a fair amount of attention, satiation has been left unattended.

Thus, the literature has not yet seen preference conditions over consumption profiles in continuous-time yielding a representation where total utility is the time integral of instant utility,[1] and where instant utility de-

[1]Kahneman, Wakker, and Sarin (1997) provide a representation of total utility as the time integral of instant utility, but their primitives are profiles of instant utility, rather than profiles of consumption.

pends on the current consumption rate and on past consumption via a state variable. Our goal is to fill this gap, by providing a representation for the so called discounted utility with satiation, or discounted satiation for short.

In our setup, we consider consumption rates; as well as consumption gulps. A consumption gulp is the limit of a consumption rate that increases in intensity but decreases in duration, while total consumption converges. Consumption gulps are a mathematical idealization, but they are essential in the description of our preference conditions. Our formulation is such that the exact same axioms, but restricted to profiles with gulps only, produce the discrete-time version of the satiation model (Baucells and Sarin, 2007).[2]

One key challenge we face is to characterize a continuous-time state variable via preference conditions. The habit formation model in discrete-time, introduced by Ryder and Heal (1973), has been recently axiomatized by Rozen (2010). Particularly meritorious is to derive a state variable—the habit level—via axioms. Rozen's formulation relies on a discrete time setup, and uses compensating sequences of consumption to derive the habit level. We use a simpler idea, namely, that the effect of past consumption on current preferences can be restored by means of a consumption gulp. The instantaneous nature of a gulp conveniently eliminates the effect that time preference could have in a restoring gulp of positive duration.

The axioms we impose capture the idea that past consumption influences current preferences, and that such influence can be summarized in a sufficient statistic or state. The state that we uncover is a stock of past consumption with decay—called the satiation level. Instant utility is proportional to the consumption rate and the marginal felicity at the satiation level.

Like all preference models, we abstract from the psychophysical determinants of preferences. In our case, however, we cannot fail to notice that satiation is a psychophysical effect that we all experience, and motivates the

[2]In discrete-time, He, Dyer, and Butler (2013) have axiomatized a version of the satiation model, but using strength of preference—as opposed to preferences over the consumption profiles—and their identification of the state variable is loose.

central notion of diminishing marginal utility.

Finally, if we relax one of our axioms—compensated stationarity—then we obtain a non-necessarily-stationary version of the satiation model. We discover that such model is preferentially equivalent to the functional described by Hindy, Huang, and Kreps (1992) (henceforth HHK).

2 Discounted Utility with Satiation

Our domain of choice are temporal profiles of consumption over a finite horizon $[0, T]$, or consumption profiles for short. As usual, we consider a single composite good. We use $X(t)$ to denote cumulative consumption up to time t. The choice set is given by

$$\mathscr{X} = \left\{X : [0, T] \to \mathbb{R}_+ \text{ increasing, right-continuous, with } X(0^-) = 0\right\}.$$

An important set of profiles is $\mathscr{X}_c \subset \mathscr{X}$, consisting of all absolutely continuous $X(t)$, and characterized by a consumption rate $x(t) = X'(t)$. A consumption gulp at t occurs when $X(t) - X(t^-) = \int_{t^-}^{t} dX(s) > 0$. Let $\mathscr{X}_d \subset \mathscr{X}$ be the set of profiles consisting of a finite number of gulps at arbitrary times. A typical element is described as $x(t) = \sum_{i=0}^{n} k_i \cdot \mathbf{1}_{t_i}$, where $k_i \geq 0$ and $0 = t_0 < t_1 < ... < t_n \leq T$ and $\mathbf{1}_{t_i}$ denotes a unit gulp at time t_i. Both $\mathscr{X}_d$ and $\mathscr{X}_c$ are closed under mixtures and additions. We use $x \in \mathscr{X}$ to denote a full profile, and let $\mathbf{0}$ be the null profile.

The utility functional we propose has the following components. Throughout, $Y_0 \geq 0$ denotes the initial satiation level, $\gamma > 0$ the satiation decay rate, and $\delta \in \mathbb{R}$ the discount rate. Given Y_0 and γ, let $u(Y) : \left[Y_0 e^{-\gamma T}, \infty\right) \to \mathbb{R}$ be a *felicity function*, which is continuous, strictly increasing, strictly concave, and continuously differentiable on $\left(Y_0 e^{-\gamma T}, \infty\right)$. Thus, $u(Y)$ could unbounded from below, provided $Y_0 > 0$.

In a static setting, a natural way to account for diminishing marginal returns is to stipulate that utility be given by $u(Y_0 + k) - u(Y_0)$, where k is the consumption during the period and Y_0 is the satiation level resulting from "retained past consumption" (Baucells and Sarin, 2007). That is, marginal utility decreases with both current and past consumption. This idea, translated into continuous-time, yields the discounted satiation model.[3]

Definition 1. *For $x \in \mathscr{X}$, the **discounted satiation** model is given by*

$$U^{\mathrm{DS}}(x) = \int_0^T e^{-\delta t} u'(Y(t)) dX(t), \text{ where} \tag{1}$$

$$Y(t) = Y_0 e^{-\gamma t} + \int_{0^-}^t e^{-\gamma(t-s)} dX(s). \tag{2}$$

$Y(t)$ is unique up to scaling; and $u(Y)$ is unique up to positive affine transformation, and such that the dual function $u^*(Y) = \delta u(Y) + \gamma u'(Y)Y$ is not constant on any open interval.

The last condition precludes thick indifference curves.[4] Note that the satiation level depends on x, but we employ $Y(t)$ and not $Y(x,t)$ to alleviate notation. Our state is equivalent to the popular form $H(t) = H_0 e^{-\gamma t} + \beta \int_{0^-}^t e^{-\gamma(t-s)} X'(t) dt$. To see this, let $Y(t) = H(t)/\beta$ and $Y_0 = H_0/\beta$. While (1) involves Lebesgue integration, it admits a dual form that can be calculated using Riemann integration.

Lemma 1. *For $x \in \mathscr{X}$, the functional (1) can be rewritten as*

$$U^{\mathrm{DS}}(x) = e^{-\delta T} u(Y(T)) - u(Y_0) + \int_0^T e^{-\delta t} u^*(Y(t)) dt.$$

In the reminder of this section, we motivate $U^{\mathrm{DS}}(x)$ by showing it exhibits a preference for savoring, not captured by discounted utility. We also show that, unlike discounted utility, the functional is continuous with respect to convergence in distribution. Finally, by replacing $e^{-\delta t}u(Y)$ for $u(Y,t)$, we obtain a generalized satiation functional that, with minor qualifications, is equivalent to the Hindy-Huang-Kreps linear functional.

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[3]Consider a consumption rate $X'(t)$. During a small time interval, $[t, t+\Delta)$, the total consumption is $k = X(t+\Delta) - X(t)$. As $\Delta \to dt$, we have that $[u(Y(t)+k) - u(Y(t))](1/k) \to u'(Y(t))$ and $X(t+\Delta) - X(t) \to X'(t)dt$. Thus, $e^{-\delta t}[u(Y(t)+k) - u(Y(t))] = e^{-\delta t}[u(Y(t)+k) - u(Y(t))](1/k)(X(t+\Delta) - X(t)) \to e^{-\delta t} u'(Y(t)) dX(t)$.

[4]For example, if $u(Y) = Y^\alpha/\alpha$, $\alpha < 1$, then $u^*(Y) = [\delta + \gamma\alpha] u(Y)$. If $\alpha = -\delta/\gamma$, then $u^*(Y) = 0$ and, by Lemma 1, preferences are represented by $U(x) = Y(T)$.

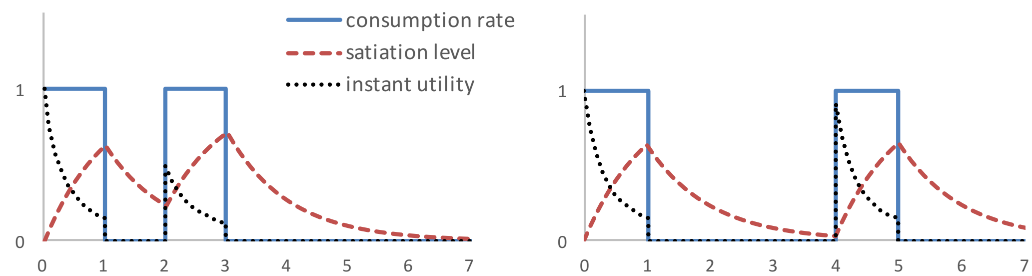

Figure 1: Spacing out consumption reduces the satiation level and may increase total utility [$\gamma = 1$, $Y_0 = 0$, $\delta = 0$, $u'(Y) = e^{-3Y}$].

2.1 Preferences for Savoring

Consider a setup whereby the consumption rate is either 0 or 1. Consider the evaluation of $X'(t) = 1_{[0,1]} + 1_{[1+s,2+s]}$, i.e., consuming during two unit time intervals with a break of length $s \geq 0$ in between. Let $Y_0 = 0$.

Consider the evolution of instant utility, given by $e^{-\delta t} u'(Y(t)) X'(t)$, over time (see Figure 1). During $[0, 1]$ instant utility is strictly positive, but decreases over time due to satiation. Indeed, $Y(t) = \left(1 - e^{-\gamma t}\right)/\gamma$ increases, and $u'(Y(t))$ decreases due to concavity. Instant utility is proportional to consumption, and hence equal to zero during the break. The break, however, reduces the satiation level to $Y(1)e^{-\gamma s}$ at time s. Thus, instant utility during the second consumption interval, $[1 + s, 2 + s]$, increases with s.

Let $\delta > 0$. In discounted utility, and due to impatience, there is no gain in postponing consumption and one would set $s = 0$. In discounted satiation, there is a trade-off between impatience and enhancing savoring during the second interval, yielding $s^* > 0$.[5]

With important qualifications, we will show in §6.1 that $U^{\text{DS}}(x)$ converges towards discounted utility as $\gamma \to \infty$.

[5] Specifically, s^* solves

$$Y(1)e^{-\gamma s^*} = -\frac{\delta}{\gamma} \frac{\int_{1+s^*}^{2+s^*} e^{-\delta t} u'(Y(t)) dt}{\int_{1+s^*}^{2+s^*} e^{-(\delta+\gamma)t} u''(Y(t)) dt}.$$

2.2 Continuity and the Discrete-Time Version

Throughout, we adopt convergence in distribution, whereby $x_m \to x$ if $X_m(t) \to X(t)$ at all points of continuity of $X(t)$ and at $t = T$. HHK argue that convergence in distribution (which on $\mathscr{X}$ agrees with convergence in the weak topology) is a natural economic notion of proximity of consumption profiles; and show that discounted utility violates continuity wrt convergence in distribution. Continuity is an important motivation behind our work (see §6.2).

According to our convergence criterion, a consumption gulp is the limit of a consumption rate of higher and higher intensity, exerted over a shorter and shorter time, i.e., $m \cdot 1_{t\in[t,t+1/m]} \to \mathbf{1}_t$ as $m \to \infty$. Also, the limit of two unit gulps $\mathbf{1}_t + \mathbf{1}_{t+\varepsilon}$ when their time separation $\epsilon \to 0$ is $2 \cdot \mathbf{1}_t$.

Lemma 2. *$U^{\text{DS}}(x)$ is continuous on $\mathscr{X}$ wrt convergence in distribution.*

Continuity allows us to define the discrete-time version, $U^{\text{DS}}(\sum_{i=0}^{n} k_i \cdot \mathbf{1}_{t_i})$, as a byproduct. For example, the evaluation of a unit gulp $U^{\text{DS}}(\mathbf{1}_t)$ can be obtained as the limit of $U^{\text{DS}}(m \cdot 1_{t\in[t,t+1/m]})$. The latter is easy to evaluate because on $\mathscr{X}_c$ both U^{DS} and Y are Riemann integrals; and continuity ensures that any sequence converging to $\mathbf{1}_t$ yields the same limit. More generally, for $x \in \mathscr{X}_d$, we obtain that $U^{\text{DS}}(x)$ agrees with the satiation model in discrete time introduced by Baucells and Sarin (2007), and given by

$$U^{\text{DS}}(x) = \sum_{i=0}^{n} e^{-\delta t_i} \left[u\left(Y(t_i^-) + k_i\right) - u\left(Y(t_i^-)\right)\right], \text{ where} \tag{3}$$

$$Y(t_i^-) = Y_0 e^{-\gamma t_i} + \sum_{t_j < t_i} k_j e^{-\gamma(t_i - t_j)}. \tag{4}$$

Finally, that $\lim_{\epsilon\to 0} U^{\text{DS}}(\mathbf{1}_t + \mathbf{1}_{t+\varepsilon}) = U^{\text{DS}}(2 \cdot \mathbf{1}_t)$ illustrates that discounted satiation satisfies the local substitution property, namely, that consumption at nearby times become perfect substitutes.

2.3 The Generalized Satiation and the HHK Functionals

The generalized satiation model results from replacing the stationary decay factor $e^{-\gamma t}$ for a *decay function* $\theta(t) : [0,T] \to (0,1]$, assumed to be continuously differentiable, strictly decreasing, and with $\theta(0) = 1$. Let

$$\Omega = \{(Y,t)\,|0 \le t \le T, Y > Y_0\theta(t)\}$$

and $\overline{\Omega} = \{(Y,t)\,|0 \le t \le T, Y \ge Y_0\theta(t)\}$. Next, we replace $e^{-\delta t}u'(Y)$ for $u_Y(Y,t)$, the partial derivative of a generalized *felicity function* $u(Y,t)$: $\overline{\Omega} \to \mathbb{R}$ continuous, strictly increasing in Y for $0 \le t < T$, strictly concave in Y for $0 < t < T$, continuously differentiable on Ω, and such that $u_t(Y,t)$ is continuous over $\overline{\Omega}$.

Definition 2. *For $x \in \mathscr{X}$, the* ***generalized satiation*** *model is given by*

$$U^{\text{GS}}(x) = \int_0^T u_Y(Y(t),t)dX(t), \text{ where} \tag{5}$$

$$Y(t) = Y_0\theta(t) + \int_{0^-}^t \frac{\theta(t)}{\theta(s)}dX(s). \tag{6}$$

The function $u(Y,t)$ is unique up to positive affine transformation; and such that its dual $u^*(Y,t) = -u_t(Y,t) - u_Y(Y,t)Y\theta'(t)/\theta(t)$ is not constant on any open interval.

The functional (5) can be rewritten as (see proof of Lemma 1)

$$U^{\text{GS}}(x) = u(Y(T),T) - u(Y_0,0) + \int_0^T u^*(Y(t),t)dt.$$

HHK (Prop. 6) introduce a utility functional, which we present here with some regularity conditions to ensure its equivalence with U^{GS}.

Definition 3. *For $x \in \mathscr{X}$, the* ***Hindy-Huang-Kreps*** *model is given by*

$$U^{\text{HHK}}(x) = \int_0^T a(Y(t),t)dX(t) + \int_0^T b(Y(t),t)dt, \tag{7}$$

where $Y(t)$ as in (6), $a(Y,t) : \Omega \to \mathbb{R}_+$ is such that $\int_{Y_0}^Y a(y,t)dy$ is a felicity function, and $b(Y,t) : \overline{\Omega} \to \mathbb{R}$ is the dual of some felicity function.

Clearly, $U^{\text{GS}}(x)$ is a special case of $U^{\text{HHK}}(x)$ after setting $a(Y,t) = u_Y(Y,t)$ and $b(Y,t) = 0$. In turn, it is always possible to write HHK in GS form. The adoption of $Y(t)$ as in (6) is key to establish the equivalence, and there are good axiomatic reasons for this choice (see §3.1).

Lemma 3. *$U^{\text{HHK}}(x)$ is preferentially equivalent to $U^{\text{DS}}(x)$. Specifically, given the functions $a(Y,t)$ and $b(Y,t)$ for U^{HHK}, if we define U^{GS} using*

$$u(Y,t) = \int_{Y_0}^{Y} a(y,t)dy + \int_t^T b\left(Y\theta(s)/\theta(t), s\right) ds,$$

then $U^{\text{HHK}}(x) = u\left(Y_0, 0\right) + U^{\text{GS}}(x)$.

As a Corollary, we have that U^{DS} is preferentially equivalent to the discounted durability model with infinite horizon (see §5).

HHK show that for a time additive utility functional on $\mathscr{X}_c$ to be continuous, it must necessarily take the form given by $U^{\text{HHK}}(x)$. Regarding sufficiency, the literature has only shown that the durability term $\int_0^T b(Y(t),t)dt$ is continuous (HHK (Prop. 7) and Bank and Riedel (2000, Prop. 3.1)). We fill this gap (see proof of Lemma 2) by establishing the continuity of U^{GS}, and hence of U^{HHK}.[6] Again, that U^{HHK} is continuous depends on adopting $Y(t)$ as in (6).

3 Behavioral Foundations of Discounted Satiation

To describe profiles we will make use of the following operators:

- Left truncation: ${}^t x$ is equal to the profile of x from t onwards, but sets consumption before t to zero, or ${}^t X(s) = \max\{0, X(s) - X(t^-)\}$.
- Right truncation: x^t is equal to the profile of x up to time t, and sets the consumption from t onwards to zero, or $X^t(s) = \min\{X(s), X(t^-)\}$.
- Shift to left: ${}^{t\leftarrow} x$ replicates on $[0, T-t]$ the profile of x on $[t, T]$.

[6]Showing that the term $\int_0^T a(Y(t),t)X'(t)dt$ is continuous is more involved because our $a(Y,t) = u_Y(Y,t)$ may go to infinity as $Y \to Y_0\theta(T)$.

- Shift to right: $x^{\to t}$ replicates on $[t,T]$ the profile of x on $[0,T-t]$.

$$^{t\leftarrow}X(s) = \begin{cases} X(s+t), s \in [0,T-t] \\ X(T), s \in (T-t,T]. \end{cases} \qquad X^{\to t}(s) \;\; = \begin{cases} 0, s \in [0,t) \\ X(s-t), s \in [t,T]. \end{cases}$$

Let $^{t}\mathscr{X} = \{^{t}x \mid x \in \mathscr{X}\}$ and $\mathscr{X}^{t} = \{x^{t} \mid x \in \mathscr{X}\}$.

Consider a binary relation $\succeq$ over $\mathscr{X}$. The relations $\succ$ and $\sim$ are defined in the usual way.

Axiom 1 (Preference Relation). *$\succeq$ is complete, transitive, and continuous wrt convergence in distribution.*

Axiom 2 (Terminal Monotonicity). *For all $t \in [0,T)$, $x \in \mathscr{X}$ and $k > 0$, $x^{t} + k \cdot \mathbf{1}_{t} \succ x^{t}$.*

Axiom 2 applies only to gulps added at the end of a profile. It implies that something is better than nothing, $\mathbf{1}_{t} \succ \mathbf{0}$, $t < T$; but does not imply that more is preferred to less. For example, the decision maker may dislike adding a gulp at the beginning of a profile because too much early consumption may lower the enjoyment of subsequent consumption.

Axiom 3 (Sensitivity to History). *Let $t < \tau < T$, $\varepsilon > 0$, and $k_1, k_2 \geq 0$ with $k_1 \neq k_2$. There are profiles $x_1, x_2 \in \mathscr{X}^{\tau}$ with $X_1(\tau^-), X_2(\tau^-) < \varepsilon$ such that $k_1 \cdot \mathbf{1}_{t} + {}^{t}x_1 \sim k_1 \cdot \mathbf{1}_{t} + {}^{t}x_2$ but $k_2 \cdot \mathbf{1}_{t} + {}^{t}x_1 \nsim k_2 \cdot \mathbf{1}_{t} + {}^{t}x_2$.*

In contrast to discounted utility, sensitivity to history rules out preferences that are completely insensitive to past consumption. In fact, it requires that a different history produced by gulps of sizes $k_1 \neq k_2$, can alter preferences over "small" continuation profiles. A side effect of Axiom 3 is to eliminate the possibility of a trivial preference on $^{t}\mathscr{X}$. Axiom 3 is technically key to uniquely pin down a state. To do so, it rules out degenerate cases such as linear felicity or a dual function that is constant on an open set (for a similar discussion, see Rozen (2010, footnote 12)).

Axiom 4 (Restoring Property). *Let $t \in (0,T]$ and $x \in \mathscr{X}$ with $X(t^-) > 0$. Then, there exists a restoring gulp $0 < C(x,t^-) < X(t^-)$, such that*

$$x^t + {}^t x_1 \succeq x^t + {}^t x_2 \quad \textit{if and only if} \quad C(x,t^-) \cdot \mathbf{1}_t + {}^t x_1 \succeq C(x,t^-) \cdot \mathbf{1}_t + {}^t x_2.$$

Moreover, the restoring gulp is additive, that is, $x_1^t + x_2^t + {}^t x_3 \succeq x_1^t + x_2^t + {}^t x_4$ if and only if $C(x_1,t^-) \cdot \mathbf{1}_t + C(x_2,t^-) \cdot \mathbf{1}_t + {}^t x_3 \succeq C(x_1,t^-) \cdot \mathbf{1}_t + C(x_2,t^-) \cdot \mathbf{1}_t + {}^t x_4$.

Thus, there is a gulp at time t that perfectly mimics the effect of consumption prior to t on continuation preferences. The condition is not too stringent, as it allows the gulp to be a function of x and t. By transitivity, if profiles x_1 and x_2 happen to agree on $C(x_1,t^-) = C(x_2,t^-)$, then
(*) $x_1^t + {}^t x_4 \succeq x_1^t + {}^t x_5$ *if and only if* $x_2^t + {}^t x_4 \succeq x_2^t + {}^t x_5$.
Hence, continuation preferences depend on one sufficient statistic.

Axiom 5 (Conditional Separability). *Let $t \in (0,T]$, $x_1, x_2, x_3 \in \mathscr{X}^t$ with $C(x_1,t^-) = C(x_2,t^-) = C(x_3,t^-)$, and $x_4, x_5, x_6 \in {}^t\mathscr{X}$.*
If $x_1^t + {}^t x_5 \sim x_2^t + {}^t x_4$ and $x_1^t + {}^t x_6 \sim x_2^t + {}^t x_5 \sim x_3^t + {}^t x_4$, then $x_2^t + {}^t x_6 \sim x_3^t + {}^t x_5$.
Moreover, if $x_1^t + {}^t x_4 \succeq x_2^t + {}^t x_4$, then $x_1^t + {}^t x_5 \succeq x_2^t + {}^t x_5$.

Axiom 5 applies two-dimensional separability to separate consumption before and after any given time point. It is a conditional version of the hexagon condition, and the last condition together with (*) is two-attribute independence (Karni and Safra, 1998, p. 394). It states that preferences over future consumption depend on past consumption via the restoring gulp, but are otherwise independent of the shape of past consumption.

Axiom 6 (Decreasing Responsiveness). *Let $t \in (0,T)$ and $x_1, x_2 \in \mathscr{X}$.*
If $x_1^t \sim x_2^t$, $C(x_1,t^-) < C(x_2,t^-)$, and $k > 0$, then $x_1^t + k \cdot \mathbf{1}_t \succ x_2^t + k \cdot \mathbf{1}_t$.

All else being equal, consumption when the state is low is preferred to consumption when the state is high. Axiom 6 captures a notion of diminishing returns and a preference for savoring.

Axiom 7 (Compensated Stationarity). *Let $t \in [0, T)$. For some $Z(t) \geq 0$ and all $x_1, x_2 \in \mathscr{X}^{T-t}$, $x_1 \succeq x_2$ if and only if $Z(t) \cdot \mathbf{1}_t + x_1^{\to t} \succeq Z(t) \cdot \mathbf{1}_t + x_2^{\to t}$.*

The stationarity axiom of discounted utility postulates that if $x_1 \succeq x_2$, then $x^t + x_1^{\to t} \succeq x^t + x_2^{\to t}$ for all histories x^t. This unconditional form cannot possibly hold if past consumption influences current preferences. Instead, we impose stationarity only if a compensatory gulp that restores the preferences at time 0 is added. We think of $Z(t)$ as a gulp that sets the state at time t equal to the level of the state at time 0. That is, $Z(t)$ offsets the decay of the state variable in the absence of consumption during $[0, t)$.

An alternative presentation of compensated stationarity is the following:
There is a preference preserving profile $\bar{x} \in \mathscr{X}$ such that for all $x_1, x_2 \in \mathscr{X}^{T-t}$, $x_1 \succeq x_2$ if and only if $\bar{x}^t + x_1^{\to t} \succeq \bar{x}^t + x_2^{\to t}$.
Of course, $Z(t)$ is simply the restoring gulp associated with $\bar{x}$ at time t.

We now present our main representation result, in two versions. Next, we discuss the key steps behind the construction of the utility functional.

Theorem 1. *$\succeq$ satisfies Axioms 1-7 if and only if $U^{\mathrm{DS}}(x)$ represents $\succeq$.*

While $U^{\mathrm{DS}}(x)$ specializes to the discrete case, the result stands alone.

Theorem 2. *$\succeq$ restricted to $\mathscr{X}_d$ satisfies Axioms 1-7 if and only if $\succeq$ is represented by $U^{\mathrm{DS}}(x)$, $x \in \mathscr{X}_d$.*

3.1 The Restoring Gulp

The goal of Axioms 1 to 4 is to derive a sufficient statistic, the restoring gulp, that captures the effect of past consumption on current preferences. The state will be equal to the restoring gulp, plus a function of time that reflects changes in the state in the absence of consumption. Axiom 4 ensures the existence, and Axiom 3 the uniqueness, of a restoring gulp. By additivity, it follows that it takes the usual linear form $C(x, t^-) = \int_{0^-}^{t^-} \theta(t, s) dX(s)$ HHK (Eq. 3). By $0 < C(x, t^-) < X(t^-)$, we have that $\theta(t, s)$ is strictly positive, decreasing in t, and increasing in s. One would think that these conditions exhaust Axioms 1 to 4. Surprisingly, we discover that $\theta(t,s)$ must take a more specific form.

Lemma 4. *Under Axioms 1 to 4, there is a decay function $\theta(t)$ such that $C(x,t^-) = \int_{0^-}^{t^-} \frac{\theta(t)}{\theta(s)} dX(s)$. Specifically, $\theta(t)$ is equal to $C(\mathbf{1}_0, t^-)$, the restoring gulp at time $t > 0$ associated with a unit gulp at time 0.*

To see that $\theta(t,s) = \theta(t)/\theta(s)$, $t > s$, we first show that if two profiles agree on $C(x_1, s^-) = C(x_2, s^-)$, and are identical on $[s,t)$, then they also agree on $C(x_1,t^-) = C(x_2,t^-)$.[7] Two such profiles are x^s and $C(x,s^-) \cdot \mathbf{1}_s$, which must share the same restoring gulp at time t, or $C(x^s, t^-) = C(C(x,s^-)\cdot \mathbf{1}_s, t^-)$. By linearity, the latter is equal to $C(\mathbf{1}_s, t^-)C(x,s^-)$. Thus, $C(x^s,t^-) = C(\mathbf{1}_s,t^-)C(x,s^-)$, or

$$\int_{0^-}^{s^-} \theta(t,\tau) dX(\tau) = \theta(t,s) \int_{0^-}^{s^-} \theta(s,\tau) dX(\tau).$$

This can only hold if $\theta(t,\tau) = \theta(t,s)\theta(s,\tau)$ for all $\tau < s < t$. Setting $\tau = 0$ and letting $\theta(t) = \theta(t,0)$ we conclude that $\theta(t) = \theta(t,s)\theta(s)$.

3.2 The Generalized Satiation Model

Using continuity, we find that for $x \in \mathscr{X}_d$, the generalized satiation model is given by (see proof of Lemma 2):

$$U^{\text{GS}}(x \in \mathscr{X}_d) = \sum_{i=0}^{n} \left[u\left(Y(t_i^-) + k_i, t_i\right) - u\left(Y(t_i^-), t_i\right)\right], \text{ where} \tag{8}$$

$$Y(t_i^-) = Y_0 \theta(t_i) + \sum_{t_j < t_i} k_j \frac{\theta(t_i)}{\theta(t_j)}. \tag{9}$$

Lemma 5. *$\succeq$ on $\mathscr{X}_d$ satisfies Axioms 1-6 if and only if $\succeq$ is represented by $U^{\text{GS}}(x)$, where $u(Y,t)$ is a felicity function that may not be differentiable.*

[7] Applying Axiom 4 to t yields $x^t + {}^t x_1 \succeq x^t + {}^t x_2$ if and only if $C(x,t^-) \cdot \mathbf{1}_t + {}^t x_1 \succeq C(x,t^-) \cdot \mathbf{1}_t + {}^t x_2$. Next, apply Axiom 4 twice, first to s and then to t, to conclude that $x^t + {}^t x_1 \succeq x^t + {}^t x_2$ if and only if $C(x,s^-)\cdot \mathbf{1}_s + {}^s x^t + {}^t x_1 \succeq C(x,s^-)\cdot\mathbf{1}_s + {}^s x^t + {}^t x_2$, if and only if $C\left(C(x,s^-)\cdot \mathbf{1}_s + {}^s x^t, t^-\right) \cdot \mathbf{1}_t + {}^t x_1 \succeq C\left(C(x,s^-)\cdot \mathbf{1}_s + {}^s x^t, t^-\right)\cdot \mathbf{1}_t + {}^t x_2$. By Axiom 3, the restoring gulp is unique, implying the recursion $C(x,t^-) = C\left(C(x,s^-)\cdot \mathbf{1}_s + {}^s x^t, t^-\right)$.

More generally, the preference represented by $U^{\text{GS}}(x)$ satisfies Axioms 1 to 6 on $\mathscr{X}$. As for the converse, note that $\mathscr{X}_d$ is dense in $\mathscr{X}$ and that if $u(Y,t)$ were differentiable, then $U_d^{\text{GS}}(x) \to U^{\text{GS}}(x)$. Thus, Axioms 1 to 6, together with some condition ensuring that $u(Y,t)$ meets Definition 2, imply the $U^{\text{GS}}(x)$ representation on $\mathscr{X}$.

Continuity plays a central role in the construction of (8). After using conditional separability, we have the partial result

$$U(x) = \sum_{i=0}^{n} u(C(x,t_i^-), k_i, t_i), \text{ with } u(C,0,t_i) = 0 \text{ for all } t_i.$$

Consider the sequence of profiles $x_m = k' \cdot \mathbf{1}_t + k \cdot \mathbf{1}_{t+1/m}$. We verify that $x_m \to x = (k'+k) \cdot \mathbf{1}_t$ as $m \to \infty$. That $\lim_{m\to\infty} U(x_m) = U(x)$ implies $u(0,k',t)+u(k',k,t) = u(0,k'+k,t)$, or $u(k',k,t) = u(0,k'+k,t)-u(0,k',t)$. Defining $u(Y,t) = u(0,Y,t)$ yields $u(Y,k,t) = u(Y+k,t) - u(Y,t)$.

The satiation level in U^{GS} is equal to the restoring gulp plus $Y_0\theta(t)$. Such function is arbitrary, because changes in $\bar{Y}(t) = Y_0\theta(t)$ can be absorbed by the time-dependent $u(Y,t)$, after writing $u(C,t) = u(Y - \bar{Y}(t), t)$. Thus, Axioms 1 to 6 do not allow us to disentangle the state decay from other effects of time on preferences. $Y_0\theta(t)$ is a default guess of the state decay.

3.3 The Strength of Compensated Stationarity

Compensated stationarity is surprisingly strong. Not only it provides the stationarity of the state, and of the felicity function, but yields as a bonus the continuous differentiability of $u(Y)$.

Lemma 6. *If $\succeq$ on $\mathscr{X}$ satisfies Axioms 1-7, then there exists*

(i) $Y_0 \geq 0$ *and* $\gamma > 0$ *such that* $\theta(t) = e^{-\gamma t}$ *and* $Z(t) = Y_0 - Y_0 e^{-\gamma t}$*;*

(ii) $\delta \in \mathbb{R}$ *and a felicity function* $u(Y)$ *such that* $u(Y,t) = e^{-\delta t} u(Y)$*.*

Theorem 1 follows from combining Lemmas 5 and 6.

To see how (i) comes about, recall that Axiom 7, allows us to preserve preference after postponing consumption, an operation we denote by

$A7^{\to t}$. There is also ${}^{t\leftarrow}A7$, the possibility of preserving preference after advancing consumption, provided we subtract a compensatory gulp, and add a restoring gulp to account for the effect of eliminating x^t.[8]

For $s < t$ and $x_1, x_2 \in \mathscr{X}$, suppose

$$x^s + {}^{(t-s)\leftarrow}({}^t x_1) \succeq x^s + {}^{(t-s)\leftarrow}({}^t x_2). \qquad (\succeq^\dagger)$$

We can shift consumption to the right by $t - s$ and preserve preference by applying $A7^{\to t-s}$.Alternatively, we can do the same by applying ${}^{s\leftarrow}A7$ first, followed by $A7^{\to t}$. That both transformations result in the same preferences is equivalent to[9]

$$C(x, s^-) = C\left(x^{\to(t-s)}, t^-\right) \text{ and} \qquad (13)$$

$$Z(t) - Z(s) = Z(t-s)C(\mathbf{1}_{t-s}, t^-). \qquad (14)$$

Thus, the restoring gulp is time invariant, hence it must decay exponentially. Indeed, insert $x = \mathbf{1}_0$ into (13) to obtain $C(\mathbf{1}_0, s^-) = \theta(s) = C(\mathbf{1}_{t-s}, t^-) = \theta(t)/\theta(t-s)$, a relation that holds if and only if $\theta(t) = e^{-\gamma t}$, for some $\gamma > 0$. Similarly, (14) can be rewritten as $Z(t) - Z(s) = Z(t-s)\theta(t)/\theta(t-s)$, which can hold only if $Z(t) = Y_0 - Y_0 e^{-\gamma t}$, for some $Y_0 \geq 0$.

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[8] Formally, ${}^{t\leftarrow}A7$ works as follows. Let $t \in (0, T)$ and $x \in \mathscr{X}$ such that $C(x, t^-) \geq Z(t)$. We have that $x^t + {}^t x_1 \succeq x^t + {}^t x_2$ *if and only if*

$$[C(x, t^-) - Z(t)] \cdot \mathbf{1}_0 + {}^{t\leftarrow}x_1 \succeq [C(x, t^-) - Z(t)] \cdot \mathbf{1}_0 + {}^{t\leftarrow}x_2. \qquad (10)$$

To see this, we first use A4 to replace x^t for $C(x, t^-) \cdot \mathbf{1}_t$, and then rewrite $C(x, t^-) \cdot \mathbf{1}_t$ as $Z(t) \cdot \mathbf{1}_t + [C(x, t^-) - Z(t)] \cdot \mathbf{1}_t$. By A7, we maintain preference if we eliminate $Z(t) \cdot \mathbf{1}_t$ and shift consumption to the left.

[9] By $A7^{\to t-s}$, $\succeq^\dagger$ *if and only if*

$$Z(t-s) \cdot \mathbf{1}_{t-s} + (x^s)^{\to(t-s)} + {}^t x_1 \succeq Z(t-s) \cdot \mathbf{1}_{t-s} + (x^s)^{\to(t-s)} + {}^t x_2. \qquad (11)$$

By A4, $\left[Z(t-s)C(\mathbf{1}_{t-s}, t^-) + C(x^{\to(t-s)}, t^-)\right] \cdot \mathbf{1}_t$ replaces $Z(t-s) \cdot \mathbf{1}_{t-s} + (x^s)^{\to(t-s)}$.

By ${}^{s\leftarrow}A7$, $\succeq^\dagger$ *if and only if* $\left[C(x, s^-) - Z(s)\right] \cdot \mathbf{1}_0 + {}^{t\leftarrow}({}^t x_1) \succeq \left[C(x, s^-) - Z(s)\right] \cdot \mathbf{1}_0 + {}^{t\leftarrow}({}^t x_2)$. By $A7^{\to t}$, the latter preference holds if and only if

$$\left[C(x, s^-) - Z(s) + Z(t)\right] \cdot \mathbf{1}_t + {}^t x_1 \succeq \left[C(x, s^-) - Z(s) + Z(t)\right] \cdot \mathbf{1}_t + {}^t x_2. \qquad (12)$$

Hence, $C(x, s^-) - Z(s) + Z(t) = Z(t-s)C(\mathbf{1}_{t-s}, t^-) + C(x^{\to(t-s)}, t^-)$, and equality that hold true for all $x \in \mathscr{X}$ if and only if (13) and (14) hold simultaneously.

Regarding (ii), we use the dual function to translate the utility representation over x as a utility representation over Y. Crucially, compensated stationarity over x implies stationarity in unconditional form over Y, resulting in $u(Y,t) = e^{-\delta t}u(Y)$. Thus, compensated stationarity allows us to separate the effect of satiation decay, captured by Y_0 and γ, from the effect of temporal preference, captured by δ.

Finally, to see that $u(Y)$ is continuously differentiable, consider a profile that sets the satiation level to $Y > Y_0$ at time 0 and keeps the satiation level constant afterwards, $x = (Y - Y_0) \cdot \mathbf{1}_0 + \gamma Y \cdot 1_{(0,T)}$. We can approximate x either from below and from above. By continuity, the total utility of either sequence must be the same, which implies that the left-side and the right-side derivatives of $u(Y)$ must be equal. Moreover, the expression for such derivative, given in Equation (A9) in the Appendix, is continuous.

3.4 Monotonicity and the Satiation Point

Recall the dual function $u^*(Y) = \gamma u'(Y)Y + \delta u(Y)$ and the representation

$$U^{\mathrm{DS}}(x) = e^{-\delta T}u(Y(T)) - u(Y_0) + \int_0^T e^{-\delta t}u^*(Y(t))dt.$$

If consumption gulps are allowed, then discounted satiation preferences satisfy *local non-satiation.* Indeed, a consumer with more budget can increase the gulp at time T, which only increases $e^{-\delta T}u(Y(x,T))$, yielding $x + k \cdot \mathbf{1}_T \succ x$, $k > 0$.

Next, consider *uniform monotonicity*, namely, *for all* $x_1, x_2 \in \mathscr{X}$ *with* $x_2 \neq \mathbf{0}$, $x_1 + x_2 \succ x_1$. Clearly, if $u^*(Y)$ is strictly increasing, then DS satisfies uniform monotonicity; and the converse holds as well.

The dual u^*, however, may not inherit the strict monotonicity of u. Thus, discounted satiation may violate uniform monotonicity. For example, let $u(Y) = Y^\alpha/\alpha$, $\alpha < 1$. We have that $u^*(Y) = (\delta + \gamma\alpha)u(Y)$ is strictly increasing only if $\alpha > -\delta/\gamma$. The following results shows the three typical cases (monotonic, pathological, and satiation point).

Proposition 1. *Assume u exhibits IRRA (i.e., twice differentiable with $-Yu''(Y)/u'(Y)$ non-decreasing). We have three cases.*

1. *If $\lim_{Y\to\infty} -Y\frac{u''(Y)}{u'(Y)} \leq 1+\delta/\gamma$, then $u^*(Y)$ is strictly increasing.*

2. *If $\lim_{Y\to 0} -Y\frac{u''(Y)}{u'(Y)} \geq 1+\delta/\gamma$, then $u^*(Y)$ is non-increasing.*

3. *Otherwise, u^* is unimodal and the satiation point $Y^S \in (0,\infty)$ solves $-Yu''(Y)/u'(Y) = 1+\delta/\gamma$.*

The second case is pathological, as the consumer prefers to postpone all consumption towards a gulp at T. The third case, however, reflects a desire for savoring, ensuring that the satiation level before time T stays below Y^S. For exponential felicity, $u(Y) = \varrho\left(1-e^{-Y/\varrho}\right)$, $\varrho > 0$, the dual is $u^*(Y) = \gamma Y e^{-Y/\varrho} + \delta\varrho\left(1-e^{-Y/\varrho}\right)$, which peaks at $Y^{\mathrm{S}} = \varrho\,(1+\delta/\gamma)$.

Suppose gulps of consumption are not allowed, only rates with some upper bound. If the felicity exhibits a satiation point, then it might be optimal to leave some cake on the table. That is, preferences become locally satiated. While this feature may not be comfortable for equilibrium analysis (and one can get around by allowing gulps, or assuming u^* is increasing), it provides prescriptive insight on savoring and consumption moderation.

4 Infinite Consumption Horizon

We now extend the axiomatic foundation to the case of consumption profiles with unbounded support. Let $\overline{\mathscr{X}}$ contain all right-continuous and non-decreasing functions, $X(t) : [0,\infty) \to [0,\infty)$. Here, $x_m \in \overline{\mathscr{X}}$ converges to $x \in \overline{\mathscr{X}}$ if and only if $X_m(t) \to X(t)$ at all points of continuity of $X(t)$.

Proposition 2. *$\succeq$ on $\overline{\mathscr{X}}$ satisfies Axioms 1 to 7, if and only if $\succeq$ is represented by $U^{\mathrm{DS}}(x)$, with the additional requirement that*

1. *$\delta > 0$.*

2. *$u(Y)$ is bounded from above.*

3. *if* $Y_0 = 0$*, then* $u(Y)$ *is continuous over* $[0, \infty)$*; and if* $Y_0 > 0$*, then* $u(Y)$ *is continuous over* $(0, \infty)$*, with* $\lim_{T\to\infty} e^{-\delta T} u\left(Y_0 e^{-\gamma T}\right) = 0$.

4. $\lim_{T\to\infty}\left(\sup_{x\in\mathscr{X}} \int_T^\infty e^{-\delta t} u^*(Y(t))dt\right) = 0$.

Similar to Koopmans (1960) and Rozen (2010), we find that when the horizon is infinite, preferences must necessarily be impatient. As in Rozen (2010), the felicity function must be bounded from above. To see why, consider the sequence $x_m = k(m) \cdot \mathbf{1}_m$, where $k(m)$ is some increasing function (e.g., $k(m) = e^{e^m}$). This sequence converges to $\mathbf{0}$ because $X(t) = 0$, for $m > t$. Hence, we must have $U^{\text{DS}}(k(m) \cdot \mathbf{1}_m) = u(Y_0 e^{-\gamma m} + k(m)) - u(Y_0 e^{-\gamma m}) \to 0$. For this to occur, u must be necessarily bounded.

Consider the conditions for $u(Y) = Y^\alpha/\alpha$, $\alpha < 1$. That u is upper bounded forces $\alpha < 0$, which implies that felicity is unbounded from below. Hence, we must set $Y_0 > 0$ and require $\lim_{T\to\infty} e^{-\delta T} u\left(Y_0 e^{-\gamma T}\right) = 0$, which is equivalent to $\alpha > -\delta/\gamma$. The latter ensures that $u^*(Y) = (\delta + \gamma\alpha)u(Y)$ is strictly increasing (see Proposition 1) and also guarantees the 4^{th} requirement.

5 The Durability Model

An important special case of $U^{\text{HHK}}(x)$ after setting $a(Y, t) = 0$ is the durability model, $U^{\text{D}}(x) = \int_0^{T^D} b(Y(t), t)dt$, with $Y(t)$ as in (6). Here, $T^D \geq T$ denotes the durability horizon, which may be finite or infinite. In U^{D}, consumption increases the level of the state, and it is the state itself that produces utility. For example, a gulp $\mathbf{1}_t$ produces utility during $[t, T^D]$.

Useful in applications is the discounted durability model with $T^D = \infty$ (HHK (§8.3] and Bank and Riedel (2001, §4)), given by

$$U^{\text{DD}}(x) = \int_0^\infty e^{-\delta t} b(Y(t))dt,$$

$Y(t)$ as in (2); and $\delta \in \mathbb{R}$ if (the consumption horizon) $T < \infty$, and $\delta > 0$ if $T = \infty$. We apply Lemma 3 and discover that (see Figure 2):

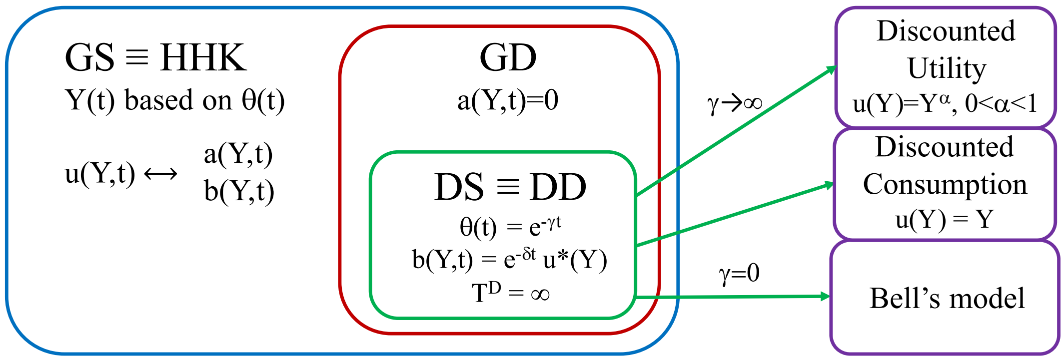

Figure 2: Relationship between the utility functionals.

Corollary 3. *U^{DD} and U^{DS} are preferentially equivalent.*

The equivalence extends to the case of $T = \infty$, provided the conditions of Proposition 2 are met. Specifically, given $U^{\mathrm{DD}}(x) = \int_0^\infty e^{-\delta t} b(Y(t))dt$, let

$$u(Y) = \int_0^\infty e^{-\delta s} b\left(Ye^{-\gamma s}\right) ds$$

to conclude that $U^{\mathrm{DD}}(x) = u(Y_0) + U^{\mathrm{DS}}(x)$. Note that $e^{-\delta T} u(Y(T)) = \int_T^\infty e^{-\delta t} b\left(Y(T)e^{-\gamma(t-T)}\right) dt$. Conversely, given U^{DS}, set $b(Y) = u^*(Y)$, plug $e^{-\delta T} u(Y(T)) = \int_T^\infty e^{-\delta t} u^*\left(Y(T)e^{-\gamma(t-T)}\right) dt$ into Lemma 1 to confirm that $U^{\mathrm{DS}}(x) = U^{\mathrm{DD}}(x) - u(Y_0)$. Thus, $e^{-\delta T} u(Y(T))$ can be seen as the durability utility gained from the lingering of satiation after T.

If (the durability horizon) $T^D < \infty$, then the durability model is time dependent. Intuitively, because consumption is enjoyed via a memory flow, if the flow is truncated at a finite time, then consumption near the horizon T^D becomes less valuable. Thus, even if b takes a stationary form, the corresponding felicity function $u(Y,t) = \int_t^T e^{-\delta s} b\left(Ye^{-\gamma(s-t)}\right) ds$ is not stationary.

6 Neighboring Utility Models

Three relevant models lie at the boundaries of U^{DS} (see Figure 2). Unless noted otherwise, u denotes a felicity function.

6.1 Discounted Utility

In this subsection, we assume $Y_0 = 0$ and that u is bounded from below and normalized to $u(0) = 0$. Intuitively, when $\gamma \to \infty$, past consumption does not influence current utility because the state variable returns to zero very fast. In discrete time, and if we assume gulps are spaced out in time, it is trivial to verify that the discounted satiation model converges towards discounted utility. Indeed, let $t_i = i\frac{T}{n}$, $i = 0, 1, ..., n$, and $x = \sum_{i=0}^{n} k_i \cdot \mathbf{1}_{t_i}$. Noting that $Y_i = (Y_{i-1} + k_{i-1})e^{-\gamma(t_i - t_{i-1})}$, $i = 1, ..., n$, we conclude that $\lim_{\gamma\to\infty} U^{\text{DS}}(x) = \sum_{i=0}^{n} e^{-\delta t_i} u(k_i)$.

With qualifications, the result also holds for the continuous-time version of discounted utility, given by

Proposition 3. *Let $x \in \mathscr{X}_c$ be piecewise continuous, and u such that $\lim_{Y\to 0} u'(Y) = \beta Y^\alpha$ for some $\beta > 0$ and $0 < \alpha \leq 1$, then,*

$$\lim_{\gamma\to\infty} \gamma^{\alpha-1} U^{\text{DS}}(x) = \beta \int_0^T e^{-\delta t}(x(t))^\alpha dt.$$

The key observation is that for large γ, $Y(t)$ converges to $x(t)/\gamma$. In the case of $u(Y) = Y^\alpha/\alpha$, $0 < \alpha < 1$, we have that instant utility $u'(Y(t))x(t)$ converges to $\gamma^{\alpha-1}u'(x(t)/\gamma)x(t)$, which in turn is equal to $\alpha u(x(t))$. Thus, it is immediate to see that

$$\lim_{\gamma\to\infty} \gamma^{\alpha-1} U^{\text{DS}}(x) = \alpha \int_0^T e^{-\delta t} u(x(t))dt, \ x \in \mathscr{X}_c.$$

This same logic extends to any felicity function whose marginal follows a power law near zero; and induces a power form for the felicity of the limiting form, with $0 < \alpha \leq 1$. Thus, if $u'(0)$ is finite, e.g., $u(Y) = \varrho(1 - e^{-Y/\varrho})$, then U^{DS} converges towards discounted consumption, which we examine next.

The limits we have obtained hold for restricted consumption profiles (discrete with spaced out gulps, or continuous rates), and restricted felicity functions (bounded from below). For arbitrary $x \in \mathscr{X}$ and general felicity functions, the $\lim_{\gamma\to\infty} U^{\text{DS}}(x)$ does not appear well defined.

6.2 Discounted Consumption

Take U^{DS} and set $u(Y) = Y$ to obtain the discounted consumption model:

$$U^{\text{DC}}(x) = \int_0^T e^{-\delta t} dX(t),\ x \in \mathscr{X}.$$

Discounted consumption does not allow us to identify the satiation level, and hence violates Axiom 3. Other than that, $U^{\text{DC}}(x)$ satisfies the rest of axioms. In fact, discounted consumption is the only time separable model satisfying continuity. To see this, subdivide $[0, 1]$ into n equal sized intervals. On each interval of duration $1/n$, alternate between consuming at a rate q_1 and q_2, i.e., let $X_n'(t) = q_1,\ t \in \left[\frac{i}{n} - \frac{1}{n}, \frac{i}{n} - \frac{1}{2n}\right)$, and $X_n'(t) = q_2,\ t \in \left[\frac{i}{n} - \frac{1}{2n}, \frac{i}{n}\right)$, $i = 1, ..., n$. Note that $X_n(t) \to X(t) = \frac{q_1+q_2}{2} t \in \mathscr{X}_c$. Finally, we evaluate x_n and x using the time separable model, and conclude that if both limits agree, then $u(q_1)/2 + u(q_2)/2 = u\left(\frac{q_1+q_2}{2}\right)$, $q_1, q_2 \geq 0$, which implies that $u(q)$ is linear.

6.3 Bell's Accumulation Model

Take U^{DS} and set $\gamma = 0$ to obtain the accumulation model (Bell, 1974),[10]

$$U^{\text{Bell}}(x) = \int_0^T e^{-\delta t} u'(X(t)) dX(t),\ x \in \mathscr{X}.$$

Instant utility is the incremental felicity over and above past accumulated consumption, multiplied by the discount factor. Concretely for $x \in \mathscr{X}_d$, $U^{\text{Bell}}(x) = \sum_{i=1}^n e^{-\delta t_i}[u(X(t_{i-1}) + k_i) - u(X(t_{i-1}))]$.

In Bell's model, because satiation fully accumulates past consumption, the restoring gulp becomes $C(x, t^-) = X(t^-)$, which Axiom 4 does not allow. Our construction relies on $C(x, t^-) < X(t^-)$, which then results in $\gamma > 0$, to derive the differentiability of u.

A second difficulty with U^{Bell} appears when $u(Y) = \varrho(1 - e^{-Y/\varrho})$. Because felicity is invariant to additive shifts and $\gamma = 0$, the initial satiation

[10] Indeed, if $\gamma = 0$, then $Y(t) = Y_0 + X(t)$. After redefining $u(X)$ as $u(X + Y_0)$, the term $u'(Y(t))$ in DS becomes $u'(X(t))$.

level becomes irrelevant. Thus, the satiation level cannot be identified, and the model violates Axiom 3. Other than that, U^{Bell} satisfies the rest of Axioms, including continuity. Thus, U^{Bell} is close in spirit to U^{DS}, but runs into technical difficulties to meet the axioms. Its behavioral characterization seems to require a separate treatment.

7 Discussion and Conclusions

Discounted utility in discrete-time is often interpreted as evaluating aggregated consumption within each period. Under this interpretation, we have that consumption taking place in two instants within the same period are perfect substitutes, but consumption taking place in different periods are complements. Hence, the model mostly satisfies the local substitution property (except when comparing consumption at the end of one period and at the beginning of the next period).

It is not difficult to see that such version of discounted utility is best seen as modification of $U^{\text{DS}}(x)$ whereby the satiation level accumulates consumption within the period, and is reset to zero at regular time intervals, just before a new period begins. Formally, let $t_i = iT/m$, $i = 0, ..., m$; and for $t \in [t_{i-1}, t_i)$, set $Y(t) = X(t) - X(t_{i-1}^-)$. Clearly, $X(t_i) - X(t_{i-1}^-)$ is the aggregated consumption during period $i = 1, ..., m$. To see that the discounted utility model, $\sum_{i=1}^{m} e^{-\delta t_i} u(X(t_i) - X(t_{i-1}^-))$ is approximately equal to $\int_0^T e^{-\delta t} u'(Y(t)) dX(t)$, suffices to observe that

$$\begin{aligned}\int_{t_{i-1}}^{t_i} u'(Y(t)) dX(t) &= \int_{t_{i-1}}^{t_i} u'(X(t) - X(t_{i-1}^-)) dX(t) \\ &= u(X(t_i) - X(t_{i-1}^-)).\end{aligned}$$

The notion that satiation accumulates and resets at regular times seems quite artificial, and important insights can be lost by not adopting a more natural evolution of the satiation level. We hope this paper awakens the interest of exploring those additional insights.

The discounted utility model with felicity unbounded from below exhibits a problematic feature, namely, that the consumer is indifferent to all consumption profiles having a zero consumption rate on some time interval of positive measure. Thus, consuming at rate zero during $(0, \varepsilon)$ and at rate 1 for the rest of time is indifferent to consuming 0 at all times. Not surprisingly, Kopylov (2010)' axiomatization of $U^{\text{DU}}(x)$ applies only to u bounded. To the best of our knowledge, discounted utility with felicity unbounded from below lacks a behavioral foundation. In contrast, discounted satiation admits felicity functions unbounded from below, popular in economics, provided $Y_0 > 0$; and we always have that $U^{\text{GS}}(\mathbf{0}) = 0$.

In summary, discounted satiation addresses some important shortcomings of the discounted utility model, and offers new insights into rational economic behavior.

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Appendix

We will write $Y(x,t)$ to be explicit about the dependence of $Y(t)$ on x. Convergence in distribution, which we adopt, is equivalent to convergence with respect to the metric $\|x_1 - x_2\| = \int_0^T |X_1(t) - X_2(t)|dt + |X_1(T) - X_2(T)|$. $(\mathscr{X}, \|\cdot\|)$ is a complete and separable metric space.

Lemma A1. *For $(\mathscr{X}, \|\cdot\|)$, we have (i) $\mathscr{X}_c$ is dense; (ii) $\mathscr{X}_d$ is dense; and (iii) if $x_m \to x$ and t is a continuity point of x, then ${}^t x_m \to {}^t x$ and $x_m^t \to x^t$.*

Proof. (i) A countable dense set is the set of piecewise linear functions that change their slopes at rational time points and have rational right-hand derivatives (Hindy, Huang, and Kreps, 1992, p. 412).

(ii) Let π denote a partition of $[0,T]$ consisting of $t_0, t_1, ..., t_n$ with $t_0 = 0$ and $t_n = T$. For any $x \in \mathscr{X}$, let

$$x_\pi(t) = X(t_0) \cdot \mathbf{1}_{t_0} + \sum_{i=1}^{n} [X(t_i) - X(t_{i-1})] \cdot \mathbf{1}_{t_i} \in \mathscr{X}_d, \tag{A1}$$

for which $X_\pi(t)$ catches up with $X(t)$ at times t_i, $i = 0, ..., n$. Clearly, as $|\pi| = \max_{i=1,\dots,n} |t_i - t_{i-1}| \to 0$, $\|x - x_\pi\| = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |X(t) - X(t_i)|dt \leq \max_{i=1,\dots,n} |t_i - t_{i-1}| \sum_{i=0}^{n-1} [X(t_{i+1}) - X(t_i)] \to 0$.

(iii) Because t is a continuity point of x, we have $X_m(t^-) \to X(t^-)$. The result then follows from the inequalities $\|{}^t x_m - {}^t x\| = \int_t^T |(X_m(t) - X_m(t^-)) - (X(t) - X(t^-))|dt + |(X_m(T) - X_m(t^-)) - (X(T) - X(t^-))| \leq \|x_m - x\| + |X_m(t^-) - X(t^-)|(1 + T - t)$ and $\|x_m^t - x^t\| = \int_0^t |X_m(s) - X(s)|ds + |X_m(t^-) - X(t^-)|(1 + T - t) \leq \|x_m - x\| + |X_m(t^-) - X(t^-)|(1 + T - t)$, the righthand sides of which converge to zero provided $\|x_m - x\| \to 0$ and $X_m(t^-) \to X(t^-)$. $\square$

We introduce the following operators. For $u(Y,t)$ differentiable,

$$u^*(Y,t) = -u_t(Y,t) - u_Y(Y,t) Y \theta'(t)/\theta(t)$$

is the *dual function* of $u(Y,t)$. For $v(Y,t)$ continuous,

$$v^{\mathrm{ID}}(Y,t) = \int_t^T v\left(Y\theta(s)/\theta(t), s\right) ds$$

is the *inverse dual function* of $v(Y,t)$. The dual and inverse dual satisfy

$$\left(u^*\right)^{\mathrm{ID}}(Y,t) = u(Y,t) - u\left(Y\theta(T)/\theta(t), T\right) \text{ and } \left(v^{\mathrm{ID}}\right)^*(Y,t) = v(Y,t).$$

Lemma A2. *For $x \in \mathscr{X}$, let $Y(x,t)$ be as in (6). Assume that $u(Y,t)$ and $v(Y,t)$ are continuously differentiable on $\overline{\Omega}$. We have:*

(i) $\int_0^T u_Y(Y(x,t),t)dX(t) = u(Y(x,T),T) - u(Y_0,0) + \int_0^T u^*(Y(x,t),t)dt.$

(ii) $\int_0^T v(Y(x,t),t)dt = v^{ID}(Y_0,0) + \int_0^T v_Y^{ID}(Y(x,t),t)dX(t).$

Proof. (i) plugging $dX(t) = dY(x,t) - Y(x,t)\frac{\theta'(t)}{\theta(t)}dt$ produces

$$\int_0^T u_Y(Y(x,t),t)dX(t) = \int_0^T u_Y(Y(x,t),t)dY(x,t) - \int_0^T u_Y(Y(x,t),t)Y(x,t)\frac{\theta'(t)}{\theta(t)}dt.$$

Inserting $u_Y(Y(x,t),t)dY(x,t) = du(Y(x,t),t) - u_t(Y(x,t),t)dt$ yields the result. For (ii), apply (i) to v^{ID} and use $\left(v^{\mathrm{ID}}\right)^{\mathrm{D}} = v$ and $v^{\mathrm{ID}}(Y,T) = 0$. □

For $u(Y,t)$ continuous and $s < t$, let $u_d^*(Y,s,t) = \left[u(Y,s) - u\left(Y\theta(t)/\theta(s), t\right)\right]/(t-s)$ be the *dual function* of $u(Y,t)$ in the discrete time.

Lemma A3. *For $x \in \mathscr{X}_d$, let $Y(x,t_i)$ and $Y(x,t_i^-)$ be as in (9). We have:*

$$\sum_{i=0}^{n}\left[u(Y(x,t_i),t_i) - u(Y(x,t_i^-),t_i)\right] = u(Y(x,t_n),t_n) - u\left(Y_0,0\right) + \sum_{i=0}^{n-1} u_d^*(Y(x,t_i),t_i,t_{i+1})(t_{i+1}-t_i).$$

Proof. Write $\sum_{i=0}^{n}\left[u(Y(x,t_i),t_i) - u(Y(x,t_i^-),t_i)\right]$ as $u(Y(x,t_n),t_n) - u\left(Y_0,0\right) + \sum_{i=0}^{n-1}\left[u(Y(x,t_i),t_i) - u(Y(x,t_{i+1}^-),t_{i+1})\right]$ and substitute $Y(x,t_{i+1}^-) = Y(x,t_i)\theta(t_{i+1})/\theta(t_i)$.
□

Proof of Lemma 1. We show that the integral $\int_0^T u_Y(Y(x,t),t)x(t)dt$ is well defined over $\mathscr{X}_c$ and equals $u(Y(x,T),T) - u\left(Y_0,0\right) + \int_0^T u^*(Y(x,t),t)dt$.

Lemma 2 below completes the proof by showing that, by continuity, this relationship extends to $\mathscr{X}$.

As $u(Y,t)$ is differentiable only for $Y > Y_0\theta(t)$, we allow for $u_Y(Y,t) \to \infty$ as $Y \to Y_0\theta(t)$. We verify that, even in this case, $\int_0^T u_Y(Y(x,t),t)dX(t)$ is well defined. For $\varepsilon > 0$, let $t_\varepsilon = \min\{t \in [0,T] | X(t) = \varepsilon\}$. Hence, $X(t) \geq \varepsilon$ for $t \in [t_\varepsilon, T]$. We apply integration by parts to $Y(x,t)$ to get

$$\begin{aligned} Y(x,t) &= Y_0\theta(t) + X(t) - \theta(t)\int_{0^-}^t X(s)d\left(\frac{1}{\theta(s)}\right) \\ &\geq Y_0\theta(t) + X(t) - \theta(t)X(t)\int_{0^-}^t d\left(\frac{1}{\theta(s)}\right) = (Y_0 + X(t))\theta(t). \end{aligned}$$

Thus, $Y(x,t) \geq (Y_0+\varepsilon)\theta(t)$ for $t \in [t_\varepsilon, T]$, and $u_Y(Y,t)$ is continuous over $\{(Y,t) | t_\varepsilon \leq t \leq T, Y \geq (Y_0+\varepsilon)\theta(t)\}$. Since both $X(t)$ and $Y(x,t)$ are absolutely continuous over $[t_\varepsilon, T]$, the integral $\int_{t_\varepsilon}^T u_Y(Y(x,t),t)dX(t)$ is well defined in the Lebesgue-Stieltjes sense. By Lemma A2-(i), we have $\int_{t_\varepsilon}^T u_Y(Y(x,t),t)dX(t) = u(Y(x,T),T) - u\left(Y(x,t_\varepsilon),t_\varepsilon\right) + \int_{t_\varepsilon}^T u^*(Y(x,t),t)dt$.

Next, we show that $\lim_{\varepsilon\to 0^+}\int_{t_\varepsilon}^T u_Y(Y(x,t),t)dX(t)$ exists. For $Y \geq (Y_0+\varepsilon)\theta(t)$, note that $u_Y(Y,t)$ is decreasing in Y, so that $u_Y(Y,t)\left(Y - Y_0\theta(t)\right) \leq u(Y,t) - u\left(Y_0\theta(t),t\right)$ and $u_Y(Y,t) \leq u_Y\left(Y_0\theta(t) + \varepsilon\theta(T),t\right)$, yielding

$$-u_Y(Y,t)Y\frac{\theta'(t)}{\theta(t)} \leq -u_Y\left(Y_0\theta(t)+\varepsilon\theta(T),t\right)Y_0\theta'(t) + \left[u(Y,t) - u\left(Y_0\theta(t),t\right)\right]\frac{\theta'(t)}{\theta(t)}.$$

Accordingly,

$$\int_{t_\varepsilon}^T u^*(Y(x,t),t)dt \leq \begin{array}{l} -\int_{t_\varepsilon}^T u_t(Y(x,t),t)dt \\ -\int_{t_\varepsilon}^T u_Y\left(Y_0\theta(t)+\varepsilon\theta(T),t\right)Y_0\theta'(t)dt \\ -\int_{t_\varepsilon}^T \left[u(Y(x,t),t) - u\left(Y_0\theta(t),t\right)\right]\frac{\theta'(t)}{\theta(t)}dt. \end{array}$$

Substituting $\int_{t_\varepsilon}^T u_Y\left(Y_0\theta(t)+\varepsilon\theta(T),t\right)Y_0\theta'(t)dt = u\left((Y_0+\varepsilon)\theta(T),T\right) - u\left(Y_0\theta(t_\varepsilon)+\varepsilon\theta(T),t_\varepsilon\right) - \int_{t_\varepsilon}^T u_t\left(Y_0\theta(t)+\varepsilon\theta(T),t\right)dt$ into the above, we get

$$\int_{t_\varepsilon}^T u_Y(Y(x,t),t)dX(t) \leq \begin{array}{l} u(Y(x,T),T) - u\left((Y_0+\varepsilon)\theta(T),T\right) \\ +u\left(Y_0\theta(t_\varepsilon)+\varepsilon\theta(T),t_\varepsilon\right) - u\left(Y(x,t_\varepsilon),t_\varepsilon\right) \\ +\int_{t_\varepsilon}^T \left(u_t\left(Y_0\theta(t)+\varepsilon\theta(T),t\right) - u_t(Y(x,t),t)\right)dt \\ -\int_{t_\varepsilon}^T \left[u(Y(x,t),t) - u\left(Y_0\theta(t),t\right)\right]\frac{\theta'(t)}{\theta(t)}dt. \end{array}$$

Since $u_t(Y,t)$ is continuous over $\overline{\Omega}$, the righthand side can be bounded from above with a constant independent of ε. Therefore, the left hand side is monotone increasing in ε and uniformly bounded from above, and hence $\int_0^T u_Y(Y(x,t),t)dX(t) = \lim_{\varepsilon\to 0}\int_{t_\varepsilon}^T u_Y(Y(x,t),t)dX(t)$ exists and takes a finite value. Letting $\varepsilon \to 0$, we get

$$\int_0^T u_Y(Y(x,t),t)dX(t) \leq \begin{cases} u(Y(x,T),T) - u\left(Y_0\theta(T),T\right) \\ + \int_0^T \left[u_t\left(Y_0\theta(t),t\right) - u_t(Y(x,t),t)\right] dt \\ - \int_0^T \left[u(Y(x,t),t) - u\left(Y_0\theta(t),t\right)\right] \frac{\theta'(t)}{\theta(t)} dt. \end{cases} \tag{A2}$$

and $\int_0^T u_Y(Y(x,t),t)dX(t) = u(Y(x,T),T) - u\left(Y_0,0\right) + \int_0^T u^*(Y(x,t),t)dt$. □

Proof of Lemma 2. We show that the functional $\int_0^T u_Y(Y(x,t),t)dX(t)$ over $\mathscr{X}_c$, together with its dual representation, extend continuously to $\mathscr{X}$.

Because $\mathscr{X}_c$ is dense, we can approach any $x \in \mathscr{X}$ with a sequence $x_m \in \mathscr{X}_c$. Given $\|x_m - x\| \to 0$, $X_m(t) \to X(t)$ almost everywhere and $X_m(T) \to X(T)$. By the Portenmanteau theorem, $Y(x_m,t)$ is uniformly bounded from above and $Y(x_m,t) \to Y(x,t)$ almost everywhere. Integration by parts, $Y(x_m,T) - Y(x,T) = X_m(T) - X(T) - \theta(T)\int_{0-}^T (X_m(t) - X(t))d\left(\frac{1}{\theta(t)}\right)$, and the triangle inequality implies $|Y(x_m,T) - Y(x,T)| \to 0$. Next, we show that

$$\lim_{m\to\infty}\int_0^T u_Y(Y(x_m,t),t)dX_m(t) = u(Y(x,T),T) - u\left(Y_0,0\right) + \int_0^T u^*(Y(x,t),t)dt. \tag{A3}$$

On the one hand, we already know from the above step that $\int_0^T u_Y(Y(x_m,t),t)dX_m(t) = u(Y(x_m,T),T) - u\left(Y_0,0\right) + \int_0^T u^*(Y_m(x,t),t)dt$. By Fatou lemma, we get

$$\lim_{m\to\infty}\int_0^T u_Y(Y(x_m,t),t)dX_m(t) \geq u(Y(x,T),T) - u\left(Y_0,0\right) + \int_0^T u^*(Y(x,t),t)dt. \tag{A4}$$

On the other hand, for $\varepsilon > 0$, let $t_{m,\varepsilon} = \min\{t \in [0,T] | X_m(t) = \varepsilon\}$ and $t_\varepsilon = \min\{t \in [0,T] | X(t) = \varepsilon\}$, for which we have $Y(x_m,t) \geq (Y_0+\varepsilon)\theta(T)$ on $[t_{m,\varepsilon}, T]$. We write

$$\int_0^T u_Y(Y(x_m,t),t)dX_m(t) = \int_0^{t_{m,\varepsilon}} u_Y(Y(x_m,t),t)dX_m(t) + \int_{t_{m,\varepsilon}}^T u_Y(Y(x_m,t),t)dX_m(t).$$

Applying (A2) to the first term, we get

$$\int_0^{t_{m,\varepsilon}} u_Y(Y(x_m,t),t)dX_m(t) \leq \begin{array}{l} u(Y(x_m,t_{m,\varepsilon}),t_{m,\varepsilon}) - u\left(Y_0\theta(t_{m,\varepsilon}),t_{m,\varepsilon}\right) \\ + \int_0^{t_{m,\varepsilon}} \left[u_t\left(Y_0\theta(t),t\right) - u_t(Y(x_m,t),t)\right]dt \\ - \int_0^{t_{m,\varepsilon}} \left[u(Y(x_m,t),t) - u\left(Y_0\theta(t),t\right)\right]\frac{\theta'(t)}{\theta(t)}dt. \end{array}$$

Applying Lemma A2-(i) to the second term, we get

$$\int_{t_{m,\varepsilon}}^T u_Y(Y(x_m,t),t)dX_m(t) = u(Y(x_m,T),T) - u(Y(x_m,t_{m,\varepsilon}),t_{m,\varepsilon}) + \int_{t_{m,\varepsilon}}^T u^*(Y(x_m,t),t)dt.$$

Notice that on $[t_{m,\varepsilon}, T]$, $u^*(Y(x_m,t),t)$ is bounded from above due to the uniform boundedness of $Y(x_m,t)$ and the boundedness of $u_Y(Y,t)$ over $Y \geq (Y_0+\varepsilon)\theta(T)$. Since $Y(x_m,t) \to Y(x,t)$ almost everywhere and $t_{m,\varepsilon} \to t_\varepsilon$, the Lebesgue dominated convergence theorem yields $\lim_{m\to\infty} \int_{t_{m,\varepsilon}}^T u^*(Y(x_m,t),t)dt = \int_{t_\varepsilon}^T u^*(Y(x,t),t)dt$, which in turn implies

$$\lim_{m\to\infty} \int_0^T u_Y(Y(x_m,t),t)dX_m(t) \leq \begin{array}{l} u(Y(x,T),T) - u\left(Y_0\theta(t_\varepsilon),t_\varepsilon\right) \\ + \int_0^{t_\varepsilon} \left[u_t\left(Y_0\theta(t),t\right) - u_t(Y(x,t),t)\right]dt \\ - \int_0^{t_\varepsilon} \left[u(Y(x,t),t) - u\left(Y_0\theta(t),t\right)\right]\frac{\theta'(t)}{\theta(t)}dt \\ + \int_{t_\varepsilon}^T u^*(Y(x,t),t)dt. \end{array}$$

Letting $\varepsilon \to 0$, we get $t_\varepsilon \to 0$ and

$$\lim_{m\to\infty} \int_{t_{m,\varepsilon}}^T u^*(Y(x_m,t),t)dt \leq u(Y(x,T),T) - u\left(Y_0,0\right) + \int_0^T u^*(Y(x,t),t)dt. \tag{A5}$$

Combining (A4) and (A5) establishes (A3). Thus, $\lim_{m\to\infty} \int_0^T u_Y(Y(x_m,t),t)dX_m(t)$ depends only on x, and not on the choice of the sequence, allowing us to define this limit as the integral $\int_0^T u_Y(Y(x,t),t)dX(t)$, which equals $u(Y(x,T),T) - u(Y_0,0) + \int_0^T u^*(Y(x,t),t)dt$.

To establish that $\int_0^T u_Y(Y(x,t),t)dX(t)$ is continuous over $\mathscr{X}$, take any sequence $x_m \in \mathscr{X} \to x \in \mathscr{X}$. Because $\mathscr{X}_c$ is dense, we can construct a sequence $\hat{x}_m \in \mathscr{X}_c$ such that $\|\hat{x}_m - x_m\| \leq 1/m$ and $\left|\int_0^T u_Y(Y(\hat{x}_m,t),t)d\hat{X}_m(t) - \int_0^T u_Y(Y(x_m,t),t)dX_m$ $1/m$. Then, $\hat{x}_m \in \mathscr{X}_c \to x \in \mathscr{X}$ and $\lim_{m\to\infty} \int_0^T u_Y(Y(\hat{x}_m,t),t)d\hat{X}_m(t) = \int_0^T u_Y(Y(x,t),t)dX(t)$ by definition. Accordingly,

$$\begin{aligned}
&\lim_{m\to\infty}\left|\int_0^T u_Y(Y(x_m,t),t)dX_m(t) - \int_0^T u_Y(Y(x,t),t)dX(t)\right| \\
\leq &\lim_{m\to\infty}\left|\int_0^T u_Y(Y(x_m,t),t)dX_m(t) - \int_0^T u_Y(Y(\hat{x}_m,t),t)d\hat{X}_m(t)\right| \\
&+ \lim_{m\to\infty}\left|\int_0^T u_Y(Y(\hat{x}_m,t),t)d\hat{X}_m(t) - \int_0^T u_Y(Y(x,t),t)dX(t)\right| \to 0,
\end{aligned}$$

which proves that $\int_0^T u_Y(Y(x,t),t)dX(t)$ is continuous over $\mathscr{X}$.

Finally, we confirm that the integral $\int_0^T u_Y(Y(x,t),t)dX(t)$ over $\mathscr{X}_d$ takes the discrete form $\sum_{i=0}^n [u(Y(t_i),t_i) - u(Y(t_i^-),t_i)]$.

To derive the discrete-time expression, for $x = \sum_{i=0}^n k_i \cdot \mathbf{1}_{t_i} \in \mathscr{X}_d$, let $x_m = \sum_{i=0}^n mk_i \cdot 1_{[t_i,t_i+1/m]} \in \mathscr{X}_c$. Then, x_m converges in distribution to x, and $Y(x_m,t)$ converges almost everywhere to $Y(x,t)$. Since $x_m(t) = 0$ outside $[t_i, t_i+1/m]$, $\int_0^T u_Y(Y(x_m,t),t)dX_m(t) = \sum_{i=0}^n \int_{t_i}^{t_i+1/m} u_Y(Y(x_m,t),t)dX_m(t)$. Applying the dual representation yields

$$\int_{t_i}^{t_i+1/m} u_Y(Y(x_m,t),t)dX_m(t) = \begin{array}{l} u\left(Y\left(x_m, t_i+1/m\right), t_i+1/m\right) \\ -u(Y(x_m,t_i),t_i) \\ +\int_{t_i}^{t_i+1/m} u^*(Y(x_m,t),t)dt. \end{array}$$

By continuity, letting $m \to \infty$ yields the discrete form. □

Proof of Lemma 3. With $\bar{a}(Y,t) = \int_{Y_0}^Y a(y,t)dy$, $\int_0^T a(Y(x,t),t)dX(t) = \int_0^T \bar{a}_Y(Y(x,t),t)dX(t)$. Applying (ii) of Lemma A2 to $\int_0^T b(Y(x,t),t)dX(t)$ yields $\int_0^T b(Y(x,t),t)dt = b^{\mathrm{ID}}(Y_0,0) + \int_0^T b_Y^{\mathrm{ID}}(Y(x,t),t)dX(t)$. Summing up the two components above, we get

$$U^{\text{HHK}}(x) = b^{\text{ID}}(Y_0, 0) + \int_0^T \left[\bar{a}_Y(Y(x,t),t) + b_Y^{\text{ID}}(Y(x,t),t)\right] dX(t).$$

which yields the desired expression by setting $u(Y,t) = \bar{a}(Y,t) + b^{\text{ID}}(Y,t)$. □

Proof of Lemma 4. We divide the proof into eight steps.

Step 1. For $t \in (0,T)$, $C(x,t^-)$ is unique.

Assume, otherwise, that for some t and x, x^t can be replaced by both k_1 and k_2, $k_1 \neq k_2$. By Axiom 3, we can find $x_1, x_2 \in \mathscr{X}$ such that $k_1 \cdot \mathbf{1}_t + {}^t x_1 \sim k_1 \cdot \mathbf{1}_t + {}^t x_2$ but $k_2 \cdot \mathbf{1}_t + {}^t x_1 \nsim k_2 \cdot \mathbf{1}_t + {}^t x_2$, which is equivalent to $x^t + {}^t x_1 \sim x^t + {}^t x_2$ but $x^t + {}^t x_1 \nsim x^t + {}^t x_2$, a contradiction.

Step 2. For $t \in (0,T)$, $C(x,t^-)$ only depends on x^t.

Assume, otherwise, that $x_1^t = x_2^t$ but $C(x_1,t^-) = k_1 \neq k_2 = C(x_2,t^-)$. By Axiom 3, we can find $x_3, x_4 \in \mathscr{X}$ such that $k_1 \cdot \mathbf{1}_t + {}^t x_3 \sim k_1 \cdot \mathbf{1}_t + {}^t x_4$ but $k_2 \cdot \mathbf{1}_t + {}^t x_3 \nsim k_2 \cdot \mathbf{1}_t + {}^t x_4$, which is equivalent to $x_1^t + {}^t x_3 \sim x_1^t + {}^t x_4$ but $x_2^t + {}^t x_3 \nsim x_2^t + {}^t x_4$. This is impossible given $x_1^t = x_2^t$.

Step 3. Regarding continuity, we have that for $t \in (0,T)$ that: (i) $C(x,t^-)$ is left-continuous in t; (ii) $\lim_{s \to t^-} C(\mathbf{1}_s, t^-) = 1$; and (iii) $\|x_m^t - x^t\| \to 0$ implies $C(x_m,t^-) \to C(x,t^-)$. As corollaries of (iii), we have: (a) $\lim_{s \to 0^+} C(\mathbf{1}_s, t^-) = C(\mathbf{1}_0, t^-)$; (b) if $\|x_m - x\| \to 0$ and t is a continuity point of x, then $C(x_m,t^-) \to C(x,t^-)$.

We argue by contradiction. For (i), if $C(x,s^-) \nrightarrow C(x,t^-)$ as $s \to t^-$, due to the boundedness of $C(x,s^-)$, we can subtract a subsequence $C(x,s_m^-)$ such that $C(x,s_m^-) \to C \neq C(x,t^-)$. By Axiom 3, we can find $x_1, x_2 \in \mathscr{X}$ such that $C(x,t^-) \cdot \mathbf{1}_t + {}^t x_1 \sim C(x,t^-) \cdot \mathbf{1}_t + {}^t x_2$ but $C \cdot \mathbf{1}_t + {}^t x_1 \nsim C \cdot \mathbf{1}_t + {}^t x_2$.

Without loss of generality, assume $C \cdot \mathbf{1}_t + {}^t x_1 \succ C \cdot \mathbf{1}_t + {}^t x_2$. Then, we find a sufficiently small $\varepsilon > 0$ such that $C \cdot \mathbf{1}_t + {}^t x_1 \succ C \cdot \mathbf{1}_t + {}^t x_2 + \varepsilon \cdot \mathbf{1}_t$. Since $C(x, s_m^-) \cdot \mathbf{1}_{s_m} \to C \cdot \mathbf{1}_t$ as $s_m \to t^-$, there exists a constant $M > 0$ such that $C(x, s_m^-) \cdot \mathbf{1}_{s_m} + {}^t x_1 \succ C(x, s_m^-) \cdot \mathbf{1}_{s_m} + {}^t x_2 + \frac{\varepsilon}{2} \cdot \mathbf{1}_t$ for all $m \geq M$. Given $s_m < t$ and ${}^{s_m}({}^t x_i) = {}^t x_i$ for $i = 1, 2$, the above is equivalent to $x^{s_m} + {}^t x_1 \succ x^{s_m} + {}^t x_2 + \frac{\varepsilon}{2} \cdot \mathbf{1}_t$. Since $x^{s_m} \to x^t$ as $s_m \to t^-$, taking the limit yields $x^t + {}^t x_1 \succeq x^t + {}^t x_2 + \frac{\varepsilon}{2} \cdot \mathbf{1}_t \succ x^t + {}^t x_2$, a contradiction to $C(x, t^-) \cdot \mathbf{1}_t + {}^t x_1 \sim C(x, t^-) \cdot \mathbf{1}_t + {}^t x_2$. Thus, $C(x, t^-)$ is left-continuous in t.

For (ii), if $C(\mathbf{1}_s, t^-) \nrightarrow 1$ as $s \to t^-$, following the argument above, we can subtract a subsequence $C(\mathbf{1}_{s_m}, t^-)$ such that $C(\mathbf{1}_{s_m}, t^-) \to C \neq 1$, and moreover, $\mathbf{1}_t + {}^t x_1 \sim \mathbf{1}_t + {}^t x_2$ but $C \cdot \mathbf{1}_t + {}^t x_1 \succ C \cdot \mathbf{1}_t + {}^t x_2 + \varepsilon \cdot \mathbf{1}_t$. Since $C(\mathbf{1}_{s_m}, t^-) \to C$, there exists a constant $M > 0$ such that $C(\mathbf{1}_{s_m}, t^-) \cdot \mathbf{1}_t + {}^t x_1 \succ C(\mathbf{1}_{s_m}, t^-) \cdot \mathbf{1}_t + {}^t x_2 + \frac{\varepsilon}{2} \cdot \mathbf{1}_t$ for all $m \geq M$. Given $s_m < t$ and $\mathbf{1}_{s_m}^t = \mathbf{1}_{s_m}$, the above is equivalent to $\mathbf{1}_{s_m} + {}^t x_1 \succ \mathbf{1}_{s_m} + {}^t x_2 + \frac{\varepsilon}{2} \cdot \mathbf{1}_t$. Since $\mathbf{1}_{s_m} \to \mathbf{1}_t$, taking the limit yields $\mathbf{1}_t + {}^t x_1 \succeq \mathbf{1}_t + {}^t x_2 + \frac{\varepsilon}{2} \cdot \mathbf{1}_t \succ \mathbf{1}_t + {}^t x_2$, a contradiction to $\mathbf{1}_t + {}^t x_1 \sim \mathbf{1}_t + {}^t x_2$.

For (iii), if $C(x_m, t^-) \nrightarrow C(x, t^-)$ as $\|x_m^t - x^t\| \to 0$, following the argument above, we can subtract a subsequence, denoted by $C(x_m, t^-)$ again, such that $C(x_m, t^-) \to C \neq C(x, t^-)$, and moreover, $C(x, t^-) \cdot \mathbf{1}_t + {}^t x_1 \sim C(x, t^-) \cdot \mathbf{1}_t + {}^t x_2$ but $C \cdot \mathbf{1}_t + {}^t x_1 \succ C \cdot \mathbf{1}_t + {}^t x_2 + \varepsilon \cdot \mathbf{1}_t$. Since $C(x_m, t^-) \to C$, there exists a constant $M > 0$ such that $C(x_m, t^-) \cdot \mathbf{1}_t + {}^t x_1 \succ C(x_m, t^-) \cdot \mathbf{1}_t + {}^t x_2 + \frac{\varepsilon}{2} \cdot \mathbf{1}_t$ for all $m \geq M$. By definition, this is equivalent to $x_m^t + {}^t x_1 \succ x_m^t + {}^t x_2 + \frac{\varepsilon}{2} \cdot \mathbf{1}_t$. Since $x_m^t \to x^t$, taking the limit yields $x^t + {}^t x_1 \succeq x^t + {}^t x_2 + \frac{\varepsilon}{2} \cdot \mathbf{1}_t \succ x^t + {}^t x_2$, a contradiction to $C(x, t^-) \cdot \mathbf{1}_t + {}^t x_1 \sim C(x, t^-) \cdot \mathbf{1}_t + {}^t x_2$. Thus, $C(x_m, t^-) \to C(x, t^-)$ as $\|x_m^t - x^t\| \to 0$.

As corollaries of (iii), (a) follows from the fact $\lim_{s \to 0^+} \|\mathbf{1}_s^t - \mathbf{1}_0^t\| = 0$, and (b) follows from (ii) of Lemma A1.

Step 4. For $t \in (0, T)$ and $x = \sum_{i=0}^{n} k_i \cdot \mathbf{1}_{t_i} \in \mathscr{X}_d$, $C(x, t^-) = \sum_{t_i < t} k_i C(\mathbf{1}_{t_i}, t^-)$.

Following uniqueness and additivity, it is apparent that $C(x_1+x_2, t^-) = C(x_1, t^-) + C(x_2, t^-)$ for $x_1, x_2 \in \mathscr{X}$. By continuity, we have $C(kx, t^-) = kC(x, t^-)$ for $k \geq 0$ (one can first show this for any rational number k and then approach k with rational numbers if k is irrational), which in turn implies for $x = \sum_{i=0}^{n} k_i \cdot \mathbf{1}_{t_i}$ that $C(x, t^-) = C(x^t, t^-) = \sum_{i=0}^{n} k_i C(\mathbf{1}_{t_i}^t, t^-) = \sum_{0 \leq t_i < t} k_i C(\mathbf{1}_{t_i}, t^-)$. The last equality uses the fact that $\mathbf{1}_{t_i}^t = \mathbf{1}_{t_i}$ if $t_i < t$ and $\mathbf{1}_{t_i}^t = \mathbf{0}$ if $t_i \geq t$.

Step 5. For $s \in (0, T)$ and $x_1, x_2 \in \mathscr{X}^s$, $C(x_1, s^-) = C(x_2, s^-)$ implies $C(x_1, t^-) = C(x_2, t^-)$ for all $t \in (s, T)$.

If, otherwise, $C(x_1, t^-) \neq C(x_2, t^-)$ for some $t \in (s, T)$, then we can find $x_3, x_4 \in \mathscr{X}$ by Axiom 3 such that $C(x_1, t^-) \cdot \mathbf{1}_t + {}^t x_3 \sim C(x_1, t^-) \cdot \mathbf{1}_t + {}^t x_4$ but $C(x_2, t^-) \cdot \mathbf{1}_t + {}^t x_3 \nsim C(x_2, t^-) \cdot \mathbf{1}_t + {}^t x_4$. By Axiom 4, this is equivalent to $x_1^t + {}^t x_3 \sim x_1^t + {}^t x_4$ but $x_2^t + {}^t x_3 \nsim x_2^t + {}^t x_4$. Since $x_1, x_2 \in \mathscr{X}^s$ and $t > s$, we have $x_i^t = x_i^s$ for $i = 1, 2$, and ${}^s({}^t x_i) = {}^t x_i$ for $i = 3, 4$. Thus, $C(x_1, s^-) \cdot \mathbf{1}_s + {}^t x_3 \sim C(x_1, s^-) \cdot \mathbf{1}_s + {}^t x_4$ but $C(x_2, s^-) \cdot \mathbf{1}_s + {}^t x_3 \nsim C(x_2, s^-) \cdot \mathbf{1}_s + {}^t x_4$, which is impossible given $C(x_1, s^-) = C(x_2, s^-)$.

Step 6. For $0 \leq s < t < T$, we have $C(\mathbf{1}_s, t^-) = \theta(t)/\theta(s)$, where $\theta(t)$ is a continuous, strictly positive and strictly decreasing function over $[0, T)$ with $\theta(0) = 1$.

This result builds on Step 5. For $x_1, x_2 \in \mathscr{X}^s$ with $s > 0$ and $X_1(s^-) > 0$, assume without loss of generality that $C(x_2, s^-) = kC(x_1, s^-)$. Due to the linearity of $C(x, s^-)$, we get $C(x_2, s^-) = C(kx_1, s^-)$, which in turn implies $C(x_2, t^-) = C(kx_1, t^-) = kC(x_1, t^-)$ for $t \in (s, T)$. Thus, $C(x_2, s^-)/C(x_1, s^-) = C(x_2, t^-)/C(x_1, t^-)$ for all $t \in (s, T)$. As a corollary, for $0 \leq s < t < T$, $C(\mathbf{1}_0, t^-)/C(\mathbf{1}_s, t^-)$ is independent of t and can be written as a continuous function $\theta(s)$, following which we get $C(\mathbf{1}_0, t^-) = \theta(s)C(\mathbf{1}_s, t^-)$. Letting $s \to 0^+$ and using the fact $\lim_{s\to 0^+} C(\mathbf{1}_s, t^-) = C(\mathbf{1}_0, t^-)$ (see (iii) of Step 3), we set $\theta(0) = \lim_{s\to 0^+} \theta(s) = 1$. Letting $s \to t$ and using the fact $\lim_{s\to t^-} C(\mathbf{1}_s, t^-) = 1$ (see (ii) of Step 3), we get $C(\mathbf{1}_0, t^-) = \theta(t)$. The strict positivity of $\theta(t)$ follows from the requirement $C(\mathbf{1}_0, t^-) > 0$ in Axiom 4 and the strict decreasing monotonicity of $\theta(t)$ follows from the requirement $C(\mathbf{1}_s, t^-) = \theta(t)/\theta(s) < 1$ for $s < t$ in Axiom 4 again.

Step 7. For $t \in (0, T)$, $C(x, t^-) = \int_{0^-}^{t^-} \frac{\theta(t)}{\theta(s)} dX(s)$.

Due to the linearity of $C(x, t^-)$, $C(x, t^-) = \sum_{0\le t_i<t} k_i C(\mathbf{1}_{t_i}, t^-) = \sum_{0\le t_i<t} k_i \theta(t)/\theta(t_i)$ for $x = \sum_{i=0}^{n} k_i \cdot \mathbf{1}_{t_i} \in \mathscr{X}_d$. For $x \in \mathscr{X}$, we employ the sequence constructed in (A1) to obtain $C(x_\pi, t^-) = \frac{\theta(t)}{\theta(t_0)} X(t_0) + \sum_{t_i<t} \frac{\theta(t)}{\theta(t_i)} [X(t_i) - X(t_{i-1})]$, which converges to $\int_{0^-}^{t^-} \frac{\theta(t)}{\theta(s)} dX(s)$ as $|\pi| \to 0$. We then differentiate between two cases. If t is a continuity point of X, then we use (iii) of Step 3 to conclude $C(x, t^-) = \lim_{|\pi|\to 0} C(x_\pi, t^-) = \int_{0^-}^{t^-} \frac{\theta(t)}{\theta(s)} dX(s)$. If t is not a continuity point of X, noticing that X is continuous almost everywhere, we can always find a sequence of continuity points t_m satisfying $t_m \to t^-$, and use (i) of Step 3 to conclude $C(x, t^-) = \lim_{m\to\infty} C(x, t_m^-) = \lim_{m\to\infty} \int_{0^-}^{t_m^-} \frac{\theta(t)}{\theta(s)} dX(s) = \int_{0^-}^{t^-} \frac{\theta(t)}{\theta(s)} dX(s)$. Accordingly, the claim holds true for all $t \in (0, T)$.

Step 8. The $\lim_{t\to T^-} \int_{0^-}^{t^-} \frac{\theta(t)}{\theta(s)} dX(s)$ exists.

As t increases to T, $\theta(t)$ decreases and is bounded from below, and hence $\lim_{t\to T^-} \theta(t)$ exists and can be defined as $\theta(T)$. Similarly, as t increases to T, $\int_{0^-}^{t^-} \frac{\theta(t)}{\theta(s)} dX(s)$ increases and is bounded by $X(T^-)$. Thus, $\lim_{t\to T^-} \int_{0^-}^{t^-} \frac{\theta(t)}{\theta(s)} dX(s)$ exists. Hence, at time 0 we use the convention $x^0 = \mathbf{0}$ to set $C(x, 0^-) = 0$; and at time T, we set $C(x, T^-) = \lim_{t\to T^-} C(x, t^-)$. □

Proof of Lemma 5. We divide the proof into six steps.

Step 1. For $n \geq 2$, let $\mathscr{X}_d(n) = \{x \in \mathscr{X} \,|x = \sum_{i=0}^{n} k_i \cdot \mathbf{1}_{t_i}, k_i \geq 0, t_i = iT/n\}$ be the set of discrete-time profiles with equally sized time intervals. Then, there exist $n+1$ continuous functions $v_{n,i}(C,k) : \mathbb{R}_+^2 \to \mathbb{R}$ with $v_{n,i}(C,0) = 0$ for $i = 0, 1, ..., n$, such that for $x \in \mathscr{X}_d(n)$,

$$U(x) = \sum_{i=0}^{n} v_{n,i}(C(x,t_i^-), k_i).$$

For $C \geq 0$ and $t \in (0,T]$, let $\mathscr{X}^t(C) = \left\{x^t | x \in \mathscr{X}, C(x,t^-) = C\right\}$. Under Axiom 5, for any $t \in (0,T]$, the preference $\succeq$ restricted to $\mathscr{X}^t(C) \times {}^t\mathscr{X}$ satisfies completeness, transitivity, continuity, independence, and the hexagon condition. We know from Karni and Safra (1998, Theorem 1) that $\succeq$ has an additive separable representation on $\mathscr{X}^t(C) \times {}^t\mathscr{X}$, unique up to a positive affine transformation. That is, there exist functions $A(C,x,t)$ and $B(C,x,t)$ such that

$$U(x) = A(C(x,t^-), x^t, t) + B(C(x,t^-), {}^t x, t). \tag{A6}$$

We normalize the representation with $U(\mathbf{0}) = 0$. We further normalize $B(C,\mathbf{0},t) = 0$ for all $t \in (0,T]$, as otherwise, we can write $A(C,x^t,t) + B(C,\mathbf{0},t)$ as a new function $A(C,x^t,t)$. This implies $U(x^t) = A(C(x,t^-),x^t,t)$. We then proceed by induction.

First, applying (A6) at t_1 yields $U(x) = A(C(x,t_1^-),x^{t_1},t_1)+B(C(x,t_1^-),{}^{t_1}x,t_1)$. Since $x^{t_1} = k_0 \cdot \mathbf{1}_0$, $C(x,t_1^-) = k_0\theta(T/n)$, and $C(x,0^-) = 0$, we see that $A(C(x,t_1^-),x^{t_1},t_1)$ depends only on k_0, n and we can define $v_{n,0}(C(x,0^-),k_0) = A(C(x,t_1^-),x^{t_1},t_1)$. Thanks to $U(\mathbf{0}) = 0$, we have $v_{n,0}(0,0) = 0$. Define $v_{n,1}(C(x,t_1^-),k_1) = U(x^{t_2}) - U(x^{t_1}) = B(C(x,t_1^-),k_1 \cdot \mathbf{1}_{t_1},t_1)$. Clearly, $v_{n,1}(C,0) = 0$ and $U(x^{t_2}) = v_{n,0}(C(x,0^-),k_0) + v_{n,1}(C(x,t_1^-),k_1)$.

Next, applying (A6) at t_2 yields $U(x) = A(C(x,t_2^-),x^{t_2},t_2)+B(C(x,t_2^-),{}^{t_2}x,t_2)$. Replacing $A(C(x,t_2^-),x^{t_2},t_2) = U(x^{t_2})$ by $v_{n,0}(C(x,0^-),k_0)+v_{n,1}(C(x,t_1^-),k_1)$, we obtain $U(x) = v_{n,0}(C(x,0^-),k_0)+v_{n,1}(C(x,t_1^-),k_1)+B(C(x,t_2^-),{}^{t_2}x,t_2)$.

Suppose we already have for some $i = 2, ..., n-1$ that

$$U(x) = \sum_{j=0}^{i-1} v_{n,j}(C(x, t_j^-), k_j) + B(C(x, t_i^-), {}^{t_i}x, t_i),$$

where $v_{n,j}(C, 0) = 0$ for $j = 1, ..., i-1$ and $B(C, \mathbf{0}, t_i) = 0$. Define $v_{n,i}(C(x, t_i^-), k_i) = U(x^{t_{i+1}}) - U(x^{t_i}) = B(C(x, t_i^-), k_i \cdot \mathbf{1}_{t_i}, t_i)$. Clearly, $v_{n,i}(C, 0) = 0$ and $U(x^{t_{i+1}}) = \sum_{j=0}^{i} v_{n,j}(C(x, t_j^-), k_j)$. We then apply (A6) at t_{i+1} to get $U(x) = A(C(x, t_{i+1}^-), x^{t_{i+1}}, t_{i+1}) + B(C(x, t_{i+1}^-), {}^{t_{i+1}}x, t_{i+1})$. Replacing $A(C(x, t_{i+1}^-), x^{t_{i+1}}, t_{i+1}) = U(x^{t_{i+1}})$ by $\sum_{j=0}^{i} v_{n,j}(C(x, t_j^-), k_j)$, we obtain

$$U(x) = \sum_{j=0}^{i} v_{n,j}(C(x, t_j^-), k_j) + B(C(x, t_{i+1}^-), {}^{t_{i+1}}x, t_{i+1}).$$

The induction stops at $i = n-1$, with which we define $v_{n,n}(C(x, t_n^-), k_n) = B(C(x, t_n^-), {}^{t_n}x, t_n)$. The fact $v_{n,n}(C, 0) = 0$ follows from $B(C, \mathbf{0}, t_n) = 0$.

Step 2. Let $\mathscr{T}_m = \{ iT/2^m | \, i = 0, 1, ..., 2^m \}$ and $\mathscr{T} = \cup_{m=1}^{\infty} \mathscr{T}_m$. Then, $\mathscr{T}$ is a countable and dense set of times. There exists a continuous function $v(C, k, t_i) : \mathbb{R}_+^2 \times \mathscr{T} \to \mathbb{R}$ with $v(C, 0, t_i) = 0$ for all $\mathscr{T}$ such that for $x = \sum_{i=0}^{n} k_i \cdot \mathbf{1}_{t_i} \in \mathscr{X}_d$, $t_i \in \mathscr{T}$, $i = 0, 1, ..., n$,

$$U(x) = \sum_{i=0}^{n} v(C(x, t_i^-), k_i, t_i).$$

Given $m \geq 1$, we have obtained by Step 1 that $U(x) = \sum_{i=0}^{2^m} v_{2^m,i}(C(x, t_i^-), k_i)$ for $x \in \mathscr{X}_d(2^m)$. We now show that a universal representation—independent of m—is available over gulps occurring on $\mathscr{T}$. Since utility representations are identical up to affine transformations, we normalize the representations over $\mathscr{X}_d(2^m)$ with $U(\mathbf{1}_0) = 1$ and $U(\mathbf{0}) = 0$ for all $m \geq 1$, so that their evaluations of common consumption profiles are identical.

The construction of the universal representation proceeds by induction. For the purpose of clarity, we use k_t to denote the size of the gulp occurring at time t. By Step 1, we have

$$\begin{aligned}U(x) =& v_{2,0}(C(x,0^-),k_0) + v_{2,1}(C(x,T/2^-),k_{T/2}) + v_{2,2}(C(x,T^-),k_T), \quad \forall x \in \mathscr{X}_d(2),\\ U(x) =& v_{4,0}(C(x,0^-),k_0) + v_{4,2}(C(x,T/2^-),k_{T/2}) + v_{4,4}(C(x,T^-),k_T)\\ &+ v_{4,1}(C(x,T/4^-),k_{T/4}) + v_{4,3}(C(x,3T/4^-),k_{3T/4}), \qquad \forall x \in \mathscr{X}_d(4).\end{aligned}$$

Their evaluations are identical on $\mathscr{X}_d(2) \cap \mathscr{X}^{T/2}$, yielding $v_{2,0}(0,k_0) = v_{4,0}(0,k_0)$ for all $k_0 \geq 0$; identical on $\mathscr{X}_d(2) \cap \mathscr{X}^T$, yielding $v_{2,1}(C,k_{T/2}) = v_{4,2}(C,k_{T/2})$ for all $C \geq 0$ and $k_{T/2} \geq 0$; identical on $\mathscr{X}_d(2)$, yielding $v_{2,2}(C,k_T) = v_{4,4}(C,k_T)$ for all $C \geq 0$ and $k_T \geq 0$. Therefore, a universal utility representation over $\mathscr{X}_d(2) \cup \mathscr{X}_d(4)$ is

$$\begin{aligned}U(x) =& v_{2,1}(C(x,0^-),k_0) + v_{2,1}(C(x,T/2^-),k_{T/2}) + v_{2,2}(C(x,T^-),k_T)\\ &+ v_{4,1}(C(x,T/4^-),k_{T/4}) + v_{4,3}(C(x,3T/4^-),k_{3T/4})\\ =& \sum_{m=1}^{2} \sum_{i \in \mathscr{N}_m \setminus \mathscr{N}_{m-1}} v_{2^m,i}(C(x,t_i^-),k_{iT/2^m}),\end{aligned}$$

where we set $\mathscr{N}_0 = \emptyset$ and $\mathscr{N}_m = \{0,1,...,2^m\}$ for $m \geq 1$. Noticing $\mathscr{N}_1 \subset \mathscr{N}_2 \subset ... \subset \mathscr{N}_m \subset ...$, induction yields $U(x) = \sum_{m=1}^{\infty} \sum_{i \in \mathscr{N}_m \setminus \mathscr{N}_{m-1}} v_{2^m,i}(C(x,t_i^-),k_{iT/2^m})$ over $\cup_{m=1}^{\infty} \mathscr{X}_d(2^m)$. Define $v(C,k_{iT/2^m},iT/2^m) = v_{2^m,i}(C,k_{iT/2^m})$ for $i \in \mathscr{N}_m \setminus \mathscr{N}_{m-1}$ with $m = 1,2,...$, we obtain a universal utility representation

$$U(x) = \sum_{m=1}^{\infty} \sum_{t_i \in \mathscr{T}_m \setminus \mathscr{T}_{m-1}} v(C(x,t_i^-),k_{t_i},t_i)$$

over $\cup_{m=1}^{\infty} \mathscr{X}_d(2^m)$, which is the set of discrete-time profiles with gulps in $\mathscr{T}$. The fact $v(C,0,t_i) = 0$ for all $t_i \in \mathscr{T}$ follows from the fact that $v_{2^m,i}(C,0) = 0$ for all $i \in \mathscr{N}_m \setminus \mathscr{N}_{m-1}$, and the continuity of $u(C,k,t_i)$ over $\mathbb{R}_+^2 \times \mathscr{T}$ follows from the continuity of the preference.

Step 3. There exists a continuous extension $v(C,k,t) : \mathbb{R}_+^2 \times [0,T] \to \mathbb{R}$ with $v(C,0,t) = 0$ for $t \in [0,T]$ such that for $x \in \mathscr{X}_d$, $U(x) = \sum_{i=0}^{n} v(C(x,t_i^-),k_i,t_i)$.

By Step 2, we know $U\left(\frac{C}{\theta(t_i)}\cdot\mathbf{1}_0+k\cdot\mathbf{1}_{t_i}\right)=v\left(0,\frac{C}{\theta(t_i)},0\right)+v(C,k,t_i)=U\left(\frac{C}{\theta(t_i)}\cdot\mathbf{1}_0\right)+v(C,k,t_i)$ for all $t_i\in\mathscr{T}$, yielding $v(C,k,t_i)=U\left(\frac{C}{\theta(t_i)}\cdot\mathbf{1}_0+k\cdot\mathbf{1}_{t_i}\right)-U\left(\frac{C}{\theta(t_i)}\cdot\mathbf{1}_0\right)$ for all $t_i\geq 0$. Thus, a natural continuous extension of $v(C,k,t_i)$ is $v(C,k,t)=U\left(\frac{C}{\theta(t)}\cdot\mathbf{1}_0+k\cdot\mathbf{1}_t\right)-U\left(\frac{C}{\theta(t)}\cdot\mathbf{1}_0\right)$. Because $\mathscr{T}$ is dense in $[0,T]$, for $x=\sum_{i=0}^n k_i\cdot\mathbf{1}_{t_i}\in\mathscr{X}_d$, we can construct a sequence $x_j=\sum_{i=0}^n k_i\cdot\mathbf{1}_{t_{ij}}$ with $t_{ij}\in\mathscr{T}$ and such that $t_{ij}\to t_i$ from the left as $j\to\infty$. By Step 2, we have $U(x_j)=\sum_{i=0}^n v(C(x,t_{ij}^-),k_i,t_{ij})$ for all j. By our choice of metric and the continuity of the preference, we have $x_j\to x$ and $U(x_j)\to U(x)$. By the left-continuity of $C(x,t^-)$ (see (i) of Step 3 in the proof of Lemma 4), $\sum_{i=0}^n v(C(x,t_{ij}^-),k_i,t_{ij})$ converges to $\sum_{i=0}^n v(C_i(x,t_i^-),k_i,t_i)$, leading us to the desired representation.

Step 4. The function v obtained in Step 3 must take the form $v(C,k,t)=u(C+k,t)-u(C,t)$, where $u(C,t):\mathbb{R}_+\times[0,T]\mapsto\mathbb{R}_+$ is continuous and strictly increasing in C.

Keeping in mind that $C(x,t^-)$ can only take value 0 for $t=0$ but can take any value in $(0,\infty)$ for $t>0$, we differentiate between two cases. For $t>0$, $k_1\cdot\mathbf{1}_{t_1}+k_2\cdot\mathbf{1}_{t_2}$ converges in distribution to $(k_1+k_2)\cdot\mathbf{1}_t$ as $t_1,t_2\to t$. Due to the continuity of the preference, we have $U(k_1\cdot\mathbf{1}_{t_1}+k_2\cdot\mathbf{1}_{t_2})=v(0,k_1,t_1)+v\left(k_1\frac{\theta(t_2)}{\theta(t_1)},k_2,t_2\right)\to U((k_1+k_2)\cdot\mathbf{1}_t)=v(0,k_1+k_2,t)$. Letting $t_1,t_2\to t$ in the above, we obtain for all $k_1,k_2\geq 0$ that $v(0,k_1,t)+v(k_1,k_2,t)=v(0,k_1+k_2,t)$. Define $u(C,t)=v(0,C,t)$. Then, $u(C,t)$ is continuous in $\mathbb{R}_+\times(0,T]$, and the above equation can be equivalently expressed as $v(C,k,t)=u(C+k,t)-u(C,t)$. For $t=0$, we have $v(0,k,0)=U(k\cdot\mathbf{1}_0)=\lim_{t\to 0^+}U(k\cdot\mathbf{1}_t)=\lim_{t\to 0^+}v(0,k,t)=u(k,0)-u(0,0)$. Finally, applying Axiom 2 to x^t implies that for all t, we get $u(C+k,t)-u(C,t)>0$, which implies that $u(C,t)$ is strictly increasing in C.

The functional form $v(C,k,t)=u(C+k,t)-u(C,t)$ is preserved if we define $u(C,t)$ as $u(C,t)+b(t)$ for some $b(t)$. Thus, we do not require

$u(0,t)=0$ anymore. The functional form of $v(C,k,t)$ is also preserved under time-dependent affine transformation of C. In particular, letting $Y(x,t^-)=Y_0\theta(t)+C(x,t^-)$, we can write $u(C,t)=u\left(Y-Y_0\theta(t),t\right)$ as a new function $u(Y,t)$, which is strictly increasing in Y, and rewrite the function v obtained in Step 3 as $v(Y,k,t)=u(Y+k,t)-u(Y,t)$.

Step 5. $u_d^*(Y,s,t)$ is not constant in Y on any open interval over (Y,s,t).

Assume, otherwise, $u_d^*(Y,s,t)$ is constant in Y over $(\underline{Y},\overline{Y})$ for any s,t with $\underline{t}<s<t<\bar{t}$. By virtue of Lemma A3, for $x\in\mathscr{X}_d\cap{}^{\underline{t}}\mathscr{X}^{\bar{t}}$, we have the dual representation $U(x)=u(Y(x,\bar{t}),\bar{t})-u(Y_0\theta(\underline{t}),\underline{t})+\sum_{i=0}^{n-1}u_d^*(Y(x,t_i),t_i,t_{i+1})(t_{i+1}-t_i)$, where $t_0,t_1,...,t_n$ is a partition of $[\underline{t},\bar{t}]$. We choose k_1,k_2 close enough and ε small enough such that for all $k_j\cdot\mathbf{1}_{\underline{t}}+x_j$ with $X_j(\bar{t}^-)<\varepsilon$, $j=1,2$, we always have $\underline{Y}<Y(k_j\cdot\mathbf{1}_{\underline{t}}+x_j,t_i)<\overline{Y}$ for all $i=0,1,...,n-1$, $j=1,2$. Then, $U(k_j\cdot\mathbf{1}_{\underline{t}}+x_j)$ is equivalent to $u(Y(k_j\cdot\mathbf{1}_{\underline{t}}+x_j,\bar{t}),\bar{t})$ up to a constant, which is further equivalent to $Y(k_j\cdot\mathbf{1}_{\underline{t}}+x_j,\bar{t})$ up to a monotone transformation. Since $Y(k_j\cdot\mathbf{1}_{\underline{t}}+x_j,\bar{t})$ is independent of k_j, we get a contradiction to Axiom 3.

Step 6. $u(Y,t)$ is strictly concave in $Y\in(Y_0\theta(t),\infty)$ for $t\in(0,T)$.

We first claim that for any $t\in(0,T)$ and any $Y_1\in(Y_0\theta(t),\infty)$, there exists $\varepsilon>0$ small enough such that we can always find $x_1,x_2\in\mathscr{X}$ satisfying $x_1^t\sim x_2^t$, $Y(x_1,t^-)=Y_1$, and $Y(x_2,t^-)=Y_2\in(Y_1,Y_1+\varepsilon]$. The construction is as follows. By Step 5, we first choose τ,s with $0<\tau<s<t$ such that $u^*(Y,\tau,s)$ is not constant in Y over a closed interval $[\underline{Y},\overline{Y}]$ within $(Y_0\theta(\tau),Y_1\theta(\tau)/\theta(t))$. Denote $Y_{\arg\min}=\arg\min_{Y\in[\underline{Y},\overline{Y}]}u^*(Y,\tau,s)$ and $Y_{\arg\max}=\arg\max_{Y\in[\underline{Y},\overline{Y}]}u^*(Y,\tau,s)$. Let

$$x_1^t=[Y_{\arg\max}-Y_0\theta(\tau)]\cdot\mathbf{1}_\tau+[Y_1\theta(s)/\theta(t)-Y_{\arg\max}\theta(s)/\theta(\tau)]\cdot\mathbf{1}_s,$$

which sets the satiation level to $Y_{\arg\max}$ at time τ and sets the satiation level to $Y_1\theta(s)/\theta(t)$ at time s. For such x_1, we have $Y(x_1,t^-) = Y_1$ and $U(x_1) = u\left(Y_1\theta(s)/\theta(t), s\right) - u(Y_0, 0) + u_d^*(Y_0, 0, \tau)\tau + u_d^*(Y_{\arg\max}, \tau, s)(s-\tau)$. Similarly, let

$$x_2^t = [Y - Y_0\theta(\tau)] \cdot \mathbf{1}_\tau + [Y_2\theta(s)/\theta(t) - Y\theta(s)/\theta(\tau)] \cdot \mathbf{1}_s,$$

which sets the satiation level to Y at time τ and sets the satiation level to $Y_2\theta(s)/\theta(t)$ at time s. For such x_2, we have $Y(x_2,t^-) = Y_2$ and $U(x_2) = u\left(Y_2\theta(s)/\theta(t), s\right) - u(Y_0, 0) + u_d^*(Y_0, 0, \tau)\tau + u_d^*(Y, \tau, s)(s - \tau)$. Let Y_2 be a function of Y, denoted by $Y_2(Y)$, such that $U(x_1) = U(x_2)$. Since $u_d^*(Y_{\arg\min}, \tau, s) < u_d^*(Y_{\arg\max}, \tau, s)$, there must be $Y_2(Y_{\arg\min}) > Y_1$. Due to the continuity of $u^*(Y, \tau, s)$, for any $Y_2 \in (Y_1, Y_2(Y_{\arg\min})]$, we can always choose $Y \in [\underline{Y}, \overline{Y}]$ such that $x_1 \sim x_2$ and $Y(x_2, t^-) = Y_2$.

We then show that $u(Y,t)$ is strict concave in $Y \in (Y_0\theta(t), \infty)$. To see this, for $Y_1, Y_2 \in (Y_0\theta(t), \infty)$ with $Y_1 < Y_2$, we let $k = (Y_2 - Y_1)/2$ and use Axiom 6 to get $u(Y_1 + 2k, t) - u(Y_1 + k, t) < u(Y_1 + k, t) - u(Y_1, t)$, i.e., $u(Y_1, t)/2 + u(Y_2, t)/2 < u(Y_1/2 + Y_2/2, t)$. In other words, $u(Y, t)$ is midpoint-concave in $Y \in (Y_0\theta(t), \infty)$. This, when coupled with continuity, implies strict concavity of $u(Y, t)$ in $Y \in (Y_0\theta(t), \infty)$.

The above establishes assertion (i). We prove (ii) in two steps.

Step 1. On $\mathscr{X}_c$, $U(x) = \int_0^T u_Y(Y(x,t),t)dX(t)$.

We have shown in Lemma 1 that $\int_0^T u_Y(Y(x,t),t)dX(t)$ is well-defined over $\mathscr{X}_c$. For $x \in \mathscr{X}_c$, let $x_\pi \in \mathscr{X}_d$ be the approximating sequence for x given by (A1). Then,

$$\begin{aligned} U(x_\pi) &= u(Y_0 + X(0), 0) - u(Y_0, 0) + \textstyle\sum_{i=1}^n \left[u(Y(x,t_i) + X(t_i) - X(t_{i-1}), t_i) - u(Y(x,t_i), t_i)\right] \\ &= u_Y(Y_0 + \xi_0 X(0), 0)X(0) + \textstyle\sum_{i=1}^n u_Y\left(Y(x,t_i) + \xi_i[X(t_i) - X(t_{i-1})], t_i\right)[X(t_i) - X(t_{i-1})], \end{aligned}$$

where $Y(x,t_i)$ is as in (9) but applied to discrete profiles, and $\xi_i \in (0,1)$ arises from the mean value theorem. As $|\pi| \to 0$, the lefthand side of the above equation converges to $U(x)$ while the righthand side converges to $\int_0^T u_Y(Y(x,t),t)dX(t)$, leading us to the desired equation.

Step 2. The utility representation $\int_0^T u_Y(Y(x,t),t)dX(t)$ meets Axioms 1 to 6 on $\mathscr{X}$.

We have shown in Lemma 1 that $\int_0^T u_Y(Y(x,t),t)dX(t)$ is continuous with respect to convergence in distribution in $\mathscr{X}$. Axiom 3 aside, we trivially verify that the representation given by (5)-(6) meets the rest of axioms. As for Axiom 3, we argue by contradiction. Assume, otherwise, there exist $k_1 \neq k_2$ such that $U(k_1 \cdot \mathbf{1}_t + {}^t x)$ and $U(k_2 \cdot \mathbf{1}_t + {}^t x)$ represent the same preference. Since both $U(k_1 \cdot \mathbf{1}_t + {}^t x)$ and $U(k_2 \cdot \mathbf{1}_t + {}^t x)$ are additive over satiation levels (see Lemma A3), we derive from Fishburn (1970, p. 54) the existence of a constant $a(k_1, k_2) > 0$, which may depend on k_1, k_2, such that $U(k_1 \cdot \mathbf{1}_t + {}^t x) = a(k_1,k_2)U(k_2 \cdot \mathbf{1}_t + {}^t x)$. Inserting $x = k \cdot \mathbf{1}_s$ into it, and taking derivative with respect to k in both sides, we obtain

$$u_Y\left(Y_0\theta(s) + k_1\theta(s)/\theta(t) + k, s\right) = a(k_1,k_2)u_Y\left(Y_0\theta(s) + k_2\theta(s)/\theta(t) + k, s\right). \tag{A7}$$

Given $k_1 \neq k_2$, we can easily find (s,k) and $(\hat{s},\hat{k})$ such that $Y_0\theta(s) + k_1\theta(s)/\theta(t) + k = Y_0\theta(\hat{s}) + k_1\theta(\hat{s})/\theta(t) + \hat{k}$ but $Y_0\theta(s) + k_2\theta(s)/\theta(t) + k = Y_0\theta(\hat{s}) + k_2\theta(\hat{s})/\theta(t) + \hat{k}$. Because $u_Y(Y,s)$ is strictly monotone, the lefthand side of (A7) takes the same value with (s,k) and $(\hat{s},\hat{k})$ but the righthand side of (A7) takes different values, a contradiction. □

Proof of Lemma 6. First, observe that we need not impose uniqueness or continuity of $Z(t)$, because it follows from Axiom 3. We use continuity to set $Z(0) = 0$ and $Z(T) = \lim_{t\to T} Z(t)$. We divide the proof into five steps.

Step 1. For $0 < s < t < T$ and $x \in \mathscr{X}^s$ with $C(x, s^-) \geq Z(s)$, we have

$$C(x,s^-) - Z(s) = C(Z(t-s) \cdot \mathbf{1}_{t-s} + x^{\to(t-s)}, t^-) - Z(t). \tag{A8}$$

We argue by contradiction. If there exists $x \in \mathscr{X}^s$ with $C(x, s^-) \geq Z(s)$ such that $k_1 = C(Z(t-s) \cdot \mathbf{1}_{t-s} + x^{\to(t-s)}, t^-) \neq C(x, s^-) - Z(s) + Z(t) = k_2$, then we apply Axiom 3 to find $x_1, x_2 \in \mathscr{X}$ such that $k_1 \cdot \mathbf{1}_t + {}^t x_1 \sim k_1 \cdot \mathbf{1}_t + {}^t x_2$ but $k_2 \cdot \mathbf{1}_t + {}^t x_1 \nsim k_2 \cdot \mathbf{1}_t + {}^t x_2$. Using Axiom 4 to replace k_1 yields

$$Z(t-s) \cdot \mathbf{1}_{t-s} + x^{\to(t-s)} + {}^t x_1 \sim Z(t-s) \cdot \mathbf{1}_{t-s} + x^{\to(t-s)} + {}^t x_2.$$

Applying a shift to the left ${}^{(t-s)\leftarrow}$ and using Axiom 7 to eliminate $Z(t-s)$ yields $x + {}^{(t-s)\leftarrow}({}^t x_1) \sim x + {}^{(t-s)\leftarrow}({}^t x_2)$. Applying a second shift to the left ${}^{s\leftarrow}$ and using (10) with ${}^{s\leftarrow}\left({}^{(t-s)\leftarrow t}x\right) = {}^{t\leftarrow}x$, we obtain

$$[C(x, s^-) - Z(s)] \cdot \mathbf{1}_0 + {}^{t\leftarrow}({}^t x_1) \sim [C(x, s^-) - Z(s)] \cdot \mathbf{1}_0 + {}^{t\leftarrow}({}^t x_2).$$

Finally, we apply a shift to the right ${}^{\to t}$ and use Axiom 7 again to get

$$[C(x, s^-) - Z(s) + Z(t)] \cdot \mathbf{1}_t + {}^t x_1 \sim [C(x, s^-) - Z(s) + Z(t)] \cdot \mathbf{1}_t + {}^t x_2,$$

a contradiction to $k_2 \cdot \mathbf{1}_t + {}^t x_1 \nsim k_2 \cdot \mathbf{1}_t + {}^t x_2$.

Step 2. For $0 < s < t < T$, $\theta(s)\theta(t-s) = \theta(t)$ and $Z(t) - Z(s) = Z(t-s)\theta(t)/\theta(t-s)$.

Applying Equation (A8) and Lemma 4 to $x = k \cdot \mathbf{1}_0$ with $k \geq Z(s)/\theta(s)$, we obtain $k\theta(s) - Z(s) = Z(t-s)\theta(t)/\theta(t-s) + k\theta(t)/\theta(t-s) - Z(t)$, or equivalently, $k\left[\theta(t)/\theta(t-s) - \theta(s)\right] = Z(t) - Z(s) - Z(t-s)\theta(t)/\theta(t-s)$. This relation holds for all $k \geq Z(s)/\theta(s)$ if and only if the term inside the bracket of the lefthand side, and hence the righthand side, is zero.

Step 3. There exist $\gamma > 0$, $Y_0 \geq 0$ such that $\theta(t) = e^{-\gamma t}$, $Z(t) = Y_0 - Y_0 e^{-\gamma t}$ for $t \in [0, T]$.

Recall that $\theta(t)$ is continuous and strictly decreasing with $\theta(0) = 1$. Letting $\tau = t - s$, the equation governing $\theta(t)$ in Step 2 can be written as $\theta(s)\theta(\tau) = \theta(s+\tau)$ for $0 \leq s, \tau \leq T, s+\tau \leq T$. According to Aczél (1966, p. 141), the unique solution is $\theta(t) = e^{-\gamma t}$ for some $\gamma > 0$. Inserting $\theta(t) = e^{-\gamma t}$ into the equation governing $Z(t)$ in Step 2, we get $Z(t) - Z(s) = Z(t-s)e^{-\gamma s}$. By applying the relation to $0 < t-s < t < T$, we also have $Z(t) - Z(t-s) = Z(s)e^{-\gamma(t-s)}$, which is equivalent to $Z(t)e^{-\gamma s} - Z(s)e^{-\gamma t} = Z(t-s)e^{-\gamma s}$. Taking differences to eliminate $Z(t-s)e^{-\gamma s}$, we obtain $Z(t)/(1-e^{-\gamma t}) = Z(s)/(1-e^{-\gamma s})$, implying the existence of a constant $Y_0 = Z(t)/(1-e^{-\gamma t}) \geq 0$.

Step 4. The function $u(Y,t)$ necessarily takes the form $u(Y,t) = e^{-\delta t}u(Y)$.

By Axiom 7, for $\tau \in [0,T)$ and $x \in \mathscr{X}^{T-\tau}$, both $U(x)$ and $U(Z(\tau) \cdot \mathbf{1}_\tau + x^{\to\tau})$ describe the same preference. By Lemma A3, both functions are additive over the satiation levels. According to Fishburn (1970, p. 54), two additive preferences are identical if and only if one of them can be expressed as a positive linear transformation of the other. Let π denote the time partition $t_0, t_1, ..., t_n$. This implies that there exist $a(\pi,\tau) > 0$ and $b(\pi,\tau)$ such that for $x = \sum_{i=0}^{n} k_i \cdot \mathbf{1}_{t_i}$ with $t_n \leq T-\tau$, $U(Z(\tau) \cdot \mathbf{1}_\tau + x^{\to\tau}) = a(\pi,\tau)U(x) + b(\pi,\tau)$. That is,

$$\sum_{i=0}^{n}[u(Y_i+k_i, t_i+\tau) - u(Y_i, t_i+\tau)] = a(\pi,\tau)\sum_{i=0}^{n}[u(Y_i+k_i, t_i) - u(Y_i, t_i)] + b(\pi,\tau).$$

This equation holds true for $k_0 = k_1 = ... = k_n = 0$, yielding $b(\pi,\tau) = 0$; and also holds for every t_i and every time partition, yielding $u(Y_i + k_i, t_i + \tau) - u(Y_i, t_i + \tau) = a(\tau)[u(Y_i + k_i, t) - u(Y_i, t_i)]$, where $a(\pi,\tau)$ only depends on τ. Denoting $f(Y,k,t) = u(Y+k,t) - u(Y,t)$, the above relationship can be expressed as $f(Y,k,t+\tau) = a(\tau)f(Y,k,t)$. According to Aczél (1966, p. 141), the unique solution to this functional equation is $f(Y,k,t) = e^{-\delta t}f(Y,k)$ for some constant δ and a continuous function $f(Y,k)$. This implies $u(Y,t) = e^{-\delta t}u(Y)$, where $u(Y) = f(0,Y)$. By terminal monotonicity, $u(Y)$ is strictly increasing.

Step 5. The function $u(Y)$ is continuously differentiable on $\left(Y_0e^{-\gamma T}, \infty\right)$.

To prove the differentiability, we just need to check that as $\varepsilon \to 0^+$ both $(u(Y+\varepsilon)-u(Y))/\varepsilon$ and $(u(Y)-u(Y-\varepsilon))/\varepsilon$ converge to the same limit. Given $Y > Y_0e^{-\gamma T}$, we first choose $t_Y \in [0, T)$ such that $Y_0e^{-\gamma t_Y} < Y$ (e.g., choose $t_Y = 0$ if $Y > Y_0$, and $t_Y = \ln(Y_0/Y)/(2\gamma) + T/2$ if $Y \leq Y_0$). Let $\lfloor z \rfloor$ denote the largest integer no bigger than z, $\overline{\Delta}_\varepsilon = \ln(1 + \varepsilon/Y)/\gamma$ and $\underline{\Delta}_\varepsilon = -\ln(1 - \varepsilon/Y)/\gamma$, which solve $(Y+\varepsilon)e^{-\gamma\overline{\Delta}_\varepsilon} = Y$ and $Ye^{-\gamma\underline{\Delta}_\varepsilon} = Y-\varepsilon$, respectively. Consider $x_Y = (Y - Y_0e^{-\gamma t_Y}) \cdot \mathbf{1}_{t_Y} + \gamma Y \cdot 1_{(t_Y,T]}$, $\overline{x}_{Y,\varepsilon} = \left(Y + \varepsilon - Y_0e^{-\gamma t_Y}\right) \cdot \mathbf{1}_{t_Y} + \sum_{i=1}^{\left\lfloor \frac{T-t_Y}{\overline{\Delta}_\varepsilon} \right\rfloor} \varepsilon \cdot \mathbf{1}_{t_Y+i\overline{\Delta}_\varepsilon}$, and $\underline{x}_{Y,\varepsilon} = \left(Y - Y_0e^{-\gamma t_Y}\right) \cdot \mathbf{1}_{t_Y} + \sum_{i=1}^{\left\lfloor \frac{T-t_Y}{\underline{\Delta}_\varepsilon} \right\rfloor} \varepsilon \cdot \mathbf{1}_{t_Y+i\underline{\Delta}_\varepsilon}$. Notice that $\overline{x}_{Y,\varepsilon}$ sets the satiation level to $Y+\varepsilon$ at the beginning and resets the satiation to $Y+\varepsilon$ with a gulp of size ε whenever the satiation level decreases to Y, and $\underline{x}_{Y,\varepsilon}$ sets the satiation level to Y at the beginning and resets the satiation level to Y with a gulp of size ε whenever the satiation level decreases to $Y-\varepsilon$. Both $\overline{x}_{Y,\varepsilon}$ and $\underline{x}_{Y,\varepsilon}$ converge to x_Y as $\varepsilon \to 0$. Then,

$$U(\overline{x}_{Y,\varepsilon}) = \left[u(Y+\varepsilon) - u\left(Y_0e^{-\gamma t_Y}\right)\right]e^{-\delta t_Y} + \left[u(Y+\varepsilon) - u(Y)\right]\sum_{i=1}^{\left\lfloor \frac{T-t_Y}{\overline{\Delta}_\varepsilon} \right\rfloor} e^{-\delta(t_Y+i\overline{\Delta}_\varepsilon)},$$

$$U(\underline{x}_{Y,\varepsilon}) = \left[u(Y) - u\left(Y_0e^{-\gamma t_Y}\right)\right]e^{-\delta t_Y} + \left[u(Y) - u(Y-\varepsilon)\right]\sum_{i=1}^{\left\lfloor \frac{T-t_Y}{\underline{\Delta}_\varepsilon} \right\rfloor} e^{-\delta(t_Y+i\underline{\Delta}_\varepsilon)},$$

yielding

$$\frac{u(Y+\varepsilon)-u(Y)}{\varepsilon} = \frac{e^{\delta t_Y}U(\overline{x}_{Y,\varepsilon}) - u(Y+\varepsilon) + u\left(Y_0e^{-\gamma t_Y}\right)}{\varepsilon\sum_{i=1}^{\left\lfloor \frac{T-t_Y}{\overline{\Delta}_\varepsilon} \right\rfloor} e^{-\delta i\overline{\Delta}_\varepsilon}},$$

$$\frac{u(Y)-u(Y-\varepsilon)}{\varepsilon} = \frac{e^{\delta t_Y}U(\underline{x}_{Y,\varepsilon}) - u(Y) + u\left(Y_0e^{-\gamma t_Y}\right)}{\varepsilon\sum_{i=1}^{\left\lfloor \frac{T-t_Y}{\underline{\Delta}_\varepsilon} \right\rfloor} e^{-\delta i\underline{\Delta}_\varepsilon}}.$$

By the continuity of $U(x)$, we have $\lim_{\varepsilon\to0^+} U(\overline{x}_{Y,\varepsilon}) = \lim_{\varepsilon\to0^+} U(\underline{x}_{Y,\varepsilon}) =$

$U(x_Y)$. We also have

$$\lim_{\varepsilon\to 0^+}\varepsilon\sum_{i=1}^{\left\lfloor\frac{T-t_Y}{\overline{\Delta}_\varepsilon}\right\rfloor} e^{-\delta i\overline{\Delta}_\varepsilon}=\lim_{\varepsilon\to 0^+}\varepsilon\sum_{i=1}^{\left\lfloor\frac{T-t_Y}{\underline{\Delta}_\varepsilon}\right\rfloor} e^{-\delta i\underline{\Delta}_\varepsilon}=\gamma Y\int_0^{T-t_Y} e^{-\delta s}ds=\frac{\gamma Y}{\delta}\left[1-e^{-\delta(T-t_Y)}\right].$$

It then follows that both $(u(Y+\varepsilon)-u(Y))/\varepsilon$ and $(u(Y)-u(Y-\varepsilon))/\varepsilon$ converge to

$$u'(Y)=\frac{1}{\gamma Y}\left[\frac{\delta}{1-e^{-\delta(T-t_Y)}}\right]\left[e^{\delta t_Y}U(x_Y)-u(Y)+u\left(Y_0e^{-\gamma t_Y}\right)\right] \tag{A9}$$

as $\varepsilon\to 0^+$, which is continuous in $Y\in\left(Y_0e^{-\gamma T},\infty\right)$. □

Proof of Theorem 1. We begin with the "if" part. The representation given by (1)-(2) meets Axioms 1 to 2 and Axioms 4 to 7 trivially. Following the proof of Lemma 5, we verify that the representation (1)-(2) also meets Axiom 3.

Next, we prove the "only if" part. Up to Lemma 6, we have already shown that under Axioms 1-6, $\succeq$ over $\mathscr{X}_d$ can be represented by (3)-(4), and moreover, with some regularity conditions, (3)-(4) extends continuously to (1)-(2) over $\mathscr{X}$. Lemma 6 further confirms that Axiom 7 yields all the required regularity, yielding the desired representation. With $u(Y,t)=e^{-\delta t}u(Y)$, $u^*(Y,t)=e^{-\delta t}u^*(Y)$. Following the proof of Lemma 5, we verify that $u^*(Y)$ is not constant on any open interval within $\left[Y_0e^{-\gamma T},\infty\right)$, which completes the proof. □

Proof of Proposition 1. We first prove the "if" part. For all $x_1,x_2\in\mathscr{X}$, it is obvious that $Y(x_1+x_2,t)\geq Y(x_1,t)$ for all $t\in[0,T]$. According to the dual representation $U(x)=e^{-\delta T}u(Y(x,T))-u(Y_0)+\int_0^T e^{-\delta t}u^*(Y(x,t))dt$, we will get $U(x_1+x_2)\geq U(x_1)$, given that $u^*(Y)$ is nondecreasing in Y. In addition, if $x_2\neq\mathbf{0}$, we have the strictly inequality $Y(x_1+x_2,T)>Y(x_1+x_2,T)$. Thanks to the strict monotonicity of $u(Y)$, we have $u(Y(x_1+x_2,T))>u(Y(x_1,T))$, yielding $U(x_1+x_2)>U(x_1)$.

To prove the "only if" part, we argue by contradiction. Assume, otherwise, $u^*(Y)$ is strictly decreasing at some point $Y > Y_0 e^{-\gamma T}$. We choose $t_Y \in [0, T)$ such that $Y_0 e^{-\gamma t_Y} < Y$ (e.g., choose $t_Y = 0$ if $Y > Y_0$, and $t_Y = \ln(Y_0/Y)/(2\gamma) + T/2$ if $Y \leq Y_0$), and consider the consumption profile $x_{Y,M} = (Y - Y_0 e^{-\gamma t_Y}) \cdot \mathbf{1}_{t_Y} + \gamma Y \cdot 1_{(t_Y, T]} + M \cdot \mathbf{1}_T$. Notice that along such profile, the consumer first sets the satiation level to Y by use of a gulp at time t_Y, and then keeps the satiation level constant during $(t_Y, T]$, and sets the satiation level to $Y + M$ at time T. According to the dual representation, we have

$$U(x_{Y,M}) = e^{-\delta T} u(Y + M) - e^{-\delta t_Y} u\left(Y_0 e^{-\gamma t_Y}\right) + u^*(Y) \int_{t_Y}^{T} e^{-\delta t} dt.$$

Let $\varepsilon > 0$ and $x_\varepsilon = \varepsilon \cdot \mathbf{1}_{t_Y} + \gamma \varepsilon \cdot 1_{(t_Y, T]}$. Then, $x_{Y,M} + x_\varepsilon = (Y + \varepsilon - Y_0 e^{-\gamma t_Y}) \cdot \mathbf{1}_{t_Y} + \gamma (Y + \varepsilon) \cdot 1_{(t_Y, T]} + M \cdot \mathbf{1}_T$, and

$$U(x_{Y,M} + x_\varepsilon) = e^{-\delta T} u(Y + \varepsilon + M) - e^{-\delta t_Y} u\left(Y_0 e^{-\gamma t_Y}\right) + u^*(Y + \varepsilon) \int_{t_Y}^{T} e^{-\delta t} dt.$$

Since $u^*(Y)$ is strictly decreasing at Y, we can choose ε small enough such that $u^*(Y + \varepsilon) < u^*(Y)$. Thanks to $\lim_{Y \to \infty} u'(Y) = 0$, we then choose M large enough such that $e^{-\delta T}[u(Y + \varepsilon + M) - u(Y + M)] < [u^*(Y) - u^*(Y + \varepsilon)] \int_{t_Y}^{T} e^{-\delta t} dt$, yielding $U(x_{Y,M} + x_\varepsilon) < U(x_{Y,M})$, a contradiction to uniform monotonicity. $\square$

Proof of Proposition 2. With $T = \infty$, we adopt the metric $\|x_1 - x_2\| = \int_0^\infty \frac{1}{2^t} \frac{|X_1(t) - X_2(t)|}{1 + |X_1(t) - X_2(t)|} dt$, which is in the same form as the one used by Rozen (2010, p. 1343), but applied to cumulative consumption. Under such metric, $x_m \to x$ if and only if $X_m(t) \to X(t)$ at all points of continuity of $X(t)$.

We begin with the "if" part. The primary goal is to prove that under $\delta > 0$ and the conditions on $u(Y)$, the utility presentation $U(x) = \int_0^\infty e^{-\delta t} u'(Y(x, t)) dX(t)$ is continuous over $(\mathscr{X}, \|\cdot\|)$. Given continuity, one can follow the ideas for $T < \infty$ to verify that the representation given by

(1)-(2) meets Axioms 1 to 7. To prove the continuity, we distinguish between two cases.

In the first case, $Y_0 = 0$ and $u(Y)$ is continuous over $[0, \infty)$. In this case, we have the dual representation for any $T < \infty$, i.e., $U(x^T) = e^{-\delta T} u(Y(x,T)) - u(Y_0) + \int_0^T e^{-\delta t} u^*(Y(x,t))dt$. Because u is concave, we have $Yu'(Y) \leq u(Y) - u(0)$, yielding $u^*(Y) \leq \gamma(u(Y) - u(0)) + u(Y)$. Since $\delta > 0$ and both $u(Y)$ and $u^*(Y)$ are bounded from above, we know that $U(x^T)$ is bounded from above. Noticing that $U(x^T)$ is increasing in T, $U(x) = \lim_{T\to\infty} U(x^T)$ is well defined and satisfies $U(x) = -u(Y_0) + \int_0^\infty e^{-\delta t} u^*(Y(x,t))dt$. Then, the continuity of this functional over $(\mathscr{X}, \|\cdot\|)$ follows straightforwardly from the fact $u(0) \leq u^*(Y) \leq \gamma(u(Y)-u(0))+u(Y)$ and the Lebesgue's dominated convergence theorem.

In the second case, $Y_0 > 0$ and $u(Y)$ is continuous over $(0, \infty)$. In this case, we allow for the possibility $\lim_{Y\to 0} u(Y) = -\infty$. For $T < \infty$, we use the dual representation to write $U(x^T) = e^{-\delta T} u(Y(x,T)) - u(Y_0) + \int_0^T e^{-\delta t} u^*(Y(x,t))dt$. Under $\lim_{T\to\infty} \left(\sup_{x\in\mathscr{X}} \int_T^\infty e^{-\delta t} u^*(Y(x,t))dt\right) = 0$, $U(x^T)$ is increasing in T and bounded from above. Thus, $U(x) = \lim_{T\to\infty} U(x^T)$ is well defined and satisfies $U(x) = -u(Y_0) + \int_0^\infty e^{-\delta t} u^*(Y(x,t))dt$. To prove the continuity, suppose that $X_m(t) \to X(t)$ at all points of continuity of $X(t)$. Since u is bounded from above, $\lim_{T\to\infty} e^{-\delta T} u\left(Y_0 e^{-\gamma T}\right) = 0$, and $\lim_{T\to\infty} \left(\sup_{x\in\mathscr{X}} \int_T^\infty e^{-\delta t} u^*(Y(x,t))dt\right) = 0$, we can choose for any $\varepsilon > 0$ a T_ε large enough such that for all $T \geq T_\varepsilon$, $\sup_{x\in\mathscr{X}} \left|e^{-\delta T} u(Y(x,T))\right| < \varepsilon$ and $\sup_{x\in\mathscr{X}} \int_T^\infty e^{-\delta t} u^*(Y(x,t))dt < \varepsilon$. We thus get

$$-\varepsilon - u(Y_0) + \int_0^T e^{-\delta t} u^*(Y(x_m,t))dt < U(x_m^T) \leq U(x_m) < -u(Y_0) + \int_0^T e^{-\delta t} u^*(Y(x_m,t))dt + \varepsilon$$

for all $T \geq T_\varepsilon$. Since $Y(x_m, t) \to Y(x,t)$ almost everywhere and $u^*(Y)$ is bounded from both below and above for $Y \geq Y_0 e^{-\gamma T} > 0$, we use the

Lebesgue's dominated convergence theorem to get

$$-\varepsilon - u(Y_0) + \int_0^T e^{-\delta t} u^*(Y(x,t))dt \leq \lim_{m\to\infty} U(x_m) \leq -u(Y_0) + \int_0^T e^{-\delta t} u^*(Y(x,t))dt + \varepsilon.$$

Letting $T \to \infty$ and $\varepsilon \to 0$, we obtain $\lim_{m\to\infty} U(x_m) = -u(Y_0) + \int_0^\infty e^{-\delta t} u^*(Y(x,t))dt = U(x)$.

Next, we prove the "only if" part. Following the argument for the case with finite horizon, we get $U(x^T) = \int_0^T e^{-\delta t} u'(Y(x,t))dX(t) = e^{-\delta T} u(Y(x,T)) - u(Y_0) + \int_0^T e^{-\delta t} u^*(Y(x,t))dt$ for all $T > 0$. We claim: (i) $\delta > 0$; (ii) $\lim_{T\to\infty} e^{-\delta T} u\left(Y_0 e^{-\gamma T}\right) = 0$; (iii) u is bounded from above; and (iv) $\lim_{T\to\infty} \left(\sup_{x\in\mathscr{X}} \int_T^\infty e^{-\delta t} u^*(Y(x,t$ 0. To prove (i), if, otherwise, $\delta \leq 0$, we have $U(\mathbf{1}_T) = e^{-\delta T}\left[u\left(Y_0 e^{-\gamma T} + 1\right) - u\left(Y_0 e^{-\gamma T}\right)\right]$, which goes to ∞ if $\delta < 0$ and goes to $u(1) - u(0^+) > 0$ if $\delta = 0$ as $T \to \infty$. Thus, $\mathbf{1}_T \to \mathbf{0}$ as $T \to \infty$ but $U(\mathbf{1}_T)$ does not converge to $U(\mathbf{0}) = 0$, a contradiction to continuity. Given $\delta > 0$ and $\lim_{T\to\infty} e^{-\delta T} u\left(Y_0 e^{-\gamma T} + 1\right) = 0$, (ii) follows immediately from the fact $\lim_{T\to\infty} U\left(\mathbf{1}_T\right) = \lim_{T\to\infty} e^{-\delta T}\left[u\left(Y_0 e^{-\gamma T} + 1\right) - u\left(Y_0 e^{-\gamma T}\right)\right] = 0$. To prove (iii), if u is not bounded from above, we can choose $k_m > 0$ for all $m \in \mathbb{N}$ such that $U(k_m \cdot \mathbf{1}_m) = e^{-\delta m}[u(Y_0 e^{-\gamma m} + k_m) - u(Y_0 e^{-\gamma m})] = m$. Then, $\|k_m \cdot \mathbf{1}_m\| \leq \int_m^\infty \frac{1}{2^t} dt \to 0$ but $U(k_m \cdot \mathbf{1}_m) \to \infty$, another contradiction to continuity. To prove (iv), we first notice that under conditions (i) to (iii), $\int_T^\infty e^{-\delta t} u^*(Y(x,t))dt = \int_T^\infty e^{-\delta t} u'(Y(x,t))dX(t) + e^{-\delta T} u(Y(x,T^-)) \geq e^{-\delta T} u(Y(x,T^-))$, which implies

$$\lim_{T\to\infty}\left(\sup_{x\in\mathscr{X}} \int_T^\infty e^{-\delta t} u^*(Y(x,t))dt\right) \geq \lim_{T\to\infty}\left(\sup_{x\in\mathscr{X}} e^{-\delta T} u(Y(x,T^-))\right) = 0. \tag{A10}$$

If, $\lim_{T\to\infty}\left(\sup_{x\in\mathscr{X}} \int_T^\infty e^{-\delta t} u^*(Y(x,t))dt\right) \neq 0$, then there exists $\varepsilon > 0$, $T_m \to \infty$, $N_m > 0$, and $x_m \in \mathscr{X}$ such that $\int_{T_m}^{T_m+N_m} e^{-\delta t} u^*(Y(x_m,t))dt > \varepsilon$. Let $\bar{x}_m = \left(Y(x_m, T_m) - Y_0 e^{-\gamma T_m}\right) \cdot \mathbf{1}_{T_m} + {}^{T_m}x_m^{T_m+N_m}$. Then, for T_m large enough, we have

$$U(\bar{x}_m) = e^{-\delta(T_m+N_m)} u\left(Y(x_m, T_m+N_m)\right) - e^{-\delta T_m} u\left(Y_0 e^{-\gamma T_m}\right) + \int_{T_m}^{T_m+N_m} e^{-\delta t} u^*(Y(x_m,t))dt.$$

As $m \to \infty$, we then get $\bar{x}_m \to \mathbf{0}$ but $\lim_{m\to\infty} U(\bar{x}_m) \geq \varepsilon$, a contradiction to continuity. Under conditions (i) to (iv), we get $U(x) = \int_0^\infty e^{-\delta t} u'(Y(x,t))dX(t) = -u(Y_0) + \int_0^\infty e^{-\delta t} u^*(Y(x,t))dt$ for all $x \in \mathscr{X}$. Finally, we remark that thanks to (A10), a sufficient condition for (iv) is that $u^*(Y)$ is bounded from above.
□

Proof of Proposition 3. Let $t > 0$ be a point of continuity of $x(t)$. Then, for any $\varepsilon > 0$, there exists a $\zeta > 0$ such that $x(s) \in (x(t) - \varepsilon, x(t) + \varepsilon)$ for all $s \in (t - \zeta, t]$. By definition, we have

$$\gamma Y(t) = Y_0 \gamma e^{-\gamma t} + \gamma \int_0^{t-\zeta} e^{-\gamma(t-s)} x(s)ds + \gamma \int_{t-\zeta}^{t} e^{-\gamma(t-s)} x(s)ds,$$

where $\gamma \int_0^{t-\zeta} e^{-\gamma(t-s)} x(s)ds \leq \gamma e^{-\gamma\zeta} \int_0^{t-\zeta} x(s)ds$ and $(x(t) - \varepsilon)\left(1 - e^{-\gamma\zeta}\right) < \gamma \int_{t-\zeta}^{t} e^{-\gamma(t-s)} x(s)ds < (x(t) + \varepsilon)\left(1 - e^{-\gamma\zeta}\right)$, respectively. Letting $\gamma \to \infty$, we obtain $x(t) - \varepsilon \leq \lim_{\gamma\to\infty} \gamma Y(t) \leq x(t) + \varepsilon$. Letting $\varepsilon \to 0$, we conclude that $\gamma Y(t) \to x(t)$. The assumption $\lim_{Y\to 0^+} u'(Y)Y^\alpha = \beta$ implies that for any $\varepsilon > 0$, there exists a constant $\zeta > 0$ such that $(1-\varepsilon)\beta Y^{-\alpha} \leq u'(Y) \leq (1+\varepsilon)\beta Y^{-\alpha}$ for all $Y \in (0, \zeta]$. Because $Y_0 = 0$ and $x(t)$ is bounded from above, we can find a constant $\overline{\gamma}$ such that $Y(t) \leq \zeta$ for all $t \in [0, T]$ and all $\gamma \geq \overline{\gamma}$. Accordingly, we have for all $\gamma \geq \overline{\gamma}$ that

$$\gamma^{-\alpha} \int_0^T e^{-\delta t} u'(Y(t))x(t)dt \in [1-\varepsilon, 1+\varepsilon] \times \beta \int_0^T e^{-\delta t} (\gamma Y(t))^{-\alpha} x(t)dt.$$

Because $x(t)$ is piecewise continuous in t, $\gamma Y(t) \to x(t)$ as $\gamma \to \infty$ almost everywhere on $[0, T]$. By Lebesgue's dominance convergence theorem, we get $\lim_{\gamma\to\infty} \gamma^{-\alpha} \int_0^T e^{-\delta t} u'(Y(t))x(t)dt \in [(1-\varepsilon)\beta, (1+\varepsilon)\beta] \times \int_0^T e^{-\delta t} x^{1-\alpha}(t)dt$. Letting $\varepsilon \to 0$ completes the proof. □