

# Expected Utility Theory with Bounded Probability Nets\*

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## Abstract

This paper develops an extension of expected utility theory by introducing various restrictions; e.g., probabilities have only decimal (or binary) fractions of finite depths not greater than a given constant, the preference relation in question may be incomplete. The basic idea is separation between measurement of utility for pure alternatives and extension to lotteries involving more risks such as plans for future events. These are formulated in an axiomatic manner. When no depth restrictions are given on permissible probabilities, the axioms determine a complete preference relation uniquely, which may be regarded as the classical EU theory. When a finite restriction is given, there are multiple preference relations compatible with the axioms, including incomparability on some lotteries. We exemplify the Allais-Kahneman-Tversky anomaly within our theory. We also connect the measurement process in our theory to the satisficing/aspiration argument due to H. Simon.

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## 1 Introduction

Although expected utility theory (*EU theory*) plays important roles in the present economics and game theory, some anomalies have been known with it since Allais [2]. In order to facilitate discussions on such anomalies, we develop an extension of *EU theory* by introducing various restrictions; e.g., probabilities are restricted only to decimal (or binary) fractions of finite depths and the preference relation may be incomplete. The classical EU theory in the sense of von Neumann-Morgenstern [20] (cf., Herstein-Milnor [9], Hammond [5], and Schmidt [17] for later developments) can be regarded as a limit case without those restrictions. Our theory pinpoints where such anomalies emerge and show how they may be avoided.

The theory is designed to incorporate the idea of “bounded rationality” due to Simon [18], [19] into EU theory; he himself criticized EU theory as a description of “super rationality”. Our theory starts with the case of “bounded rationality”, and goes to the limit case of “super rationality”. It has some affinity to his argument of *satisficing/aspiration*, which will be briefly discussed in Section 7.1. Here, we explain our development.

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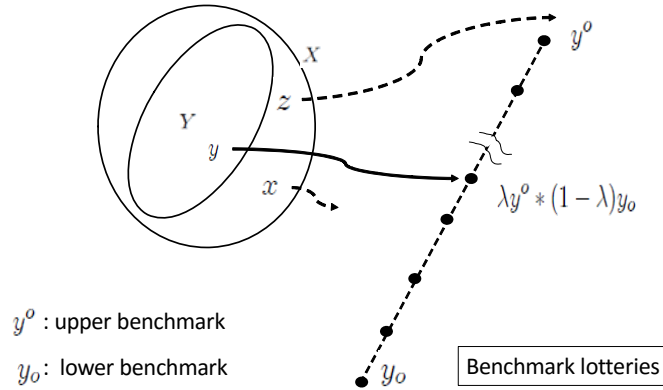


Figure 1: Measurement with the benchmark scale

A hint for the development is found in von Neumann-Morgenstern [20]. They divided their motivating argument into

- (i) measurement of utility in terms of probability; and
- (ii) extension to lotteries involving risks such as plans for future events,

but this separation is not reflected in their mathematical development. We separate and formulate these steps in a way of mathematical induction, i.e., from a simple base case to more complex cases. This enables us to examine Simon's [19] criticism.

To illustrate the above separation and to test our theory, we use an example in Kahneman-Tversky [11], which will be discussed in more details in Section 6. Consider three alternatives  $y^o, y, y_o$  with strict preferences  $y_o \prec y \prec y^o$  for the decision maker; if he chooses  $y^o, y$  or  $y_o$ , then he would get 4,000\$, 3,000\$, or 0\$, respectively<sup>1</sup>. Taking  $y^o$  and  $y_o$  as *benchmarks*, he evaluates  $y$  in terms of those benchmarks and probabilities; he makes a thought-experiment to find some probability  $\lambda$  so that  $y$  is indifferent to the lottery of  $y^o$  with  $\lambda$  and  $y_o$  with  $1 - \lambda$ , i.e.,

$$y \sim \lambda y^o * (1 - \lambda) y_o \quad (1)$$

The right-hand is called a *benchmark lottery*. Here, his preferences are not purely innate in his mind. He may find his preferences by reflecting upon past experiences, behavioral criteria, and beliefs/knowledge (see Section 7.1). This thought-experiment is the start of our theory.

The above thought-experiment starts with a coarse probability net, e.g.,  $\Pi_1 = \{\frac{0}{10}, \frac{1}{10}, \dots, \frac{10}{10}\}$ . If he finds no satisfactory  $\lambda$  in  $\Pi_1$ , he goes to the next net  $\Pi_2 = \{\frac{0}{10^2}, \frac{1}{10^2}, \dots, \frac{10^2}{10^2}\}$ . Once he finds some  $\lambda_y$  in  $\Pi_l$  for (1), he regards this  $\lambda_y$  as the "utility value" from  $y$  relative to the benchmarks  $y^o$  and  $y_o$ . He would stop at some step even though he finds no satisfactory  $\lambda$ , maybe, because he gets tired. Now, he has a set  $Y$  of such pure alternatives  $y$  with probability  $\lambda_y$ , which is a subset of the set of possible pure alternatives  $X$ . Those form a *base facet*  $F = \langle Y; y^o, y_o; \{\lambda_y\}_{y \in Y} \rangle$ . This is illustrated by Fig.1. This measurement structure is described by three axioms B1-B3, together with one basic axiom B0.

The decision maker can consciously choose a  $\lambda$  for (1). On the other hand, some lotteries are

<sup>1</sup>In the experiments reported in [11], the money amount was measured in the Israel pounds at that time (the median net monthly income for a family is about 3,000 Israel pounds - [11], p.264).

given to him, e.g.,  $\frac{25}{100}y * \frac{75}{100}y_o$ , which is not a benchmark lottery. Still, he looks for  $\lambda$  so that

$$\frac{25}{100}y * \frac{75}{100}y_o \sim \lambda y^o * (1 - \lambda)y_o. \quad (2)$$

The left lottery is one alternative in the Kahneman-Tversky example. This search differs from the above thought-experiment, since  $y$  is already evaluated in (1). Let  $\lambda_y = \frac{85}{100}$  be the probability given by (1). In the classical EU theory,  $\lambda_y y^o * (1 - \lambda_y)y_o$  is substituted for  $y$  in the left lottery, because they are indifferent for him. This leads him to a quite different situation:

$$\frac{25}{100}y * \frac{75}{100}y_o \sim \frac{2125}{10000}y^o * \frac{8875}{10000}y_o. \quad (3)$$

Now, he thinks about probability values of the 4th decimal place, while the lottery in (1) involves only probabilities of the 2nd decimal place. Probability values including the 4th decimal places are, perhaps, too precise for ordinary people for decision making; incidentally, the significance level for hypothesis testing in statistics is typically 5% or 1%.

We treat and/or restrict those “complicated” probabilities in the following manner. First, we consider the sets of lotteries  $L_0(Y), L_1(Y), \dots$ , where  $L_k(Y)$  the set of lotteries over  $Y$  with probability values of at most the  $k$ -th place. Lotteries  $\frac{25}{100}y * \frac{75}{100}y_o$  and  $\frac{2125}{10000}y^o * \frac{8875}{10000}y_o$  belong to  $L_2(Y)$  and  $L_4(Y)$ . We formulate step (ii) as a composition from lotteries from  $L_{k-1}(Y)$  to a lottery in  $L_k(Y)$ . We also give an upper bound  $\rho$  for those  $k$ 's.

The case of  $\rho = \infty$  has no restrictions on the above compositions. Then, the *complete* preference relation  $\succsim$  is *uniquely* determined over the union  $\cup_{k=0}^{\infty} L_k(Y)$  by the axioms and a given facet  $F = \langle Y; y^o, y_o; \{\lambda_y\}_{y \in Y} \rangle$ . This can be regarded as the classical EU theory, interpreted as Simon’s “super rationality”. Here, completeness and uniqueness on  $\succsim$  are derived, even though those are not contained in the axioms.

The case of small finite  $\rho$  is relevant to “bounded rationality”. In the above example, when  $\rho$  is the smallest, i.e.,  $\rho = 2$ , the lottery  $\frac{2125}{10000}y^o * \frac{8875}{10000}y_o$  is not permitted. Yet, step (ii) has some implications. In the case of  $\rho < \infty$ , the axioms allow *multiple* and *incomplete* preference relations over  $L_\rho(Y)$ . The smallest preference relation, which we call the *canonical preference relation*  $\succsim_c$ , is uniquely determined, but exhibits incomparability outside over some subset  $\mathbf{C}(F; \rho)$  of  $L_\rho(Y)$ . These cases are summarized in Tables 1.1 and 1.2.

Table 1.1		Table 1.2
$\rho = \infty$	$\implies$	$\rho < \infty$
$\mathbf{C}(F; \rho) = L_\rho(Y)$		$\mathbf{C}(F; \rho) \subsetneq L_\rho(Y)$
unique and complete $\succsim$		multiple and incomplete $\succsim$

In Section 5, we study the structure of incompatibility involved in the canonical relation  $\succsim_c$  for  $\rho < \infty$ . Then, we apply our theory to the Allais-Kahneman-Tversky anomaly in Section 5; specifically we compare our results with the experimental result due to Kahneman-Tversky [11]. We meet some conceptual subtlety in behavioral implications of incomparability. For this, we assume that even when two lotteries are incomparable, the decision maker could make a choice though it may be arbitrary, provided that he is asked to choose one in an experimental setting. When  $\rho = 2$  (the smallest case), our theoretical result has a prediction compatible with their experimental results. When  $\rho = 3$ , it cannot avoid the anomaly, and when  $\rho \geq 4$ , the original anomaly returns.

In Section 7.1, we formulate how a base facet  $F = \langle Y; y^o, y_o; \{\lambda_y\}_{y \in Y} \rangle$  is derived from the criterion of satisficing/aspiration due to Simon [19]. Specifically, we derive a possible  $\lambda_y$  in (1)

from it. From this point of view, the probability  $\lambda_y$  measured in (1) is not an exact measurement but an, possibly rough, approximation<sup>2</sup>. Here, we emphasize that the satisficing/aspiration is used to derive components for utility theory, rather than decision-making/behavior itself, which differs from the standard use of satisficing/aspiration (cf., Rubinstein-Salant [15]).

In Section 7.2, we give a remark on multiple base facets. In this case, incomparability across the facets appears even when no bounds are given for probability nets. This corresponds to the literature of expected utility theory without the completeness axiom from Aumann [4] (cf., Dubra, *et al.* [6] and its references) where a representation theorem is studied.

The paper is written as follows: Section 2 gives the concept of a bounded probability net. Section 3 gives a formulation of a basic facet together with basic axioms for measurement, and another axiom for extension. Section 4 discusses the canonical preference relation  $\succsim_c$ , and Section 5 studies incomparability involved in  $\succsim_c$ . Section 6 examines an anomalous example due to Kahneman-Tversky. Section 7 gives two remarks on a derivation of a basic facet from the viewpoint of satisficing/aspiration and on multiple base facets. Section 8 gives comments on further possible studies.

## 2 Bounded Probability Nets and Bounded Lotteries

We define the concept of probability nets, and then the sets of lotteries with probability coefficients up to a given depth. These sets are connected from shallow ones to deeper ones; with this, we can conduct mathematical induction arguments on those sets.

### 2.1 Probability nets

Let  $\ell$  be an integer with  $\ell \geq 2$ . This  $\ell$  is the base for describing probability values; we take  $\ell = 10$  in the example in this paper. The *probability nets*  $\Pi_k, k = 0, 1, \dots$  are defined by

$$\Pi_k = \left\{ \frac{\nu}{\ell^k} : \nu = 0, \dots, \ell^k \right\} \text{ for any } k \geq 0. \quad (4)$$

Here,  $\Pi_1 = \left\{ \frac{\nu}{\ell} : \nu = 0, \dots, \ell \right\}$  is the set of basic probability units, and  $\Pi_0 = \{0, 1\}$  for convenience. It holds that  $\Pi_k \subseteq \Pi_{k+1}$  for any  $k \geq 0$ .

The parameter  $k$  describes how precise probabilities the decision maker can use. He thinks about his preferences over  $X$  using the probability nets  $\Pi_k$  from a shallow depth to a greater depth. As stated in Section 1, probabilities are used in two ways: One is to evaluate pure alternatives, and the other is to describe situations including risks.

We use the limit  $\Pi_\infty := \cup_{k=0}^\infty \Pi_k$  as the reference case<sup>3</sup>. When the decision maker uses  $\Pi_\infty$ , he has no upper bound of depths for probability nets. We write  $\Pi_\rho$  either for  $\Pi_\infty$  or for  $\Pi_k$ .

A probability value  $\lambda$  is of *depth*  $k$ , denoted by  $\delta(\lambda) = k$ , iff  $\lambda \in \Pi_k - \Pi_{k-1}$ . If  $\delta(\lambda) = k$ , then

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<sup>2</sup>This is similar to the literatures of imprecise probability (cf., Augustin *et al.* [1]) and of similarity (cf., Rubinstein [14], Schmeidler-Gilboa [8]) in that probabilities are imprecise or those with small differences are treated as “similar”. In our treatment, imprecision/similarity is involved in the process of finding probabilities to have indifference, but probabilities themselves are restricted rather than being imprecise.

<sup>3</sup>The set  $\Pi_\infty = \cup_{k=0}^\infty \Pi_k$  is a proper subset of the set of all rationals  $\mathbb{Q}$ . When the base  $\ell = 10$ ,  $\Pi_\infty$  is the set of finite decimal fractions, but  $\frac{1}{3}$  is expanded as a repeated decimal fractions.

$\lambda$  is expressed as

$$\frac{\nu_1}{\ell^1} + \frac{\nu_2}{\ell^2} + \cdots + \frac{\nu_k}{\ell^k}, \text{ where } 0 \leq \nu_t < \ell \text{ for } t = 1, \dots, k \text{ and } \nu_k > 0. \quad (5)$$

If  $\nu_k = 0$ , then  $\delta(\lambda) < k$ , though  $\lambda$  still belongs to  $\Pi_k$ . If  $\delta(\lambda) = m < k$ , then  $\nu_m > 0$  and  $\nu_{m+1} = \cdots = \nu_k = 0$ .

The following lemma lists basic properties of  $\Pi_k$  ( $k \geq 0$ ).

**Lemma 2.1.(1):** if  $\alpha_1, \dots, \alpha_s \in \Pi_k$  and  $\sum_{t=1}^s \alpha_t \leq 1$ , then  $\sum_{t=1}^s \alpha_t \in \Pi_k$ .

**(2):**  $\lambda \in \Pi_k$  implies  $1 - \lambda \in \Pi_k$ ; and  $\delta(\lambda) = k$  implies  $\delta(1 - \lambda) = k$ ;

**(3):**  $\delta(\lambda \cdot \lambda') \leq \delta(\lambda) + \delta(\lambda')$ ; and  $\delta(\lambda + \lambda') \leq \max\{\delta(\lambda), \delta(\lambda')\}$ .

**Proof.** We show only the first statement of (3). Let  $\delta(\lambda) = k$  and  $\delta(\lambda') = k'$ . Their last terms  $\frac{\nu_k}{\ell^k}$  and  $\frac{\nu'_{k'}}{\ell^{k'}}$  are not zero. Consider the multiplication of those  $\ell$ -ary fractions. It is expressed also as an  $\ell$ -nary fraction of at most length  $k + k'$ ; when  $\nu_k$  and  $\nu'_{k'}$  contain factors making  $\ell$ , the length is less than  $k + k'$ . Thus,  $\delta(\lambda \cdot \lambda') \leq k + k' = \delta(\lambda) + \delta(\lambda')$ . ■

An example for the strict inequality for first statement of (3) is; when  $\lambda = \frac{2}{10}$  and  $\lambda' = \frac{5}{10}$ , we have  $\delta(\lambda \cdot \lambda') = \delta(\frac{1}{10}) = 1 < \delta(\lambda) + \delta(\lambda') = 2$ . Thus, the depth may not be monotone with multiplication. When  $\ell$  is a prime number, we could avoid this. However, we allow  $\ell$  not to be a prime, since the base  $\ell$  is 10 in the experimental example we refer to.

We show that for each finite  $k \geq 1$ ,  $\Pi_k$  is obtained from  $\Pi_{k-1}$  by taking weighted sums of elements in  $\Pi_{k-1}$  with equal weights. This lemma is basic for our method connecting shallow probability nets with deeper ones.

**Lemma 2.2(Composition of probabilities).** Let  $1 \leq k < \infty$ . Then  $\Pi_k = \{\sum_{t=1}^{\ell} \frac{1}{\ell} \lambda_t : \lambda_1, \dots, \lambda_{\ell} \in \Pi_{k-1}\}$ .

**Proof.** It is easy to see that the right-hand set is included in  $\Pi_k$ . Consider the opposite inclusion. Let  $\lambda \in \Pi_k$  with  $\delta(\lambda) \geq 1$ . Then,  $\lambda$  is expressed as (5). We let  $\lambda_1 = \dots = \lambda_{\nu_1} = 1$ ,  $\lambda_{\nu_1+1} = \frac{\nu_2}{\ell^1} + \dots + \frac{\nu_k}{\ell^{k-1}}$ , and  $\lambda_t = 0$  for  $t = \nu_1 + 2, \dots, \ell$ . Those are in  $\Pi_{k-1}$ , and  $\lambda = \sum_{t=1}^{\ell} \frac{1}{\ell} \lambda_t$ . ■

We can use the binary operation rather than the  $\ell$ -ary operation:

$$\Pi_{k+1} = \{\alpha\beta_1 + (1 - \alpha)\beta_2 : \alpha \in \Pi_1 \text{ and } \beta_1, \beta_2 \in \Pi_k\} \text{ for any } k \text{ (} 0 \leq k < \infty \text{)}. \quad (6)$$

This is proved by induction on  $k \geq 0$ . However, the corresponding result to (6) does not hold for lotteries with bounded probability nets, which will be stated in Remark 2.1.

## 2.2 Lotteries with bounded probabilities

Let  $X$  be a finite set of pure alternatives. For any  $k < \infty$ , we define  $L_k(X)$  by

$$L_k(X) = \{f : f : X \rightarrow \Pi_k \text{ and } \sum_{x \in X} f(x) = 1\}. \quad (7)$$

For any integer  $k \geq 0$ , it holds that  $L_k(X) \subseteq L_{k+1}(X)$ , since  $\Pi_k \subseteq \Pi_{k+1}$ . We define  $L_{\infty}(X) = \cup_{k=1}^{\infty} L_k(X)$ . In the following,  $k$  is used to denote a finite number, but  $\rho$  is either finite or  $\infty$ .

We define the *depth of a lottery*  $f$  in  $L_{\rho}(X)$  to be  $k$  iff  $f \in L_k(X) - L_{k-1}(X)$ , which we denote by  $\delta(f) = k$ . We have the following observations, which explains the relationship between

the depth of a probability and that of a lottery.

**Lemma 2.3.(1):** if  $f \in L_k(X)$ , then  $\delta(f(x)) \leq k$  for all  $x \in X$ ;

(2):  $\delta(f) = k$  if and only if  $f \in L_k(X)$  and  $\delta(f(x)) = k$  for some  $x \in X$ .

**Proof.** (1) follows from (7). Consider (2). The *if* part follows from (1), since  $f \notin L_{k-1}(X)$ . Consider the *only-if* part. Let  $\delta(f) = k$ , i.e.,  $f \in L_k(X) - L_{k-1}(X)$ . If  $\delta(f(x)) < k$  for all  $x \in X$ , then  $f \in L_{k-1}(X)$  by (5). Thus,  $\delta(f(x)) = k$  for some  $x \in X$ . ■

In fact, the set  $L_k(X)$  can be constructed inductively, which will play a crucial role in our theory. We formulate a connection from  $L_k(X)$  to  $L_{k+1}(X)$  in a very restricted manner. Specifically, for the weight vector  $\mathbf{e} = (\frac{1}{\ell}, \dots, \frac{1}{\ell})$  and an  $\ell$  vector of lotteries  $\mathbf{f} = (f_1, \dots, f_\ell)$  in  $L_k(X)^\ell := L_k(X) \times \dots \times L_k(X)$ , we define the new function  $h : X \rightarrow \Pi_{k+1}$  by

$$h(x) = \sum_{t=1}^{\ell} \frac{1}{\ell} \times f_t(x) \text{ for all } x \in X. \quad (8)$$

We denote  $h$  by  $\mathbf{e} * \mathbf{f}$  or  $\sum_{t=1}^{\ell} \frac{1}{\ell} * f_t$ . This  $h$  belongs to  $L_{k+1}(X)$ . In fact, the converse holds, which is stated as Lemma 2.4. A proof is given in the Appendix.

**Lemma 2.4 (Composition of lotteries).** Let  $k \geq 0$ . Then  $L_{k+1}(X) = \{f : f = \mathbf{e} * \mathbf{f} \text{ for some } \mathbf{f} \in L_k(Y)^\ell\}$ .

Any  $f \in L_{k+1}(X)$  is composed of one-degree lower lotteries in  $L_k(X)$  with equal weights. By this lemma, we can think that any lotteries can be constructed from the pure alternatives in  $X$ .

**Remark 2.1.** Lemma 2.4 cannot be changed into the form of binary combination  $f = \alpha f_1 * (1 - \alpha) f_2$  with  $\alpha \in \Pi_1$  and  $f_1, f_2 \in L_{k-1}(X)$  when  $|X| \geq 3$ . For example,  $f = \frac{3}{10} y^o * \frac{4}{10} y * \frac{3}{10} y^o \in L_1(X)$  cannot be expressed by a binary combination in terms of  $f_1, f_2 \in L_0(X)$  and  $\alpha \in \Pi_1$ .

### 3 An EU System with a Base Facet

Here, we introduce the concept of a base facet, which is interpreted as being found/decided in the mind of the decision maker. We impose three axioms on a base facet in addition to one basic axiom. Then, we extend one axiom and give the final axiom for the extension step. When  $\rho = \infty$ , a given base facet and the axioms uniquely determine the complete preference relation, which corresponds to the classical EU theory. When  $\rho < \infty$ , the preference relation is not uniquely determined and allows incomparabilities.

#### 3.1 Base facets

For any given  $Y \subseteq X$ , we consider  $L_k(Y) = \{f : f : Y \rightarrow \Pi_k(Y) \text{ and } \sum_{y \in Y} f(y) = 1\}$ . Lemmas 2.3 and 2.4 are also applied to this  $L_k(Y)$ . The upper bound for  $k$  is given as  $\rho$ ; if  $\rho = \infty$ , then  $L_\rho(Y) = \cup_{k=0}^{\infty} L_k(Y)$ , and if  $\rho = k < \infty$ , then  $L_\rho(Y) = L_k(Y)$ .

Let  $\succsim$  be a binary relation over  $L_\rho(Y)$ . We define the strict relation  $\prec$  and indifference relation  $\sim$  by

$$\begin{aligned} f \prec g &\iff_{def} f \succsim g \text{ and not } g \succsim f; \\ f \sim g &\iff_{def} f \succsim g \text{ and } g \succsim f. \end{aligned} \quad (9)$$

We assume the following axiom on the binary relation  $\succsim$  as basic, throughout the paper.

**Axiom B0.(1)(Reflexivity):**  $f \succsim f$  for any  $f \in L_\rho(Y)$ ;

**(2):(Transitivity):** for any  $f, g, h \in L_\rho(Y)$ , if  $f \succsim g$  and  $g \succsim h$ , then  $f \succsim h$ .

Reflexivity is rather for convenience sake. Transitivity is essential for measurement: We do not assume completeness: for any  $f, g \in L_\rho(X)$ ,  $f \succsim g$  or  $g \succsim f$ ; instead, we study incompleteness for  $\rho < \infty$ .

Now, we go to the measurement step (i): The result of this step is formulated by the concept of a base facet, which has a status parallel to the induction base of the principle of mathematical induction. A *base facet* is given as  $F = \langle Y; y^o, y_o; \{\lambda_y\}_{y \in Y} \rangle$ , where  $Y$  is a subset of  $X$ , containing  $y^o, y_o$ , called *benchmarks*; and  $\{\lambda_y\}_{y \in Y}$  is a family in  $\Pi_\rho$  indexed by  $y \in Y$ . Throughout the following, we assume that  $Y$  is a finite set and that  $l_o := \max\{\delta(\lambda_y) : y \in Y\} \leq \rho \leq \infty$ . The meaning of the components in  $F$  are fixed by the following three axioms. The first is:

**Axiom B1(Benchmarks):**  $y_o \prec y \prec y^o$  for all  $y \in Y - \{y^o, y_o\}$ .

In fact, it is enough to assume that  $y_o \succsim y \succsim y^o$  for all  $y \in Y$ , but to avoid unnecessary complications, we adopt the above form of B1.

We call a lottery of the form  $\lambda y^o * (1 - \lambda)y_o$  with  $\lambda \in \Pi_\rho$  a *benchmark lottery*. We denote the set of all benchmark lotteries by  $\mathbf{B}_\rho(y^o; y_o)$ , which we call the *benchmark scale*. In Fig.1.1, the benchmark scale is depicted where the dots correspond to the benchmark lotteries. The preferences are monotone along the benchmark scale  $\mathbf{B}_\rho(y^o; y_o)$ . Note that when  $\rho = \infty$ ,  $\mathbf{B}_\infty(y^o; y_o)$  is expressed as  $\cup_{k=0}^\infty \mathbf{B}_k(y^o; y_o)$ .

**Axiom B2<sup>o</sup>(Monotonicity along the benchmark scale):** for  $\lambda, \lambda' \in \Pi_\rho$ ,

$$\lambda < \lambda' \text{ implies } \lambda y^o * (1 - \lambda)y_o \prec \lambda' y^o * (1 - \lambda')y_o.$$

Using this scale, the decision maker tries to measure a given alternative  $x \in X$ , and  $Y$  is the set of pure alternatives in  $X$  that he succeeds in measuring. Each  $\lambda_y$  is a result of his measurement of  $y \in Y$ .

**Axiom B3(Measurement with benchmarks):**  $y \sim \lambda_y y^o * (1 - \lambda_y)y_o$  for each  $y \in Y$ .

In Fig.1.1, each  $y \in Y$  corresponds to a dot in the benchmark scale. The remaining alternatives in  $X$  do not go to those dots (see Section 7.1).

The above three axioms describe how a pure lottery is measured in terms of the benchmark scale  $\mathbf{B}_\rho(y^o; y_o)$ . In Section 7.1, we will discuss how these are derived from the criterion of satisficing/aspiration due to Simon [18].

Lemma 3.1 states that  $F = \langle Y; y^o, y_o; \{\lambda_y\}_{y \in Y} \rangle$  fixes the preference relation  $\succsim$  over  $Y$ .

**Lemma 3.1 (Measurement Lemma).** Let  $F = \langle Y; y^o, y_o; \{\lambda_y\}_{y \in Y} \rangle$  be a base facet in  $\mathcal{L} = \langle L_\rho(X); \succsim \rangle$  with  $\rho \geq l_o := \max\{\delta(\lambda_y) : y \in Y\}$  and Axioms B0, B1, B2<sup>o</sup>, B3.

**(1)(Completeness over  $Y$ ):**  $x \succsim y$  or  $y \succsim x$  for any  $x, y \in Y$ ;

**(2)(Uniqueness):** For each  $y \in Y$ , if  $y \sim \alpha y^o * (1 - \alpha)y_o$ , then  $\alpha = \lambda_y$ ; and  $\lambda_{y_o} = 0, \lambda_{y^o} = 1$ .

**Proof (1):** Let  $y, z \in Y$ . By B3,  $y \sim \lambda_y y^o * (1 - \lambda_y)y_o$  and  $z \sim \lambda_z y^o * (1 - \lambda_z)y_o$ . Then either  $\lambda_y > \lambda_z$ ,  $\lambda_y = \lambda_z$  or  $\lambda_y < \lambda_z$ . In the first case,  $\lambda_y y^o * (1 - \lambda_y)y_o \prec \lambda_z y^o * (1 - \lambda_z)y_o$  by B2<sup>o</sup>; so by B0.(2)(transitivity), we have  $y \prec z$ . The third case is symmetric. In the second case,  $y \sim \lambda_y y^o * (1 - \lambda_y)y_o \sim z$ . Thus,  $y \sim z$  by B0.

(2): Consider its contrapositive. Let  $\alpha > \lambda_y$ . Then, by B2<sup>o</sup>,  $\alpha y^o * (1 - \alpha)y_o \prec \lambda_y y^o * (1 - \lambda_y)y_o$ . The case  $\alpha < \lambda_y$  is symmetric. Hence,  $\lambda_y$  is unique. Since  $y^o \sim \lambda_{y^o} y^o * (1 - \lambda_{y^o})y_o$ , we have  $\lambda_{y^o} = 1$  by B2<sup>o</sup>. Similarly,  $\lambda_{y_o} = 0$ . ■

Lemma 3.1 implies the existence of a utility function  $u_o : Y \rightarrow R$  defined by  $u_o(y) = \lambda_y$  all  $y \in Y$ ; by Axioms B2<sup>o</sup> and B0, it represents the relation  $\succsim$  over  $Y$ , i.e., for any  $y, y' \in Y$ ,

$$u_o(y) \leq u_o(y') \text{ if and only if } y \succsim y'. \quad (10)$$

This representation result will be extended below.

### 3.2 Classical case

Here, we mention the additional extension axiom in the case  $\rho = \infty$ , since the axiomatic system with this addition becomes the classical EU theory. It exhibits the clear-cut structure of expected utility, which, however, sacrifices a rich structure in the case  $\rho < \infty$ . To understand this, here we consider the case  $\rho = \infty$ .

We add one extension axiom: For  $\mathbf{f} \in L_\infty(Y)^\ell$  and  $\mathbf{g} \in \mathbf{B}_\infty(y^o; y_o)^\ell$ , we write  $\mathbf{f} \sim \mathbf{g}$  when  $f_t \sim g_t$  for all  $t = 1, \dots, \ell$ .

**Axiom B4<sup>∞</sup>(Extension for  $\rho = \infty$ ):** for any  $\mathbf{f} \in L_\infty(Y)^\ell$  and  $\mathbf{g} \in \mathbf{B}_\infty(y^o; y_o)^\ell$ , if  $\mathbf{f} \sim \mathbf{g}$ , then  $\mathbf{e} * \mathbf{f} \sim \mathbf{e} * \mathbf{g}$ .

That is, when  $f_t \in L_\infty(Y)$  and  $g_t \in \mathbf{B}_\infty(y^o; y_o)$  are indifferent for all  $t = 1, \dots, \ell$ , the composed lottery  $\mathbf{e} * \mathbf{f}$  is also indifferent to the composed lottery  $\mathbf{e} * \mathbf{g}$ , which is also a benchmark lottery.

We define the *expected utility function*  $u_{eu}$  over the full domain  $L_\infty(Y)$  by

$$u_{eu}(h) = \sum_{y \in Y} h(y)u_o(y) \text{ for any } h \in L_\infty(Y). \quad (11)$$

This function will play an important role even for  $\rho < \infty$ . When the domain is restricted to  $L_\rho(Y)$  ( $\rho < \infty$ ), we use the same symbol  $u_{eu}$ , which causes no confusion. We define the binary relation  $\succsim_{eu}$  over  $L_\infty(Y)$  by

$$f \succsim_{eu} g \text{ if and only if } u_{eu}(f) \leq u_{eu}(g). \quad (12)$$

Thus, this is the preference relation determined by the expected utility function  $u_{eu}$ , which we call the *eu-preference relation*.

When a base facet  $F$  is given, Axiom B0, B1, B2<sup>o</sup>, B3 and B4<sup>∞</sup> imply the EU theorem in the classical sense.

**Theorem 3.1(EU Theorem for  $\rho = \infty$ ).** A preference relation  $\succsim$  over  $L_\infty(Y)$  satisfies B0, B1, B2<sup>o</sup>, B3, and B4<sup>∞</sup> if and only if  $\succsim$  coincides with  $\succsim_{eu}$ .

We can directly prove this theorem by induction over  $L_\infty(Y)$ ; the induction argument becomes possible by Lemma 3.4. Here, we choose an indirect proof in order to make comparisons with the case  $\rho < \infty$ . First, we prove that a base facet  $F$  with Axiom B0, B1, B2<sup>o</sup>, B3, B4<sup>∞</sup> uniquely determine the preference relation  $\succsim$ . Second, the eu-preference relation  $\succsim_{eu}$  satisfies those axioms. Hence, we have Theorem 3.1.

**Lemma 3.2.** For each  $f \in L_\infty(Y)$ , there is a unique  $\lambda_f \in \Pi_\infty$  such that  $f \sim \lambda_f y^o * (1 - \lambda_f)y_o$ .



**Proof.** Let  $k = 0$ . Then,  $L_0(Y) = Y$ . By Lemma 3.1, for any  $y \in Y$ , there is a unique  $\lambda_f \in \Pi_\infty$  ( $= \cup_{t < \infty} \Pi_t$ ) such that  $y \sim \lambda_y y^o * (1 - \lambda_y) y_o$ .

Suppose the induction hypothesis that the assertion holds for  $k$ . Take any  $f \in L_{k+1}(Y)$ . By Lemma 2.4, we have an  $\mathbf{f} \in L_k(Y)^\ell$  such that  $\mathbf{e} * \mathbf{f} = f$ . By the induction hypothesis, there is a  $\mathbf{g} = (g_1, \dots, g_\ell)$  such that  $\mathbf{f} \sim \mathbf{g}$  and  $g_t = \lambda_{f_t} y^o * (1 - \lambda_{f_t}) y_o$ . Applying Axiom B4 $^\infty$ , we have  $f = \mathbf{e} * \mathbf{f} \sim \mathbf{e} * \mathbf{g} = \sum_{t=1}^\ell \frac{1}{10} * g_t = \sum_{t=1}^\ell \frac{1}{10} * (\lambda_{f_t} y^o * (1 - \lambda_{f_t}) y_o) = (\sum_{t=1}^\ell \frac{1}{10} \lambda_{f_t}) y^o * (1 - \sum_{t=1}^\ell \frac{1}{10} \lambda_{f_t}) y_o$ . Hence,  $\lambda_f$  is uniquely determined to be  $\sum_{t=1}^\ell \frac{1}{10} \lambda_{f_t}$  by B2. ■

**Lemma 3.3.** The eu-relation  $\lesssim_{eu}$  satisfies B0, B1, B2 $^\circ$ , B3 and B4 $^\infty$ .

**Proof.** It is easy to see that  $\lesssim_{eu}$  satisfies B0, B1, B2 $^\circ$ , B3. We see that it satisfies B4 $^\infty$ . It suffices to show that  $u_{eu}$  satisfies B4 $^\infty$ . Suppose  $u_{eu}(f_t) = u_{eu}(g_t)$  for  $t = 1, \dots, \ell$ . Since  $g_t = \alpha_t y^o * (1 - \alpha_t)$  for some  $\alpha_t \in \Pi_\infty$ ,  $u_{eu}(g_t) = \alpha_t$ . It holds that  $u_{eu}(\mathbf{e} * \mathbf{f}) = u_{eu}(\sum_{t=1}^\ell \frac{1}{\ell} * f_t) = \sum_{y \in Y} \sum_{t=1}^\ell (\frac{1}{\ell} * f_t)(y) = \sum_{t=1}^\ell \sum_{y \in Y} (\frac{1}{\ell} * f_t)(y) = \sum_{t=1}^\ell \frac{1}{\ell} \sum_{y \in Y} f_t(y) = \sum_{t=1}^\ell \frac{1}{\ell} u_{eu}(f_t)$ , and similarly, that  $u_{eu}(\mathbf{e} * \mathbf{g}) = \sum_{t=1}^\ell \frac{1}{\ell} u_{eu}(g_t)$ . Since  $u_{eu}(f_t) = u_{eu}(g_t)$  for  $t = 1, \dots, \ell$ , we have  $u_{eu}(\mathbf{e} * \mathbf{f}) = u_{eu}(\mathbf{e} * \mathbf{g})$ . ■

Here, we emphasize the following implication:

**I0:** the preference relation  $\lesssim$  is uniquely determined and is complete.

This is because every lottery  $f \in L_\infty(Y)$  can be measured by the benchmark scale  $\mathbf{B}_\infty(y^o; y_o)$  since substitution of  $\lambda_y y^o * (1 - \lambda_y) y_o$  for  $y$  is free for any  $y \in Y$  by B4 $^\infty$ . In case  $\rho < \infty$ , this implication does not hold, which will be discussed below.

### 3.3 Extending the preferences given by a base facet

When  $\rho < \infty$ , Axiom B2 $^\circ$  is not enough. We extend it to the following way:

**Axiom B2(Monotonicity):** Let  $y, z \in Y$  and  $\lambda_y > \lambda_z$ . For any  $f, g \in L_\rho(Y)$

with  $f(x) = g(y)$  for all  $x \neq y, z$ , if  $f(y) > g(y)$ , then  $g \prec f$ .

That is, if  $f$  is obtained from  $g$  by transferring some probability from a less preferred  $z$  to a more preferred  $y$ , keeping the other part the same, then,  $f$  is preferred to  $g$ . Axiom B2 $^\circ$  is a special case of B2.

Now, Axiom B4 $^\infty$  is extended. We write  $\mathbf{f} \lesssim \mathbf{g}$  when  $f_t \lesssim g_t$  for all  $t = 1, \dots, \ell$ , and stipulate  $\rho - 1 = \infty$  when  $\rho = \infty$ .

**Axiom B4(Extension of preferences):** for any  $\mathbf{f} \in L_{\rho-1}(Y)^\ell$ ,  $\mathbf{g} \in \mathbf{B}_{\rho-1}(y^o; y_o)^\ell$

(1) : if  $\mathbf{f} \lesssim \mathbf{g}$ , then  $\mathbf{e} * \mathbf{f} \lesssim \mathbf{e} * \mathbf{g}$ ; and (2) : if  $\mathbf{g} \lesssim \mathbf{f}$ , then  $\mathbf{e} * \mathbf{g} \lesssim \mathbf{e} * \mathbf{f}$ ,

and (1), (2) have the strict parts, e.g., in (1), if  $\mathbf{f} \lesssim \mathbf{g}$  and  $f_t \prec g_t$  for some  $t$ ,

then  $\mathbf{e} * \mathbf{f} \prec \mathbf{e} * \mathbf{g}$ .

Here, preferences are taken into account together with strict parts, while only indifferences are considered in B4 $^\infty$ . Hence, B4 for  $\rho = \infty$  is an extension of B4 $^\infty$ .

Thus, Axiom B2 $^\circ$  is a special case of B2 as well as B4 $^\infty$  is a special case of B4 for  $\rho = \infty$ . On the other hand, Theorem 3.1 has the following implication, in addition to the interpretation I0 for  $\rho = \infty$ ,

**I1:** when  $\rho = \infty$ , Axioms B2 and B4 are derived from B0, B1, B2 $^\circ$ , B3, and B4 $^\infty$ .

Thus, Axioms B2 and B4 are redundant for  $\rho = \infty$ . Also, as emphasized in I0, the preference relation  $\succsim$  is uniquely determined and is complete. For  $\rho < \infty$ , neither uniqueness nor completeness is obtained (except some trivial case). We need really the extended axioms B2 and B4 for our full arguments. We study those results in the subsequent sections.

Before going to the next section, we check the consistency of our axiomatic system  $\langle L_\rho(X); \succsim; F; \text{B0-B4} \rangle$ . Consistency means that when  $L_\rho(X)$  and  $F$  are given with  $\rho \geq l_o = \max\{\delta(\lambda_y) : y \in Y\}$ , there exists a preference relation  $\succsim$  satisfying Axioms B0-B4. For this claim, we use the function  $u_{eu}$  in (11). This  $u_{eu}$  has the largest domain  $L_\infty(Y)$ , but for  $\rho < \infty$ , we regard  $u_{eu}$  as defined over  $L_\rho(Y)$ .

An implication from Lemma 3.3 is the consistency of the axiomatic system  $\langle L_\rho(X); \succsim; F; \text{B0-B4} \rangle$  with  $0 < \lambda_y < 1$  for all  $y \in Y - \{y^o, y_o\}$  for each  $\rho$  ( $l_o \leq \rho \leq \infty$ ). Indeed, the EU preference relation  $\succsim_{eu}$  restricted to  $L_\rho(X)$  satisfies B0-B4 with  $0 < \lambda_y < 1$  for all  $y \in Y - \{y^o, y_o\}$ <sup>4</sup>.

A fact proved in the proof of Lemma 3.2 will be used, and is stated separately:

$$u_{eu}(\mathbf{e} * \mathbf{f}) = \sum_{t=1}^{\ell} \frac{1}{\ell} u_{eu}(f_t) \text{ for any } \mathbf{f} \in L_k(Y)^\ell, k \geq 0. \quad (13)$$

## 4 The Canonical Preference Relation $\succsim_c$ and Central Domain

When  $\rho = \infty$ , the extension step (ii) generates the unique relation  $\succsim_{eu}$  in the constructive manner. In case  $\rho < \infty$ , however, multiple relations are allowed, and they may not be constructive in the intended sense of (ii). To capture this intention, we consider the smallest relation satisfying Axioms B0-B4, called the *canonical relation*  $\succsim_c$ , and define the domain in terms of  $\succsim_c$ , called the *central domain*  $\mathbf{C}(F; \rho)$ . We argue that any lotteries in this domain of lotteries are reached by a finite number of applications of Axiom B4. For  $\rho = \infty$ , the canonical relation  $\succsim_c$  coincides with the eu-relation  $\succsim_{eu}$ , and the central domain  $\mathbf{C}(F; \rho)$  is  $L_\rho(Y)$  itself. In the following, we assume  $\rho < \infty$  and  $0 < \lambda_y < 1$  for  $y \in Y - \{y^o, y_o\}$  for a base facet  $F$ , giving remarks on case  $\rho = \infty$ , time to time.

We denote, by  $\mathcal{P}(F; \rho)$ , the set of all preference relations satisfying B0-B4 with a given base facet  $F$  and a bound  $\rho$ . Since  $\langle L_\rho(X); \succsim; F; \text{B0-B4} \rangle$  is consistent,  $\mathcal{P}(F; \rho)$  is nonempty. In fact,  $\mathcal{P}(F; \rho)$  consists of multiple relations, some of which satisfy completeness but some others exhibit incomparabilities. The smallest relation  $\succsim_c$  plays a crucial role in the subsequent analysis. We start with the following lemma: For two relations  $\succsim$  and  $\succsim'$  in  $\mathcal{P}(F; \rho)$ , we define: for all  $f, g \in L_\rho(Y)$ ,

$$f \prec^o g \Leftrightarrow (f \prec g) \& (f \prec' g) \text{ and } f \sim^o g \Leftrightarrow (f \sim g) \& (f \sim' g)$$

$$f \succsim^o g \Leftrightarrow f \prec^o g \text{ or } f \sim^o g.$$

This  $\succsim^o$  is said to be the *common part* of  $\succsim$  and  $\succsim'$ .

**Lemma 4.1(Common part of two relations).** The common part  $\succsim^o$  of  $\succsim$  and  $\succsim'$  satisfies

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<sup>4</sup>If we weaken B1 to B1<sup>o</sup> :  $y_o \succsim y \succsim y^o$  for all  $y \in Y$ , for this lemma, we could drop the condition that  $0 < \lambda_y < 1$  for all  $y \in Y - \{y^o, y_o\}$ .

Axioms B0-B4 with the same  $F^5$ .

**Proof.** We verify that  $\succsim^o$  satisfies B0-B4. Axioms B1 and B2 are about the strict parts of  $\succsim$  and  $\succsim'$ , which are preserved in  $\succsim^o$ . Similarly, Axioms B0.(1) and B3 are about the indifference parts of  $\succsim$  and  $\succsim'$ , which are preserved in  $\succsim^o$ . For Axiom B0.(2), let  $f \succsim^o g$  and  $g \succsim^o h$ . Consider  $(f \prec g) \wedge (f \prec' g)$  and  $(g \sim h) \wedge (g \sim' h)$ . Then,  $(f \prec h) \wedge (f \prec' h)$ ; so, we have  $f \succsim^o g$ . In the other cases, we have  $f \succsim^o h$  similarly.

Consider Axiom B4.(1). Suppose  $\mathbf{f} \in L_{\rho-1}(Y)$ ,  $\mathbf{g} \in L_{\rho-1}(Y)$ , and  $\mathbf{f} \succsim^* \mathbf{g}$ . The last means  $f_t \succsim^o g_t$  for all  $t = 1, \dots, \ell$ . This is equivalent to  $[(f_t \prec g_t) \wedge (f_t \prec' g_t)]$  or  $[(f_t \sim g_t) \wedge (f_t \sim' g_t)]$  for  $t = 1, \dots, \ell$ . This implies  $(f_t \succsim g_t) \wedge (f_t \succsim' g_t)$  for  $t = 1, \dots, \ell$ . By Axiom B4.(1) for  $\succsim$  and  $\succsim'$ , it holds that  $[(\mathbf{e} * \mathbf{f} \prec \mathbf{e} * \mathbf{g}) \wedge (\mathbf{e} * \mathbf{f} \prec' \mathbf{e} * \mathbf{g})]$  or  $[(\mathbf{e} * \mathbf{f} \sim \mathbf{e} * \mathbf{g}) \wedge (\mathbf{e} * \mathbf{f} \sim' \mathbf{e} * \mathbf{g})]$ , which means  $\mathbf{e} * \mathbf{f} \succsim^o \mathbf{e} * \mathbf{g}$ . Axiom B4.(2) is parallel. ■

Observe that  $L_\rho(Y)$  is a finite set since both  $Y$  and  $\rho$  are finite; so,  $\mathcal{P}(F; \rho)$  is also a finite set. By Lemma 4.1, the common part  $\succsim_c$  of all the relations in  $\mathcal{P}(F; \rho)$  belongs to  $\mathcal{P}(F; \rho)$ , too, which is the *canonical relation*. Thus,  $\succsim_c$  satisfies B0-B4 for  $F$  and  $\rho$ . It is the smallest with respect to the set-theoretical inclusion in  $\mathcal{P}(F; \rho)$ ; i.e.,

$$\text{for any } \succsim \text{ in } \mathcal{P}(F; \rho) \text{ and } f, g \in L_\rho(Y), \quad f \succsim_c g \text{ implies } f \succsim g. \quad (14)$$

To study the canonical relation  $\succsim_c$ , we define the following set of lotteries: for  $k = 0, 1, \dots, \rho$ ,

$$\mathbf{C}_k(F; \rho) := \{f \in L_k(Y) : f \sim_c \lambda y^o * (1 - \lambda) y_o \text{ for some } \lambda \in \Pi_\rho\}. \quad (15)$$

This is the set of lotteries indifferent to benchmark lotteries with respect to  $\succsim_c$ . By Axioms B2, the probability weight  $\lambda$  in the right-hand side in (15) is unique for each  $f \in \mathbf{C}_k(F; \rho)$ ; thus we denote this unique  $\lambda$  by  $\lambda_f$ . Observe  $\mathbf{C}_k(F; \rho) \subseteq \mathbf{C}_{k+1}(F; \rho)$  for  $k < \rho$  and  $Y \cup \mathbf{B}_k(y^o; y_o) \subseteq \mathbf{C}_k(F; \rho)$  for  $k \leq \rho$ . We denote  $\mathbf{C}_\rho(F; \rho)$  simply by  $\mathbf{C}(F; \rho)$ , which is the *central domain*. For  $\rho < \infty$ ,  $\mathbf{C}(F; \rho) \subsetneq L_\rho(Y)$  if and only if  $\{y^o, y_o\} \subsetneq Y$ , which will be shown in Theorem 4.2.

For  $\rho = \infty$ ,  $\mathcal{P}(F; \rho)$  consists of only the eu-preference relation  $\succsim_{eu}$ ; so, the canonical relation  $\succsim_c$  is  $\succsim_{eu}$  itself, and  $\mathbf{C}(F; \rho)$  is  $L_\infty(Y)$  by Lemma 3.2.

Scrutinizing Axioms B0-B4, we find that Axioms B3 and B4 give are the source for indifferences to benchmark lotteries. Axiom B3 gives the base cases for  $y \in Y$ , and Axiom B4 extends such indifferences to deeper layers. Lemma 4.2 states that this statement holds as long as the canonical relation  $\succsim_c$  is concerned.

**Lemma 4.2(Constructiveness).** Let  $f \in L_\rho(Y) - Y \cup \mathbf{B}_\rho(y^o; y_o)$ . Let  $k < \rho$ . Then  $f \in \mathbf{C}_{k+1}(F; \rho)$  if and only if

$$f = \mathbf{e} * \mathbf{f} \text{ for some } \mathbf{f} = (f_1, \dots, f_\ell) \in \mathbf{C}_k(F; \rho)^\ell \text{ and } \lambda_{f_t} \in \Pi_{\rho-1} \text{ for } t = 1, \dots, \ell. \quad (16)$$

It follows from Lemma 4.2 that for any  $f \in \mathbf{C}(F; \rho)$ , there is some  $g \in \mathbf{B}_\rho(y^o; y_o)$  with  $f \sim_c g$  such that it is constructed from the base fact  $F = \langle Y; y^o; y_o; \{\lambda_y\}_{y \in Y} \rangle$  with B3 by a finite number of applications of Axiom B4. This can be seen in the reverse manner. Consider

<sup>5</sup>Each relation  $\succsim$  is a subset of  $L_\rho(Y)^2$ . Hence, it may appear to natural to define the common part by the intersection  $\succsim \cap \succsim' = \{(f, g) : f \succsim g \text{ and } f \succsim' g\}$ . However, this does not capture the common part of preference relations. For example, let  $A = \{a, b\}$ ,  $a \sim b$  and  $a \prec' b$  are expressed as  $\succsim = \{(a, b), (b, a)\}$  and  $\succsim' = \{(a, b)\}$ . In this case,  $\succsim \cap \succsim' = \{(a, b)\}$  meaning that  $a$  is strictly preferred to  $b$ , which does not capture our intention of “common part”.

any  $f \in \mathbf{C}_{k+1}(F; \rho)$ . Lemma 4.2 states that  $f$  is decomposed into  $\mathbf{f} = (f_1, \dots, f_\ell) \in \mathbf{C}_k(F; \rho)^\ell$  with  $f = \mathbf{e} * \mathbf{f}$ . Consider each  $f_t$ . If  $f_t \in Y$ , the process stops. If  $f_t \in \mathbf{B}_\rho(y^o; y_o) - Y$ , we can decompose  $(h_1, \dots, h_\ell) \in \mathbf{B}_{\rho-1}(y^o; y_o)^\ell$  by Lemma 2.4. If  $f_t \in L_\rho(Y) - Y \cup \mathbf{B}_\rho(y^o; y_o)$ , then, by Lemma 2.4,  $f_t$  is decomposed into  $(h_1, \dots, h_\ell) \in \mathbf{C}_{k-1}(F; \rho)^\ell$ . We continue this decomposition process until we stop at an element in  $Y$ . After all, this lemma guarantees that the constructive nature of the canonical relation  $\succsim_c$  over the central domain  $\mathbf{C}(F; \rho)$ .

**Proof of Lemma 4.2.(If):** Let (16) holds. Then  $f_t \sim_c \lambda_{f_t} y^o * (1 - \lambda_{f_t}) y_o$  for  $t = 1, \dots, \ell$ . We write  $g_t = \lambda_{f_t} y^o * (1 - \lambda_{f_t}) y_o$  for  $t = 1, \dots, \ell$ . Then,  $\mathbf{g} \in \mathbf{B}_{\rho-1}(y^o; y_o)$ . We can apply Axiom B4, and obtains  $f = \mathbf{e} * \mathbf{f} \sim_c \mathbf{e} * \mathbf{g}$ . Here,  $\mathbf{e} * \mathbf{g}$  is a benchmark lottery in  $\mathbf{B}_\rho(y^o; y_o)$ . Thus,  $f \in \mathbf{C}_{k+1}(F; \rho)$ .

**(Only-If):** We let  $f \in \mathbf{C}_{k+1}(F; \rho)$ . We define  $b(\lambda_f) := \lambda_f y^o * (1 - \lambda_f) y_o$ . We suppose, on the contrary, that (16) does not hold for  $f$ .

Define  $Z = \{h \in L_\rho(Y) - Y \cup \mathbf{B}_\rho(y^o; y_o) : h \sim_c b(\lambda_f) \text{ but (16) does not hold for } h\}$ . Then,  $f \in Z$ . Now, we consider the binary relation  $\succsim_*$  obtained from  $\succsim_c$  by eliminating the indifferences between any elements in  $Z$  and  $b(\lambda_f)$ : that is,  $\succsim_*$  is defined as:

$$\succsim_* = \succsim_c - \{(h, b(\lambda_f)), (b(\lambda_f), h) : h \in Z\}. \quad (17)$$

We show that this relation  $\succsim_*$  satisfies B0-B4. Once this is shown, since  $f \in Z$ , the canonical  $\succsim_c$  is not the smallest preference relation in  $\mathcal{P}(F; \rho)$ , which is a contradiction to the definition of  $\succsim_c$ ; so (16) should hold for  $f$ .

Let us see that  $\succsim_*$  satisfies B0-B4. The elimination (17) does not affect B0.(i)(reflexivity) since  $h \in Z$  and  $b(\lambda_f) \notin Z$ . Axiom B0.(ii)(transitivity) will be considered below. Since B1 is about preferences over  $Y$ . Since B2 is about the strict part of  $\succsim_c$ , it is not affected by (17). Since  $h \in Z$  does not belong to  $Y \cup \mathbf{B}_\rho(y^o; y_o)$ , it is not in  $Y$ ; and  $h \neq y$  for any  $y \in Y$ . Hence, B3 is not affected either.

Let us show that  $\succsim_*$  satisfies B4. Consider any  $h \in Z$ . Let us see that the indifference  $h \sim_c b(\lambda_f)$  cannot be the conclusion of B4. By our assumption for  $h$ , there is no  $\mathbf{h} \in \mathbf{C}_k(F; \rho)^\ell$  such that  $\mathbf{h} \sim_c \mathbf{g}$  some  $\mathbf{g} \in \mathbf{B}_{\rho-1}(y^o; y_o)^\ell$ . Hence, if  $h \in L_{k+1}(Y)$  is decomposed into of  $\mathbf{h} \in L_k(Y)^\ell$ , then we do not have  $\mathbf{g} \in \mathbf{B}_{\rho-1}(y^o; y_o)^\ell$  with  $\mathbf{h} \sim_c \mathbf{g}$ . So, the conclusion of B4,  $h \sim_c b(\lambda_f)$  ( $= \mathbf{e} * \mathbf{g}$ ), cannot be obtained. Thus, the elimination of  $h \sim_c b(\lambda_f)$  by (17) does not affect the conclusion of B4. Hence, the instance of B4 for  $h$  holds for the preference relation  $\succsim_*$ .

Finally, we show that  $\succsim_*$  satisfies B0.(ii)(transitivity). Consider an instance of transitivity:

$$w \succsim_c u \ \& \ u \succsim_c v \ \text{imply} \ w \succsim_c v. \quad (18)$$

We show that this instance holds also for  $\succsim_*$ . If both  $w$  and  $v$  differ from  $b(\lambda_f)$ , then (17) does not affect the truth value of (18). If  $w = v$ , then  $w \succsim_* v$  remains by reflexivity for  $\succsim_c$ . Hence, the remaining case is that  $w \neq v$  and ( $w = b(\lambda_f)$  or  $v = b(\lambda_f)$ ).

If strict preference  $w \prec_c v$  holds, then  $(w, v)$  is neither  $(h, b(\lambda_f))$  nor  $(b(\lambda_f), h)$  for any  $h \in Z$ ; so (17) does not affect this instance, and (18) holds for  $\succsim_*$ . Thus, we can assume  $w \sim_c v$ . Thus, the instance implies that  $w \sim_c u \ \& \ u \sim_c v$  and  $w \sim_c v$ .

Consider the case where at least one of  $w, u$  and  $v$  is  $b(\lambda_f)$ . Now, let us assume that two of  $w, u, v$  are  $b(\lambda_f)$ . If  $w = u = b(\lambda_f)$ , then the instance holds for  $\succsim_*$ : indeed, if  $v \in Z$ , the instance holds for  $\succsim_*$  in the trivial sense since the premise  $u = b(\lambda_f) \sim_c v$  is eliminated; and if  $v \notin Z$ ,

then the conclusion  $w = b(\lambda_f) \sim_c v$ , which is not eliminated by (17); so, the instance holds for  $\succsim_*$ . Consider the case where  $w \sim_c u = v = b(\lambda_f)$ . If  $w \in Z$ , then the premise  $w \sim_c u = b(\lambda_f)$  is eliminated, and so, the instance holds for  $\succsim_*$ . If  $w \notin Z$ , then the conclusion  $w \sim_c v$  is not eliminated. Thus, the instance (18) holds for  $\succsim_*$ .

Now, we consider assume exactly one of  $w, u$  and  $v$  is  $b(\lambda_f)$ . Let  $w \sim_c u = b(\lambda_f) \sim_c v$ . If  $w, v \notin Z$ , the instance is not affected by (17). If  $w \in Z$  or  $v \in Z$ , then the instance holds trivially for  $\succsim_*$ .

Let  $w = b(\lambda_f) \sim_c u \sim_c v$ . If  $u \in Z$  or  $v \in Z$ , then the instance holds trivially for  $\succsim_*$ . Let  $u \notin Z$  and  $v \notin Z$ . Then,  $w = b(\lambda_f) \sim_c v$  is not affected by (17). Finally, let  $w \sim_c u \sim_c v = b(\lambda_f)$ . If  $w \in Z$  or  $u \in Z$ , then the instance holds trivially for  $\succsim$ . Let  $w \notin Z$  and  $u \notin Z$ . Then,  $w \sim_c v = b(\lambda_f)$  is not affected by (17). ■

The following is the AKT example, which will be discussed in Section 5.

**Example 4.1 (AKT example):** Consider a base facet  $F = \langle Y; y^o, y_o; \{\lambda_y\}_{y \in Y} \rangle$  with  $Y = \{y^o, y, y_o\}$  and  $\lambda_y = \frac{85}{10^2}$ . Let  $\rho = 2$ . In this case,

$$\begin{aligned} \mathbf{C}_0(F; \rho) &= Y; \mathbf{C}_1(F; \rho) = \mathbf{B}_1(y^o; y_o) \cup \{y\}; \\ \text{and } \mathbf{C}_2(F; \rho) &= \mathbf{B}_2(y^o; y_o) \cup \{y\}. \end{aligned} \quad (19)$$

These follow from (15). In particular,  $y \sim_c \frac{85}{10^2} y^o * \frac{15}{10^2} y_o$  is expressed in  $\mathbf{C}(F; \rho) = \mathbf{C}_2(F; \rho)$ . However, the lottery  $\frac{25}{10^2} y * \frac{15}{10^2} y_o$  cannot be expressed in  $\mathbf{C}_2(F; \rho)$ . In Section 5, we will discuss how such lotteries are evaluated.

The following theorem states that over the central domain  $\mathbf{C}(F; \rho)$ , all the preference relations in  $\mathcal{P}(F; \rho)$  coincide with the canonical relation  $\succsim_c$ ; and *a fortiori*, the eu-preference relation  $\succsim_{eu}$  coincides with  $\succsim_c$ . Thus, this theorem corresponds to Theorem 3.1 for  $\rho = \infty$ . For  $\rho < \infty$ , this theorem has some immediate implications over the remaining domain  $L_\rho(Y) - \mathbf{C}(F; \rho)$ , which will be stated in Theorem 4.2.

**Theorem 4.1 (Uniqueness of  $\succsim_c$  over the central domain).** Let  $\succsim$  be any preference relation in  $\mathcal{P}(F; \rho)$ . Then, for any  $f, g \in \mathbf{C}(F; \rho)$ , **(1):**  $f \succsim g \iff f \succsim_c g$ ; and **(2):**  $\lambda_f = u_{eu}(f)$ .

**Proof.(1):** The one direction is already mentioned in (14). The converse would hold if  $\succsim_c$  is complete over  $\mathbf{C}(F; \rho)$ . Indeed, let  $f, g \in \mathbf{C}(F; \rho)$ . By (15),  $f \sim_c \lambda_f y^o * (1 - \lambda_f) y_o$  and  $g \sim_c \lambda_g y^o * (1 - \lambda_g) y_o$ . By B2,

$$f \succsim_c g \iff \lambda_f y^o * (1 - \lambda_f) y_o \succsim \lambda_g y^o * (1 - \lambda_g) y_o \iff \lambda_f \leq \lambda_g.$$

Hence,  $\succsim_c$  is complete over  $\mathbf{C}(F; \rho)$ .

**(2):** We prove by induction over  $k = 0, \dots, \rho$  that  $\lambda_f = u_{eu}(f)$  for any  $f \in \mathbf{C}_k(F; \rho)$ . Let  $k = 0$ . Then,  $f \in L_0(Y) = Y$ . For this  $f = y$ , we have  $\lambda_f = u_{eu}(f)$  by Lemma 3.1.(2). Now, suppose that  $\lambda_f = u_{eu}(f)$  for  $f \in \mathbf{C}_k(F; \rho)$  and  $k < \rho$ . Let  $f \in \mathbf{C}_{k+1}(F; \rho)$ . Then  $f \sim_c \lambda_f y^o * (1 - \lambda_f) y_o$ . By Lemma 4.2, there is a  $\mathbf{f} = (f_1, \dots, f_\ell)$  such that  $f_t \in \mathbf{C}_k(F; \rho)$  and  $\lambda_{f_t} \in \Pi_{\rho-1}$  for each  $t = 1, \dots, \ell$ . Hence, by B4,

$$\lambda y^o * (1 - \lambda) y_o \sim_c f = \mathbf{e} * \mathbf{f} \sim_c \sum_{t=1}^{\ell} \frac{1}{\ell} * (\lambda_{f_t} y^o * (1 - \lambda_{f_t}) y_o) = \left( \sum_{t=1}^{\ell} \frac{1}{\ell} \lambda_{f_t} \right) y^o * \left( 1 - \sum_{t=1}^{\ell} \frac{1}{\ell} \lambda_{f_t} \right) y_o.$$

Hence,  $\lambda_f \sim_c \sum_{t=1}^{\ell} \frac{1}{\ell} \lambda_{f_t} = \sum_{t=1}^{\ell} \frac{1}{\ell} u_{eu}(f_t) = u_{eu}(\sum_{t=1}^{\ell} \frac{1}{\ell} * f_t) = u_{eu}(f)$  by (13). ■

Now, we have implications of the above theorem for the remaining part  $L_\rho(Y) - \mathbf{C}(F; \rho)$ .

**Theorem 4.2 (Outside the central domain  $\mathbf{C}(F; \rho)$ ).**

(1): For any  $f \in L_\rho(Y)$ , if  $\delta(u_{eu}(f)) > \rho$ , then  $f \notin \mathbf{C}(F; \rho)$ .

(2): Then,  $\mathbf{C}(F; \rho) = L_\rho(Y)$  if and only if  $Y = \{y^o, y_o\}$ .

**Proof.** (1): We show the contrapositive. Let  $f \in \mathbf{C}(F; \rho)$ . Then,  $\lambda_f \in \Pi_\rho$ . Since  $u_{eu}(f) = \lambda_f$  by Theorem 4.1.(2), we have  $\delta(u_{eu}(f)) = \delta(\lambda_f) \leq \rho$ .

(2): Let  $Y = \{y^o, y_o\}$ . Since  $\lambda_{y^o} = 1$  and  $\lambda_{y_o} = 0$ , we have  $\delta(y^o) = \delta(y_o) = 0$  and  $l_o = 0$ . By (15),  $\mathbf{C}_\rho(F; \rho) = \mathbf{B}_\rho(y^o; y_o) = L_\rho(Y)$ .

Conversely, suppose some  $y \in Y$  with  $y \neq y^o, y_o$ . By B1,  $0 < \lambda_y < 1$ . Then  $\delta(\lambda_y) > 0$ . Take the lottery  $f = \frac{1}{\lambda_y} y * (1 - \frac{1}{\lambda_y}) y_o \in L_\rho(Y)$ . Then,  $\delta(f) = \rho + \delta(\lambda_y) > \rho$ . Hence,  $f \notin \mathbf{C}_\rho(F; \rho)$ . Thus,  $\mathbf{C}_\rho(F; \rho) \subsetneq L_\rho(Y)$ . ■

The converse of the first claim does not necessarily hold: A counter example is constructed by adjusting Example 4.1 to  $\lambda_y = \frac{5}{10}$  and  $\rho = 1$ . Then,  $\mathbf{C}(F; 1) = \{\alpha y^o * (1 - \alpha) y_o : \alpha \in \Pi_1\} \cup \{y\}$ , and  $h = \frac{2}{10} y * \frac{8}{10} y_o \notin \mathbf{C}(F; 1)$ . However,  $u_{eu}(h) = \frac{5}{10} \cdot \frac{2}{10} = \frac{1}{10}$ ; so  $\delta(u_{eu}(h)) = 1$ .

Before studying the behavior of  $\preceq_c$  over the remaining part  $L_\rho(Y) - \mathbf{C}(F; \rho)$ , we mention affine-uniqueness of representations. We say that  $u : \mathbf{C}(F; \rho) \rightarrow R$  represents  $\preceq$  in the EU sense iff it for all  $f, g \in \mathbf{C}(F; \rho)$ ,  $u(f) \leq u(g) \iff f \preceq g$ ; and

$$u(\lambda y_o * (1 - \lambda) y^o) = \lambda u(y^o) + (1 - \lambda) u(y_o) \quad \text{for any } \lambda \in \Pi_\rho. \quad (20)$$

This keeps linearity over the benchmark lotteries. We obtain the affine-uniqueness for representing utility functions. This theorem holds also for  $\rho = \infty$ .

**Theorem 4.3 (Affine-uniqueness):** Let  $u, v$  be two real-valued functions on the central domain  $\mathbf{C}(F; \rho)$  representing the same  $\preceq$ . Then, there are  $a > 0$  and  $b$  such that  $v(f) = au(f) + b$  for all  $f \in \mathbf{C}(F; \rho)$ .

**Proof:** We define  $a = \frac{v(y^o) - v(y_o)}{u(y^o) - u(y_o)}$  and  $b = -\frac{v(y^o) - v(y_o)}{u(y^o) - u(y_o)} u(y_o) + u(y_o)$ . Then, it holds that  $au(y^o) + b = v(y^o)$  and  $av(y_o) + b = v(y_o)$ . Let  $f \in \mathbf{C}(F; \rho)$  with  $f \sim \lambda_f y_o * (1 - \lambda_f) y^o$ . Then, by (20),  $au(f) + b = au(\lambda_f y_o * (1 - \lambda_f) y^o) + b = a(\lambda_f u(y^o) + (1 - \lambda_f) u(y_o)) + b = \lambda_f v(y^o) + (1 - \lambda_f) v(y_o) = v(\lambda_f y^o * (1 - \lambda_f) y_o) = v(f)$ . ■

## 5 Incomparability in the Canonical Relation $\preceq_c$

Here, we study incomparability included in the canonical relation  $\preceq_c$  over  $L_\rho(Y) - \mathbf{C}(F; \rho)$ .

For any  $f \in L_\rho(Y)$ , we define the *greatest lower bound*  $\underline{\lambda}_f$  and *least upper bound*  $\bar{\lambda}_f$  by

$$\underline{\lambda}_f = \max\{\lambda_g : g \preceq_c f \text{ \& } g \in \mathbf{C}(F; \rho)\}; \text{ and } \bar{\lambda}_f = \min\{\lambda_g : f \preceq_c g \text{ \& } g \in \mathbf{C}(F; \rho)\}. \quad (21)$$

These are abbreviated as *GLB* and *LUB* for  $f$ . By the definition of  $\mathbf{C}(F; \rho)$ , we can replace  $\mathbf{C}(F; \rho)$  by  $\mathbf{B}_\rho(y^o; y_o)$  in (21). Since  $y_o \preceq_c f \preceq_c y^o$  by B2 and B0, these  $\underline{\lambda}_f$  and  $\bar{\lambda}_f$  are well-defined. Also, it holds that  $\underline{\lambda}_f \leq \bar{\lambda}_f$  for any  $f \in L_\rho(Y)$ . but there is no  $\lambda \in \Pi_\rho$  for any  $f \in L_\rho(Y) - \mathbf{C}(F; \rho)$  such that  $f \sim_c \lambda y^o * (1 - \lambda) y_o$ . Thus, we have

$$\underline{\lambda}_f = \bar{\lambda}_f \iff f \in \mathbf{C}(F; \rho); \text{ and } \underline{\lambda}_f < \bar{\lambda}_f \iff f \in L_\rho(Y) - \mathbf{C}(F; \rho). \quad (22)$$

The second holds since there is no  $\lambda \in \Pi_\rho$  for any  $f \in L_\rho(Y) - \mathbf{C}(F; \rho)$  such that  $f \sim_c \lambda y^o * (1 - \lambda) y_o$ . This is crucial for consideration of the AKT anomaly in Section 6.

Let  $f, g \in L_\rho(Y)$ . There are three cases to be considered:

**IC:**  $f, g \in \mathbf{C}(F; \rho)$ ; **IA:**  $f \in L_\rho(Y) - \mathbf{C}(F; \rho)$  &  $g \in \mathbf{C}(F; \rho)$ ; and **IB:**  $f, g \in L_\rho(Y) - \mathbf{C}(F; \rho)$ .

Case IC is already treated in Theorem 4.1;  $f, g$  are comparable. Case IA is covered by the following theorem. This is crucial for the AKT example.

**Theorem 5.1(Incomparability A):** Let  $f \in L_\rho(Y) - \mathbf{C}(F; \rho)$  and  $g \in \mathbf{C}(F; \rho)$ . Then,  $f$  and  $g$  are incomparable with respect to  $\succsim_c$  if and only if  $\underline{\Delta}_f y^o * (1 - \underline{\Delta}_f) y_o \prec_c g \prec_c \bar{\lambda}_f y^o * (1 - \bar{\lambda}_f) y_o$ .

**Proof.(If):** We show the contrapositive. Suppose that  $f$  and  $g$  are comparable, i.e.,  $f \succsim_c g$  or  $g \succsim_c f$ . If  $f \succsim_c g$ , then  $\bar{\lambda}_f y^o * (1 - \bar{\lambda}_f) y_o \succsim_c g$  by (21), which denies the latter part of the claim. If  $g \succsim_c f$ , then  $g \succsim_c \underline{\Delta}_f y^o * (1 - \underline{\Delta}_f) y_o$  by (21), which denies the latter part again.

**(Only-if):** Consider the contrapositive, again. Suppose the negation of the latter part. Since  $g$  and  $\underline{\Delta}_f y^o * (1 - \underline{\Delta}_f) y_o$  are in  $\mathbf{C}(F; \rho)$ , they are comparable by B0.(0) and B2. Thus, the negation of the latter part is written as  $g \succsim_c \underline{\Delta}_f y^o * (1 - \underline{\Delta}_f) y_o$  or  $\bar{\lambda}_f y^o * (1 - \bar{\lambda}_f) y_o \succsim_c g$ . Hence,  $g \prec_c f$  or  $f \prec_c g$  by (21). This means that  $f$  and  $g$  are comparable. ■

Since the central domain  $\mathbf{C}(F; \rho)$  is explicitly given, it is not difficult to see whether  $f \notin \mathbf{C}(F; \rho)$  or not. However, the calculations of  $\underline{\Delta}_f$  and  $\bar{\lambda}_f$  for  $f \notin \mathbf{C}(F; \rho)$  need additional arguments for each  $f$ . We give some argument in the AKT example in Section 6.

To study incomparability in Case IB, we introduce the following concept: When the premise of Axiom B2 holds for  $f, g$  in  $L_\rho(Y)$ , we write  $g \rightarrow f$ . We say that  $f$  is *monotonically accessible from  $g$* , denoted by  $g \rightarrow_M f$  iff there are  $f_0, \dots, f_m$  in  $L_\rho(Y)$  such that  $g = f_0 \rightarrow f_1 \rightarrow \dots \rightarrow f_m = f$ . Applying Axiom B2 to this sequence, we have

$$\text{if } g \rightarrow_M f, \text{ then } g \prec_c f. \quad (23)$$

We say that  $f$  and  $g$  are *monotonically inaccessible* iff neither  $g \rightarrow_M f$  nor  $f \rightarrow_M g$ .

The following theorem gives a condition for comparability in IB; its negation is a condition for incomparability in IB.

**Theorem 5.2(Incomparability B):** Let  $f, g \in L_\rho(Y) - \mathbf{C}(F; \rho)$ . Let  $f, g$  be monotonically inaccessible. Then,  $f \succsim_c g$  if and only if  $f \succsim_c h \succsim_c g$  for some  $h \in \mathbf{C}(F; \rho)$ .

**Proof.** The *if* part follows from B0.(2). Consider the *only-if* part. Suppose  $f \succsim_c g$  but that

$$\text{there is no } h \in \mathbf{C}(F; \rho) \text{ such that } f \succsim_c h \succsim_c g. \quad (24)$$

We derive a contradiction. Since  $L_\rho(Y)$  is finite, there is a finite sequence of distinct  $f_1, \dots, f_k$  such that  $f = f_1 \succsim_c f_2 \succsim_c \dots \succsim_c f_k = g$  and each  $f_{t+1}$  is immediately next to  $f_t$  with respect to  $\succsim_c$ . By (24), all of those lotteries are in  $L_\rho(Y) - \mathbf{C}(F; \rho)$ . Since  $f$  and  $g$  are monotonically inaccessible, there is a  $t$  such that  $f_t$  and  $f_{t+1}$  are monotonically inaccessible, for, otherwise, each accessibility could be extended to the accessibility between  $f$  and  $g$ . We take such a  $(f_t, f_{t+1})$ , and define a binary relation  $\succsim^*$  by

$$\succsim^* = \succsim_c - \{(f_t, f_{t+1})\}. \quad (25)$$

Then, we prove that  $\succsim^*$  satisfies B0-B4. This is a contradiction to the fact, (14), that  $\succsim_c$  is the smallest relation satisfying B0-B4; and thus,  $f \succsim_c g$  implies that  $g \succsim_c h \succsim_c f$  for some  $h \in \mathbf{C}(F; \rho)$ .

B0: Since  $f_t$  and  $f_{t+1}$  are distinct, Reflexivity is not affected by (25) for  $\succsim^*$ . Transitivity is seen as follows: Since  $f_t$  and  $f_{t+1}$  are immediately next by construction, the preference  $f_t \succsim_c f_{t+1}$  is a conclusion of transitivity only if one of the premise is  $f_t \succsim_c f_t$  or  $f_{t+1} \succsim_c f_{t+1}$ . In either case,  $f_t \succsim_c f_{t+1}$  appears in its premise. Hence, transitivity holds for  $\succsim^*$  in the trivial sense by elimination of  $(f_t, f_{t+1})$ .

B1: This is not directly related to the preferences. Hence, it is not affected by (25).

B2,B3: Those are about lotteries in  $Y \cup \mathbf{B}_\rho(y^o; y_o) \subseteq \mathbf{C}(F; \rho)$ . Since  $f_t, f_{t+1}$  are not benchmark lotteries, those axioms are not affected by (25).

B4: This is about a pair of lotteries where at least one is a benchmark lottery. However, neither  $f_t$  nor  $f_{t+1}$  is a benchmark lottery. Hence, B4 is not affected by (25).■

It is useful to combine Theorems 4.1, 5.1 and 5.2 to have a representation in terms of the set-valued function:

$$\Lambda(f) := \{\underline{\lambda}_f, \bar{\lambda}_f\} \text{ for any } f \in L_\rho(Y). \quad (26)$$

We introduce the order relation:  $\Lambda(f) \leq \Lambda(g) \iff \lambda \leq \lambda'$  for all  $\lambda \in \Lambda(f)$  and  $\lambda' \in \Lambda(g)$ . We would like to check where the following equivalence holds:

$$f \succsim_c g \iff \Lambda(f) \leq \Lambda(g). \quad (27)$$

The right-hand side,  $\Lambda(f) \leq \Lambda(g)$ , gives a partial ordering. In the second assertion of Theorem 5.3, we need the additional condition “monotonically inaccessible”. Hence, (27) holds partially. A counter example will be given in the very end of Section 6.2. Still, (27) is useful to see incomparability involved in  $\succsim_c$ .

**Theorem 5.3(Partial representation):(1):** (27) holds if  $(f \in L_\rho(Y) \ \& \ g \in \mathbf{C}(F; \rho))$  or  $(f \in \mathbf{C}(F; \rho) \ \& \ g \in L_\rho(Y))$ .

**(2):** (27) holds if  $f, g \in L_\rho(Y) - \mathbf{C}(F; \rho)$  and they monotonically inaccessible.

**Proof.**(1): Let  $f \in L_\rho(Y)$  and  $g \in \mathbf{C}(F; \rho)$ . If  $f \in \mathbf{C}(F; \rho)$ , then we have the equivalence by Theorem 4.1. Let  $f \in L_\rho(Y) - \mathbf{C}(F; \rho)$  and  $g \in \mathbf{C}(F; \rho)$ . By Theorem 5.1,  $f \succsim_c g \iff \bar{\lambda}_f \leq \lambda_g$ , i.e.,  $\Lambda_c(f) \leq \Lambda_c(g)$ , and  $g \succsim_c f \iff \lambda_g \leq \underline{\lambda}_f$ , i.e.,  $\Lambda_c(g) \leq \Lambda_c(f)$ .

(2): Consider the second assertion, i.e., let  $f, g \in L_\rho(Y) - \mathbf{C}(F; \rho)$  be monotonically inaccessible. By Theorem 5.2,  $f \succsim_c g \iff f \succsim_c h \succsim_c g$  for some  $h \in \mathbf{C}(F; \rho)$ . Then, by the case of IA, we have  $\bar{\lambda}_f \leq \lambda_h \leq \underline{\lambda}_g$ , which implies  $\Lambda_c(f) \leq \Lambda_c(g)$ . Suppose  $\Lambda_c(f) \leq \Lambda_c(g)$ , i.e.,  $\bar{\lambda}_f \leq \underline{\lambda}_g$ . Then, we take  $h = \bar{\lambda}_f y^o * (1 - \bar{\lambda}_f) y_o$ . Then,  $f \succsim_c h$ , and also,  $h \succsim_c \underline{\lambda}_g y^o * (1 - \underline{\lambda}_g) y_o \succsim_c g$ .■

As stated in Section 1, the above incomparability does not correspond to incomparability considered in the literature of the EU theory without the completeness axiom (cf., Aumann [4], Deburá *et al.* [6]). In ours, case  $\rho = \infty$  leads to completeness, as shown in Section 3. It will be briefly mentioned in Section 7.2 that when there are multiple base facets, incompleteness would be natural even when  $\rho = \infty$ , which case corresponds to that literature.

## 6 Alleis-Kahneman-Tversky Anomaly

Here, we look at the Alleis-Kahneman-Tversky anomaly from the the viewpoint of incomparability studied Section 5. Specifically, we study at an example and the experimental result in Kahneman-Tversky [11]. A subtlety involved here is how to interpret the incomparability shown



in the theory as revealed choices by experimental subjects. We also discuss also the calculation method for the GLB and LUB for a lottery.

## 6.1 AKT experiment

In Kahneman-Tversky [11], 95 subjects were asked to choose one from lotteries  $a$  and  $b$ , and one from  $c$  and  $d$ . In the first problem, 20% chose  $a$ , and 80% chose  $b$ , and in the second, 65% out of the same subjects chose  $c$ ; and the remaining did  $d$ .

$$\begin{aligned} a &= \frac{8}{10}4000 * \frac{2}{10}0; [20\%]; & b &= 3000 \text{ with probability } 1; [80\%]. \\ c &= \frac{2}{10}4000 * \frac{8}{10}0; [65\%]; & d &= \frac{25}{100}3000 * \frac{75}{100}0; [35\%]. \end{aligned}$$

They observed that the case of modal choices ( $a \prec b \wedge c (\succ d)$ ), abbreviated as  $b \wedge c$ , contradicts the classical EU theory. Indeed, under EU theory, these revealed preferences are expressed in terms of expected utilities as:

$$\begin{aligned} 0.8u(4000) + 0.2u(0) &< u(3000) \\ 0.2u(4000) + 0.8u(0) &> 0.25u(3000) + 0.75u(0) \end{aligned} \tag{28}$$

Multiplying 4 to the second inequality, we have the opposite inequality of the first, a contradiction. This is one variant of many examples reported since Allais [2]. We call this the *Allais-Kahneman-Tversky (AKT-) anomaly*.

In the above reported percentages, the three choices, other than  $b \wedge c$ , are not really negligible. The other case contradicting the classical EU theory is  $a \wedge d$ . The case  $b \wedge c$  may be regarded as more important in that more people showed the choice of  $b$ , which is interpreted as the concavity (risk-averseness) of  $u$ . Yet, the portion of subjects choosing  $a$  is not very negligible, and the portions between  $c$  and  $d$  are more equal. Hence, the anomalous case  $b \wedge c$  is one possible case, though it is modal.

Table 6.1

	$c : 65\%$	$d : 35\%$
$a : 20\%$	$a \wedge c : \text{EU: } 0 // \mathbf{13} // 20$	$a \wedge d : \text{anomaly: } 20 // \mathbf{7} // 0$
$b : 80\%$	$b \wedge c : \text{anomaly: } 65 // \mathbf{52} // 45$	$b \wedge d : \text{EU: } 15 // \mathbf{28} // 35$

No more information is reported in [11] about the above choices<sup>6</sup>. We consider three possible distributions of the answers over the four cases: In Table 5.1, the first, second, or third, entry in each cell is the percentage derived by assuming that the percentage for  $b \wedge c$  is 65%, 52% or 45%. The first 65% is the maximum possible percentage for  $b \wedge c$ , which implies 0% for  $a \wedge c$  since the sum of 65%, and the other two %'s in %In  $a \wedge d$  and  $b \wedge d$ . The second entry 52%, the choices of  $b$  and  $c$  are assumed to be stochastically independent, and in the third, 45% is the minimum possibility for  $b \wedge c$ . Although the percentages in  $a \wedge c$ ,  $a \wedge d$ , and  $b \wedge d$  are less frequent than  $b \wedge c$ , these distributions show divergence in revealed choices.

<sup>6</sup>In the problem 1 and 2 in [11], pp.265-266, more information is given about the the percentage of the modal choice. Unfortunately, this information seems incorrect (contradictory to the other information). We do not refer to these examples.

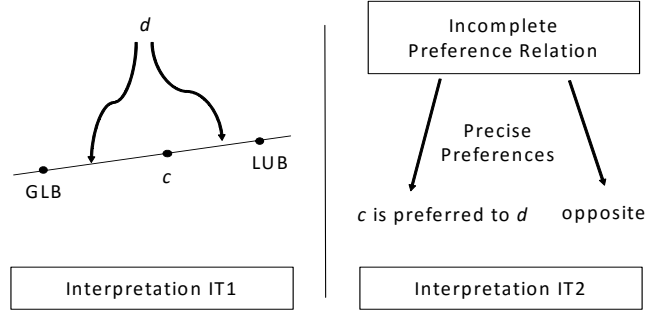


Figure 2: Two interpretations of revealed choices

Hence, we meet the problem: there are cases compatible/incompatible with EU. Also, both choices of  $a$  and  $b$  are observed. In the following, we consider mainly the row for  $b$ , and later, we mention the results for the row for  $a$ .

Now, consider Table 6.1 in our theory. Recall  $F = \langle Y; y^o, y_o; \{\lambda_y\}_{y \in Y} \rangle$  in Example 4.1, where  $Y = \{y^o, y, y_o\}$  with  $y^o = 4000, y = 3000, y_o = 0$ , and  $\lambda_y = \frac{85}{10^2}$ . Here, the assumption of  $\lambda_y = \frac{85}{10^2}$  implies that the cases for  $b$ , i.e., the row of  $b$ , are considered. Later, the other cases determined by  $\lambda_y = \frac{75}{10^2}$  will be considered.

**Case  $\rho = 2$ .** Throughout the following argument, we consider the canonical relation  $\succsim_c$ . Now, we compare the choices between  $c = \frac{2}{10}y^o * \frac{8}{10}y_o$  and  $d = \frac{25}{10^2}y * \frac{75}{10^2}y_o$ . In Section 6.2, the following inequalities will be checked:

$$\underline{\lambda}_d = \frac{16}{10^2} < \lambda_c = \frac{2}{10} < \bar{\lambda}_d = \frac{25}{10^2}, \quad (29)$$

where  $\underline{\lambda}_d, \bar{\lambda}_d$  are the GLB and LUB of  $d$ . Thus, by Theorem 5.1,  $c$  and  $d$  are incomparable with respect to the canonical relation  $\succsim_c$ . These GLB and LUB are unchanged even if we take  $\lambda_y = \frac{84}{10^2}, \dots, \frac{80}{10^2}$ .

Here, we interpret this incomparability with the locations of those GLB,  $\underline{\lambda}_d, \lambda_c$ , and LUB  $\bar{\lambda}_d$  in (29) as meaning that  $d$  rather than  $c$  would be chosen if  $d$  is regarded as falling in the interval between  $\frac{16}{10^2}$  and  $\frac{2}{10}$ ; and  $c$  would be chosen if it is in the interval between  $\frac{2}{10}$  and  $\frac{25}{10^2}$ . Thus, the dividing point is  $\frac{2}{10}$ , and both choices,  $c$  and  $d$ , are quite possible. This is depicted in the left of Fig.2. The numbers in the cells of the second row of Table 6.1 are consistent with the above incomparability results.

One question is why/how every subject chose one out of  $c$  and  $d$ , though those are incomparable for him. In the following, we consider two possible interpretations. The first one is:

**IT1:** a subject could make an arbitrary choice from  $c$  and  $d$ , provided he is asked to choose one.

Even if they are incomparable for him, but since he is asked to choose one, he would make a choice in an arbitrary manner. If the subjects fall uniformly into those two intervals  $(\text{GLB}, c)$  and  $(c, \text{LUB})$ , the ratio is  $(19 - 16) : (24 - 20) = 3 : 4$ , while  $65 : 35$  is observed in Table (29). This depends upon the uniformity assumption, but may be modified by taking the subsequent result for the row for  $a$ .

The other possible interpretation is to take the existence of many preference relations in

$\mathcal{P}(F; \rho)$  behind the canonical relation  $\succsim_c$  into account. Then, incomparability between  $c$  and  $d$  is interpreted as:

**IT2:** he would choose  $c$  (or  $d$ ) as if he has a more precise preference  $c$  over  $d$  regarding  $d$  as  $\lambda y^o * (1 - \lambda)y_o$  with  $\lambda$  from  $\frac{17}{10^2}$  to  $\frac{19}{10^2}$  (respectively, from  $\frac{21}{10^2}$  to  $\frac{24}{10^2}$ )<sup>7</sup>.

That is, he could have actually more precise preferences than  $\succsim_c$ . Here,  $\frac{20}{10^2}y^o * \frac{80}{10^2}y_o$  is the dividing point for the choice of  $c$  for the choice of  $c$  or  $d$ . The row of Table 6.1 is obtained by assuming that the 95 subjects' preferences are distributed between these two possible cases.

Interpretation IT1 is more faithful to our theory, but IT2 may be applied to more decisive people. One relevant question is how incomparability is examined in an experiment; for this, the conditional “provided he is asked to choose either” should be eliminated, but we do not have a clear-cut alternative. Instead, the GLB and LUB can be examined in an experiment, since they are in comparable lotteries.

Let us our attention to the first row of Table 6.1. For the choice  $a$  having 20% of the subjects, we change  $\lambda_y = \frac{85}{10^2}$  to  $\lambda_y = \frac{75}{10^2}$ . Then, (29) is now

$$\underline{\lambda}_d = \frac{14}{10^2} < \lambda_c = \frac{20}{10^2} < \bar{\lambda}_d = \frac{24}{10^2} \quad (30)$$

Under the uniformity assumption, the predicting ratio for the choices  $c$  and  $d$  is  $(19 - 14) : (23 - 20) = 5 : 3 = 6.25 : 3.75$ . This is closer to 65% : 35% than for the above case of  $b$ . If  $\lambda_y = \frac{65}{10^2}$ , then  $\underline{\lambda}_d = \frac{12}{10^2}$  and  $\bar{\lambda}_d = \frac{21}{10^2}$ . Hence, more choices of  $c$  would be predicted.

Now, consider the case  $\rho = 3$  and the case  $\rho \geq 4$ .

**Case  $\rho = 3$ :** Consider the case  $\lambda_y = \frac{85}{10^2}$ . In this case,  $\underline{\lambda}_d = \frac{20}{10^2}$  and  $\bar{\lambda}_d = \frac{25}{10^2}$ . Here,  $\underline{\lambda}_d = \frac{20}{10^2}$  is calculated as follows: The substitution of  $\frac{8}{10}y^o * \frac{2}{10}y_o$  for  $y$  in  $d = \frac{25}{10^2}y * \frac{75}{10^2}y_o$  generates  $\frac{2}{10}y^o * \frac{8}{10}y_o \prec_c d$ . Hence, these mean that the decision maker would choose  $d$ . The theory meets the anomaly.

Now, let  $\lambda_y = \frac{75}{10^2}$ . Then,  $\underline{\lambda}_d = \frac{165}{10^3}$  and  $\bar{\lambda}_d = \frac{20}{10^2}$ . In this case, the theory implies that the decision maker chooses  $c$ , which predicts the EU result.

**Case  $\rho \geq 4$ :** In this case,  $\underline{\lambda}_d = \bar{\lambda}_d = \frac{2125}{10^4}$ , since the substitution of  $\frac{85}{10^2}y^o * \frac{15}{10^2}y_o$  for  $y$  in  $\frac{25}{10^2}y * \frac{75}{10^2}y_o$  is allowed. This is the same as the classical EU theory, in either  $\lambda_y = \frac{85}{10^2}$  or  $\frac{75}{10^2}$ .

In sum, in either case where  $b$  or  $a$  is chosen, only the assumption  $\rho = 2$  is compatible with Table 6.1. In the case  $\rho \geq 3$ , a subject needs to go to the decimal fractions of place 3, which is unusual with our ordinary thinking about probabilities. In this sense, the explanation of the AKT anomaly in terms of our theory is compatible with our ordinary-life practices.

## 6.2 Switchback method to calculate GLB and LUB

Here, we calculate the GLB  $\underline{\lambda}_d = \frac{16}{10^2}$  and LUB  $\bar{\lambda}_d = \frac{25}{10^2}$  of  $d = \frac{25}{10^2}y * \frac{75}{10^2}y_o$  in the AKT example with  $\lambda_y = \frac{85}{10^2}$  and  $\rho = 2$ . For this, we think about how a preference between  $d$  and  $g \in \mathbf{B}_2(y^o; y_o)$  is obtained. First, observe that  $d$  has no indifferent lottery in  $\mathbf{B}_2(y^o; y_o)$  with respect to  $\succsim_c$ . Hence, if a strict preference between  $d$  and  $g$  is obtained, it must come through applications of B2 and/or B4. We may have two steps:

<sup>7</sup>and if it is  $\frac{20}{10^2}$ , then he is indifferent from  $c$  and  $d$ , in which case he would give either  $c$  or  $d$ , provided he should make a choice.

**Backward Step:** Lottery  $d = \frac{25}{10^2}y * \frac{75}{10^2}y_o$  is in  $L_2(Y)$ , and also, indifference  $y \sim_c \frac{85}{10^2}y^o * \frac{15}{10^2}y_o$  is expressed in  $L_2(Y)$ . We return to  $L_1(Y)$  from  $L_2(Y)$  by reducing  $d$  and  $\frac{85}{10^2}y^o * \frac{15}{10^2}y_o$  to  $\frac{2}{10}y * \frac{8}{10}y_o$  and  $\frac{8}{10}y^o * \frac{2}{10}y_o$ . Those are less preferred to  $d$  and  $y$  by B2.

**Forward Step:** Axiom B4 is now applied to those reduced lotteries.

This method is “switchback” in that we once return to a shallower world and then go to deeper lotteries.

The GLB  $\underline{\lambda}_d = \frac{16}{10^2}$  of  $d = \frac{25}{10^2}y * \frac{75}{10^2}y_o$  is obtained by this switchback method. Indeed, denoting  $h = \frac{85}{10^2}y^o * \frac{15}{10^2}y_o$ , we have, by B2,

$$d' = \frac{2}{10}y * \frac{8}{10}y_o \prec_c d \text{ and } h' = \frac{8}{10}y^o * \frac{2}{10}y_o \prec_c h \sim_c y,$$

Those  $d'$  and  $h'$  are the best lower approximations of  $d$  and  $h$  in  $L_1(Y)$ . We let  $\mathbf{h}' = (h', h', y_o, \dots, y_o)$  and  $\mathbf{d}' = (y, y, y_o, \dots, y_o)$ . Applying Axiom B4, we obtain  $\frac{2}{10}h' * \frac{8}{10}y_o \prec_c \frac{2}{10}y * \frac{8}{10}y_o = d'$ . The left lottery is  $\frac{16}{10^2}y^o * \frac{84}{10^2}y_o$ , since  $h' = \frac{8}{10}y^o * \frac{2}{10}y_o$ . Hence,  $\underline{\lambda}_d = \frac{16}{10^2}$ .

The verification of  $\bar{\lambda}_d = \frac{25}{10^2}$  needs only the backward step: We have  $\frac{25}{10^2}y * \frac{75}{10^2}y_o \prec_c \frac{25}{10^2}y^o * \frac{75}{10^2}y_o$  simply by B2. However, we can show that  $\frac{25}{10^2}$  for the second lottery is the smallest probability for this preference. Thus,  $\bar{\lambda}_d = \frac{25}{10^2}$ . In fact, the switchback method gives a higher probability  $\frac{27}{10^2}$ .

Theorem 5.3.(2) would not hold without the monotonic inaccessibility condition:  $d^* := \frac{24}{10^2}y * \frac{76}{10^2}y_o \prec_c \frac{25}{10^2}y * \frac{75}{10^2}y_o = d$  by B2, but  $\underline{\lambda}_{d^*} = \underline{\lambda}_d = \frac{16}{10^2}$  and  $\bar{\lambda}_{d^*} = \frac{24}{10^2} < \bar{\lambda}_d = \frac{16}{10^2}$ . We do not have  $\Lambda(d^*) < \Lambda(d)$ , but  $d^* \prec_c d$ .

## 7 Two Remarks on Base Facets

Here, we give a remark on the derivation of a base facet  $F$  by using Simon’s [18] notion of “satisficing/aspiration” and also a remark on multiple base facets.

### 7.1 Simon’s satisficing/aspiration

Our theory starts with the description of a base facet  $F = \langle Y; y^o, y_o; \{\lambda_y\}_{y \in Y} \rangle$ . As emphasized,  $\lambda_y$  does not represent the innate evaluation of  $y$  existing in his mind; instead, it is derived by his reflections on beliefs/knowledge including past experiences and evaluation criteria in his mind. This is based on Simon’s [19] “bounded rationality”. More specifically, the derivation of  $\lambda_y$  can be done by his notion of satisficing/aspiration. We give a brief account of this derivation.

Let  $y^o, y_o$  be given benchmark lotteries. The evaluation system is given as a *satisficing quintuple*  $\langle y^o, y_o; \psi; l_o; \pi \rangle$ , where  $\psi : X \times \Pi_{l_o} \rightarrow R_+$  is a satisficing function;  $l_o$  is the upper limit (natural number) for  $\Pi_k$  to be used for measurement, and  $\pi$  is an *aspiration level* in  $R_+$ . We assume that for each  $x \in X$ ,  $\psi(x, \lambda)$  is a single-peaked (in the weak sense) function of  $\lambda \in \Pi_{l_o}$ . The intended meaning of the value  $\psi(x, \lambda)$  ( $\lambda \in \Pi_k$ ) is the *satisficing degree* of the proposition:

$$“x \sim \lambda y^o * (1 - \lambda) y_o” . \quad (31)$$

The value  $\psi(x, \lambda)$  is compared with the aspiration level  $\pi$ , and he would accept his evaluation  $\lambda y^o * (1 - \lambda) y_o$  for  $x$  if and only if  $\psi(x, \lambda) \geq \pi$ .

Now, the decision maker starts using  $\Pi_1, \dots, \Pi_{l_o}$  in order to evaluate each  $x \in X$ . When he uses the net  $\Pi_1$ , the satisficing function  $\psi$  is regarded as a function over  $\Pi_1$ , and when he goes to the net  $\Pi_2$ ,  $\psi$  is a function over  $\Pi_2$ , and so on. He continues evaluating each  $x$  using the probability net  $\Pi_t$  until he succeeds in finding some  $\lambda$  with  $\psi_t(\lambda, x) \geq \pi$  or meets the bound  $l_o$ . We define, for each  $t = 0, 1, \dots, l_o$ ,

$$Y_t = \{y \in X - \cup_{s < t} Y_s : \psi(\lambda, y) \geq \pi \text{ for some } \lambda \in \Pi_t\}. \quad (32)$$

For  $y \in X - \cup_{s < t} Y_s$ ,  $\Pi_t$  is the first net where he finds a satisfactory  $\lambda$  so that  $\psi(\lambda, y) \geq \pi$ . We define  $Y = \cup_{t \leq l_o} Y_t$ . For each  $y \in Y_t$  ( $t \leq l_o$ ), we choose any  $\lambda_y$  satisfying  $\psi_t(\lambda_y, y) \geq \pi$ . Then,  $\delta(\lambda_y) = t$ . Thus, we have a base facet  $F = \langle Y; y^o, y_o; \{\lambda_y\}_{y \in Y} \rangle$ .

For each  $y \in Y_t$  ( $t \leq l_o$ ), we can take the smallest and largest  $\lambda$  satisfying  $\psi(\lambda, y) \geq \pi$ , denoted by  $\lambda_{*y}$  and  $\lambda_y^*$  respectively. These are called, the *lower* and *upper evaluations* of  $y \in Y$ . For each  $y \in Y_t$ , since  $\psi(\lambda, y)$  is single-peaked with respect to  $\lambda \in \Pi_t$ ,

$$\lambda_{*y} \leq \lambda \leq \lambda_y^* \text{ if and only if } \psi(\lambda, y) \geq \pi. \quad (33)$$

Thus, it holds that  $\lambda_{*y} \leq \lambda_y \leq \lambda_y^*$  for all  $y \in Y$ .

Furthermore, we can evaluate the difference between  $\lambda_y^* - \lambda_{*y}$  for each  $y \in Y_t$ . Since  $t$  is the first depth to have  $\psi(\lambda, y) \geq \pi$  for some  $\lambda \in \Pi_t$ , we have  $\lambda_{*y} = \frac{\nu}{\ell^{t-1}} + \frac{\nu'}{\ell^t}$  for and  $\lambda_y^* = \frac{\nu}{\ell^{t-1}} + \frac{\nu''}{\ell^t}$  for some  $\nu$  ( $0 \leq \nu < \ell^{t-1}$ ) and  $\nu', \nu''$  ( $0 < \nu' \leq \nu'' < \ell$ ). Hence, for each  $y \in Y_t$ ,

$$\lambda_y^* - \lambda_{*y} < \left(\frac{\nu}{\ell^{t-1}} + \frac{\ell-1}{\ell^t}\right) - \left(\frac{\nu}{\ell^{t-1}} + \frac{1}{\ell^t}\right) = \frac{\ell-2}{\ell^t}. \quad (34)$$

In the case  $t = 2$ , this difference is already with  $\frac{\ell-2}{\ell^2}$ ; and if  $\ell = 2$ , then  $\lambda_y^* = \lambda_{*y} = \lambda_y$ ; and if  $\ell = 3$ , then is  $\frac{\ell-2}{\ell^2} = \frac{1}{9}$ ; and if  $\ell = 10$ , then  $\frac{\ell-2}{\ell^2} = \frac{8}{100}$ .

In Fig.2, the decision maker finds a  $\lambda$  satisficing the aspiration level  $\pi$  when  $\rho = 2$ . The lower and upper evaluations  $\lambda_{*y}$  and  $\lambda_y^*$  are given as  $\frac{83}{10^2}$  and  $\frac{87}{10^2}$ . Although the maximum satisficing value is taken at  $\frac{85}{10^2}$ , this is one of the values for satisficing. In the AKT example, we assumed  $\lambda_y = \frac{85}{10^2}$ . This is approximated by the lower  $\frac{83}{10^2}$  and upper  $\frac{87}{10^2}$ . Rather, the decision maker is simply satisfied by thinking that such a value is around the interval from  $\frac{83}{10^2}$  to  $\frac{87}{10^2}$ . In fact, the same conclusion (30) based on  $\lambda_y = \frac{85}{10^2}$  and  $\rho = 2$  hold. More generally, the discussions in Section 6.2 are converted here.

Here, we can incorporate imprecision into our arguments: Perhaps, we should think about the whole argument here including those different values  $\lambda_y$ , rather than think about each specific value  $\lambda_y = \frac{85}{10^2}$ .

## 7.2 Vertical and horizontal extensions of a base facet

We have assumed that the benchmarks  $y^o$  and  $y_o$  are given. The choice of the lower  $y_o$  could naturally be chosen as the logically worst case such as in Kaneko-Nakamura [12], or the present status quo. In the latter, as suggested the prospect theory of in Kahneman-Tversky [11], the choice of  $y_o$  is dependent upon a situation in question and upon the viewpoint of the researcher. On the other hand, the choice of  $y^o$  is of more temporary nature. In general, the choices of benchmarks  $y^o$  and  $y_o$  are not really fixed. Then, there are different benchmarks than the present benchmarks. We can think about some extensions of the choices of benchmarks.

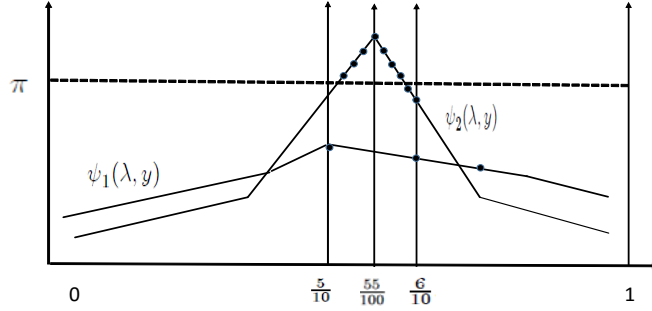


Figure 3: Satisficing function and aspiration

One possible extension is a *vertical extension*: We take another pair of benchmarks  $y^{oo}$  and  $y_{oo}$ . Then, we may assume  $y_{oo} \succsim y_o < y^o \succsim y^{oo}$ . The new set of measurable pure alternatives is given as  $Y(y^{oo}; y_{oo})$ . Then, the set  $Y$  based on  $y^o$  and  $y_o$  may be assumed to be included in  $Y(y^{oo}; y_{oo})$ . This extension may sound straightforward, but if we take the depths of probabilities carefully, the extension or connection between the original system and the new system may not be so simple. For example, if the connections of  $y^o$  and  $y_o$  with  $y^{oo}$  and  $y_{oo}$  are complicated, the probabilistic measurement for  $y$  in the new system may require a probability depth much more than the original  $\lambda_y$ . This may suggest that we may have multiple preference systems even for similar target problems.

Another possible extension is a *horizontal extension*. For example,  $y_o$  is the present status quo for a student. Now, he faces a big choice problem of a new carrier  $y^o$ , a large and good company, or a  $y^{oo}$ , a graduate school for his interest. We assume that either includes great success in his carrier. However, he cannot make comparisons between  $y^o$  and  $y^{oo}$ .

He can restrict himself within the pure alternatives between  $y_o$  and  $y^o$  (or  $y^{oo}$ ). Another possible alternative  $y^{o'}$  ( $y^{oo'}$ ) between  $y_o$  and  $y^o$  ( $y^{oo}$ ) is that his success is mild. Then,  $y^{o'}$  is comparable with  $y^o$ ; and likewise,  $y^{oo'}$  is comparable with  $y^{oo}$ . Here, it is the point that we may choose the benchmarks for some target problems, but it does not work for a different target problem. These include incomparabilities, which are different from those discussed in Section 5.

In case  $\rho = \infty$ , each base facet leads to completeness within it, but across the facets, we may have incomparabilities. This incompleteness may correspond to the literature of the EU theory without the completeness axiom since Aumann [4]. This remains to be an open problem.

## 8 Conclusions and Possible Extensions

We have presented the expected utility theory with bounded probability nets. The set of available probabilities is restricted to the form of  $\ell$ -ary fractions up to the given depth  $\rho$ . The theory has two steps: measurement and extension steps. The measurement step is formulated in terms of a base facet  $F = \langle Y; y^o, y_o; \{\lambda_y\}_{y \in Y} \rangle$ , in addition to a basic axiom B0 and the three axioms B1, B2, B3 on  $F$  and  $\succsim$ . The extension step is based on Axiom B4. When  $\rho = \infty$ , B2 and B4 can

be weaker forms  $B2^\rho$  and  $B4^\infty$ , and these together with Axioms B0,B1,B3 determine the unique and complete preference relation over the lotteries  $L_\infty(Y)$ ; this corresponds to the classical EU theory. Our main concern is the bounded case  $\rho < \infty$ , in particular, small  $\rho$ .

In the bounded case, we need the full axiomatic system, B0-B4, which, however, determine a preference relation to be neither unique nor complete. To study this case, we have considered the canonical relation  $\succsim_c$ , which is uniquely determined by those axioms over the central domain  $\mathbf{C}(F; \rho)$  of lotteries. However,  $\succsim_c$  shows incomparabilities outside the central domain. In Section 5, we studied conditions for incomparabilities.

We applied the results in Section 5 to the AKT anomaly in Section 6. We have considered for which  $\rho$ , the prediction made by our theory is compatible with the experimental result reported in Kahneman-Tversky results[11], and we obtained the result that  $\rho = 2$  is best to explain their experimental results. This is based on the interpretation of incomparabilities meaning that arbitrary choices are possible within some bounds, provided that subjects are asked to choose one between two alternative lotteries.

In Section 7, we have considered the measurement step from the viewpoint of Simon's [19] satisficing/aspiration. We showed that our concept of a base facet is derived by his satisficing/aspiration argument. That is, the probabilistic evaluation  $\lambda_y$  of pure alternative  $y$  comes from the thought-experiment by the decision maker to find an indifferent benchmark lottery; this thought-experiment is formulated as the decision-making process with satisficing/aspiration. Thus, the base facet expresses the result from such thought-process.

This process synthesizes his past experiences, behavioral criteria, and beliefs/knowledge. It is related to also the problem of how "probability" is interpreted; in particular, the basic probability concept can be interpreted from the frequentist viewpoint of probability (cf., Hu [10]). This is also related to the foundation of the inductive game theory (cf., Kaneko-Kline [13]). This is an open problem.

A more specific problem is a representation theorem of a preference relation satisfying Axioms B0-B4 for  $\rho < \infty$ . The canonical relation  $\succsim_c$  is represented by the expected utility function over the central domain  $\mathbf{C}(F; \rho)$ , but outside  $\mathbf{C}(F; \rho)$ , we have incompatibilities. We conjecture that the preferences may be extended to complete relations so that incomparable preferences are expressed by opposite utility values. This result, if it is obtained, could help us interpret the AKT anomaly in terms of utility functions.

Another problem is to connect our theory to the literature of subjective utility/probability from Savage [16], Anscomb-Aumann [3], to that without the completeness axiom (cf., Galaabaatar-Karni [7]). These are still open problems.

## Appendix:

**Proof of Lemma 2.4.** We prove that if  $f \in L_k(X)$ , then  $f = \mathbf{e} * \mathbf{f}$  for some  $\mathbf{f} \in L_{k-1}(Y)^\ell$ . First, we describe a sketch of the proof, and then give a detailed proof.

Consider the support  $S = \{x \in X : f(x) > 0\}$  of  $f$ . It is pointed out in (5) that for each  $x \in S$ ,  $f(x)$  can be expressed as  $\sum_{m=1}^k \frac{v_m(x)}{\ell^m}$ , where  $0 \leq v_m(x) < \ell$  for all  $m$ . Let  $T_m := \{(m, x) : v_m(x) > 0\}$  for  $m = 1, \dots, k$ , where each  $(m, x)$  determines the probability  $\frac{v_m(x)}{\ell^m}$ , and let  $T := \cup_{m=1}^k T_m$ . Then  $\sum_{(m,x) \in T} \frac{v_m(x)}{\ell^m} = \sum_{x \in S} f(x) = 1$ . In order to construct an  $\mathbf{f} =$

$(f_1, \dots, f_\ell) \in L_{k-1}(Y)^\ell$  having  $f = \mathbf{e} * \mathbf{f}$ , we partition  $\sum_{(m,x) \in S} \frac{v_m(x)}{\ell^m}$  into  $\ell$  portions so that each has the same sum  $\frac{1}{\ell}$ . However, this partition may not be possible if we keep the form of  $\frac{v_m(x)}{\ell^m}$  for each element; e.g., if  $\ell = 10$  and  $v_1(x) = 2$ , then  $\frac{v_1(x)}{\ell^1} = \frac{2}{10}$  exceeds  $\frac{1}{\ell} = \frac{1}{10}$ . We avoid this by dividing  $\frac{v_m(x)}{\ell^m}$  into  $v_m(x)$  elements with equal weights  $\frac{1}{\ell^m}$ . In the previous example, we divide  $\frac{2}{10}$  into  $\frac{1}{10}$  and  $\frac{1}{10}$ . Keeping this in mind, we construct  $\mathbf{f} = (f_1, \dots, f_\ell) \in L_{k-1}(Y)^\ell$ .

Since  $S$  is a finite set, we can take a total order  $\triangleleft$  over the support  $S$  of  $f$ . Then, we define the lexicographic order  $\triangleleft$  over  $T$  by  $(m, x) \triangleleft (m', x') \iff m < m'$  or  $(m = m') \& (x \triangleleft x')$ . Then, we define  $\eta_\tau = \sum_{(m,x) \triangleleft \tau} v_m(x)$  for  $\tau = (m, x) \in T$ , and  $I = \{1, \dots, \sum_{\tau \in T} \eta_\tau\}$ . Now, we construct a partition  $\pi$  of  $I$  to have  $\mathbf{f} = (f_1, \dots, f_\ell)$ . Before this, we need another partition  $\{I_\tau : \tau \in T\}$ . For each  $\tau \in T$ , we let  $I_\tau = \{\eta_{\tau^o} + 1, \dots, \eta_\tau\}$ , where  $\tau^o$  is the immediate predecessor of  $\tau$  with respect to  $\triangleleft$  and  $\tau^o = 0$  if  $\tau$  is the first element in  $T$ . Using and adjusting this partition  $\{I_\tau\}_{\tau \in T}$ , we will define the target partition  $\pi$  of  $I$ . We associate each element in  $I_\tau = I_{(m,x)}$  with the probability  $\frac{1}{\ell^m}$ . When  $n \in I_\tau = I_{(m,x)}$ , we denote  $m$  by  $m(n)$ .

The idea of defining the target partition  $\pi$  is as follows: The elements in  $T$  are ordered with  $\triangleleft$ , and the very first element in  $I$  is from  $I_{(1,x)}$  for some  $x \in X$  (supposing  $I_{(1,x)} \neq \emptyset$ ), and is associated with  $\frac{1}{\ell}$ . If  $|I_{(1,x)}| = 1$ , the first set of the partition  $\pi$  is  $I_{(1,x)}$  itself. However, if  $|I_{(1,x)}| > 1$ , then  $I_{(1,x)}$  is divided into  $\{\{1\}, \dots, \{|I_{(1,x)}|\}\}$ , which are now subsets of  $\pi$ . Then, we go to the second  $I_{(1,x')}$ , and repeat the previous construction of the sub-partition corresponding to  $I_{(1,x')}$ . We continue this process following the order  $\triangleleft$  until we have no other  $I_{(1,x')}$ . Then the process goes to the first element in  $I_{(2,x)}$ . This process can proceed, since  $\sum_{(m,x) \in T} \frac{v_m(x)}{\ell^m} = \sum_{x \in S} f(x) = 1$ .

The general definition of the process is as follows: Let  $s = 1, \dots, \ell - 1$ . We can find the unique  $\tau_s = (m_s, x_s) \in T$  so that

$$\sum_{(m,x) \triangleleft \tau_s^o} \frac{v_m(x)}{\ell^m} < \frac{s}{\ell} \leq \sum_{(m,x) \triangleleft \tau_s} \frac{v_m(x)}{\ell^m},$$

where  $\tau_s^o$  is the immediate predecessor of  $\tau_s$  and  $\tau_s^o = 0$  if  $s = 1$ . Since  $1 - \sum_{(m,x) \triangleleft \tau_s^o} \frac{v_m(x)}{\ell^m}$  takes the form of  $\frac{v}{\ell^m}$  with  $m \leq m_s$ , we find a unique  $\sigma_s \in I_\tau = \{\eta_{\tau^o} + 1, \dots, \eta_\tau\}$  so that

$$\sum_{(m',x') \triangleleft \tau^o} \frac{v_{m'}(x')}{\ell^{m'}} + \frac{\sigma_s - \eta_{\tau^o}}{\ell^m} = \frac{s}{\ell}.$$

Additionally, we let  $\pi_s = \{\sigma_{s-1} + 1, \dots, \sigma_s\}$  for  $s = 1, \dots, \ell$ , where  $\sigma_0 = 0$ . Then, it follows that

$$\sum_{n \in \pi_s} \frac{1}{\ell^{m(n)}} = \frac{1}{\ell}. \quad (35)$$

Using this partition  $\pi = \{\pi_s : s = 1, \dots, \ell\}$  of  $I$ , we construct lotteries  $f_1, \dots, f_\ell \in L_{k-1}(X)$ .

Now, let  $s = 1, \dots, \ell$  be fixed. We will define  $f_s$  based on  $\pi_s$ . Since  $\pi_s$  may include  $n$  and  $n'$  having different depths  $m(n) \neq m(n')$ , we should take such depths into account. Formally, we define  $f_s$  by

$$f_s(x) = \sum_{m=1}^k \frac{|I_{(m,x)} \cap \pi_s|}{\ell^{m-1}} \text{ for all } x \in X. \quad (36)$$

Each value  $f_s(x)$  has at most the depth  $k - 1$ . Also, by (35), we have

$$\sum_{x \in X} f_s(x) = \sum_{x \in X} \sum_{m=1}^k \frac{|I_{(m,x)} \cap \pi_s|}{\ell^{m-1}} = \ell \times \sum_{n \in \pi_s} \frac{1}{\ell^{m(n)}} = 1.$$



These mean  $f_s \in L_{k-1}(X)$  for all  $s = 1, \dots, \ell$ .

Finally, we should show that  $f(x) = \sum_{s=1}^{\ell} \frac{1}{\ell} \times f_s(x)$  for any  $x \in X$ . Let  $x \in X$  be fixed. As remarked above, the value  $f(x)$  is divided into several pieces and those pieces appear in several elements in  $\pi$ . In the construction  $\pi$  and the definition (36) of  $f_1, \dots, f_\ell$ , however, we counted all the fractions included in  $f(x)$  keeping their degrees (minus 1). Indeed, we

$$\sum_{s=1}^{\ell} \frac{1}{\ell} \times f_s(x) = \sum_{s=1}^{\ell} \sum_{m=1}^k \frac{|I_{(m,x)} \cap \pi_s|}{\ell^{m-1}} = f(x). \quad (37)$$

Thus,  $f = \mathbf{e} * \mathbf{f}$ . ■

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