Asymmetric Common Value Auctions with Applications to Auctions with Resale

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Abstract

We study a model of common value auctions in which bidders' private information are independently and asymmetrically distributed. We provide two sufficient conditions under which a first-price auction generates higher revenue than a second-price auction (for a selected equilibrium). A necessary condition is given for this revenue-ranking result to hold in general. We illustrate that in a first-price auction, the seller may benefit from the asymmetry of the bidder information, unlike in the case of private value auctions (as in Cantillon 2007).

We further establish a strategic equivalence between our model of common value auctions and an independent and private value (IPV) auction model with resale, when the resale mechanism satisfies a sure-trade property and the common value is the sure-trade price. Using this strategic equivalence and the revenue-ranking result for the common value auctions, we provide an alternative proof of the revenue-ranking result of Hafalir and Krishna (2007) in the IPV auctions with resale. The revenue ranking holds when the offer-maker is fixed or is contingent on the auction outcome. We illustrate that the opposite revenue-ranking may arise (i) when one of the distribution functions does not satisfy the regularity property, or (ii) when the resale mechanism involves repeated offers and delay costs, or (iii) when the Coase Conjecture holds as in Gul, Sonnenschein, and Wilson (1986) and Ausubel and Deneckere (1992).

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1 Introduction

In this paper, we study the effects of asymmetry of the bidders on the revenue in a common value auction model. Many important spectrum auctions held in countries all over the world and participated by communication companies have raised billions of dollars. These auctions are often considered as common value auctions and participants of such auctions tend to have information disparities. How such information disparities affect the degree of competition among the bidders and the seller's revenue in various auction formats are important questions that deserve a careful study.

We consider a common value auction model with two bidders in which bidders' private information are independently and asymmetrically distributed. We provide two sufficient conditions under which a first-price auction generates higher revenue than a second-price auction with a selected equilibrium. The conditions are related to the submodular property of the common value function. The submodular property says that when one bidder's private signal is higher, the other bidder's private signal has less marginal impact on the common value.

Our study of common value auctions has important implications for asymmetric private value auctions if resale is allowed. In fact, resale is an important source of common value among the bidders. This idea is quite intuitive, and has been discussed in Lebrun (2007). We will provide a theoretical examination for this intuition. We show that an independent private value (IPV) auction with resale is strategically equivalent to a common value auction. In a simple environment a seller has no way of knowing the difference between the two from the bidding behavior in the auctions, nor can an econometrician from the bidding data. The strategic equivalence is established in a very general manner. The resale stage is described by a general trade mechanism between a buyer and a seller with two-sided asymmetric information. If the trade mechanism satisfies a sure-trade property, then the first-price auction with resale is strategically equivalent to a first-price common value auction with the common value defined by the sure-trade price. The sure-trade property is first proposed by Hafalir and Krishna (2007)[1]. We use a more general description of the trade mechanism to allow for dynamic bargaining with delay costs. In such a trade mechanism, transaction can occur at many different prices, one of which is called the sure-trade price. The sure-trade property is a very weak condition. It requires that trade must occur with probability one when the trade surplus is the maximum possible amount, and the transaction price is called the sure-trade price. The sure-trade property rules out the no-trade equilibrium in which there cannot be strategic equivalence between the auction with resale and the

[1] Hafalir and Krishna (2007) showed that the condition is sufficient for the symmetry property of the bidding strategy in first-price auctions with resale. We show that the condition is sufficient for the strategic equivalence. In more general models (such as affiliated signals), the condition is also sufficient for strategic equivalence, even though the Hafalir and Krishna (2007) property may fail. The strategic equivalence property seems to hold in more general environments than is treated in this paper. This will be explored in a separate paper.

common value auction. We show that an incentive efficient trade mechanim in the sense of Myerson and Satterthwaite (1983) satisfies the sure-trade property. We would expect a trade mechanism with a reasonable degree of efficiency to possess the sure-trade property. When there are delay costs in repeated offers, the sure-trade price is the first offer price and later offers are not involved in the definition of the common value.

Laffont and Vuong (1996) showed that for any fixed number of bidders, any symmetric affiliated values model is observationally equivalent to some symmetric affiliated private values model. The strategic equivalence in our paper means that when bidders anticipate trading activities after the auction, the bidding data is observationally equivalent to a common value auction in which the common value is defined by the trading prices. Haile (2001) studied the empirical evidence of the effects of resale in the U.S. forest timber auctions.[2]

The strategic equivalence of the auction with resale and the common value auction allows us to apply the ranking results for the common value auctions to the case of auctions with resale. Hafalir and Krishna (2007) have shown that in auctions with resale with a pair of weak-strong bidders, the first-price auction has higher revenue than the second-price auction when valuations are independent, regular and the resale market is either a monopoly or a monopsony. Our approach yields an alternative proof of this result. When it is not a weak-strong pair, the ranking result holds when the offer-maker is fixed or contingent on winning the auction. The offer-making bidder can be chosen by any random process with or without contingency on winning the auctions. We give an example showing that the result may fail without regularity. We also obtain necessary conditions for the ranking result to hold in a general manner.

One important insight from our approach is that the revenue ranking property of auctions with resale depends on the bargaining power of the offer-maker. We give a simple example of a two-offer monopoly resale mechanism. The valuations of the bidders are all uniformly distributed. The opposite ranking is obtained when the monopolist has a high delay cost, while the buyer has no delay cost. The opposite ranking is due to the weakened bargaining power of the monopoly seller. Furthermore, when the Coase conjecture (1972) holds, the opposite ranking also holds. The validity of the Coase conjecture has been shown in Gul, Sonnenschein and Wilson (1986)[3], when the uninformed party makes the offers, the bargaining interval converges to zero, and the equilibrium is stationary. If we allow alternating offers, it has also been shown in Ausubel and Deneckere (1992) as a consequence of the Silence Theorem.

We restrict our study to the case with two bidders, as there are well-known difficulties in analyzing the equilibrium bid of first-price auctions with asymmetric distributions when there are more bidders. At this stage, many issues need

[2] His model of resale is different from our specifications here. In his model, there is no asymmetry among bidders before auctions, and trade occurs after the auction because of information differences after the auction. In our model, bidders are asymmetric before auctions.

[3] For the literature on the Coasian conjecture and theorems, see Coase (1972), Bulow (1982), Stokey (1981), Cramton (1984), Fudenberg, Levine, and Tirole (1985), Ausubel and Deneckere (1987, 1989a, 1989b, 1992), and Gul, Sonnenschein and Wilson (1992).

to be understood first in the bilateral context. In establishing the ranking result for the common value auctions, we have to deal with an issue of multiple equilibria. It is well-known that there is a continuum of equilibria in second-price common value auctions (with continuous distributions). For the comparison to make sense, we need to deal with the equilibrium selection issue. The equilibrium we select is motivated by later applications to auctions with resale. It is the one that is reduced to the dominant strategy equilibrium in private value auctions or the only robust equilibrium in auctions with resale in Hafalir and Krishna (2007) when the common value auction arises from auctions with resale. We also justify the equilibrium selection by a refinement concept allowing for a small private value component in valuations. There is a unique second-price auction equilibrium when the private value component is present. As the private value component goes to 0 in the limit, we get the selected equilibrium under certain symmetric error conditions.[4]

The nature of competition among the bidders in common value auctions can be quite different from private value auctions. Cantillon (2006) studied the effect of asymmetry on private value first-price auctions. In private value auctions, asymmetry among the bidders tends to reduce competition. As a result, revenues are reduced by asymmetry. We will show that the opposite is often true for common value auctions. There is in fact more competition when there are information disparities. When the signals received by the bidders are independent, and the common value is of a separable form, the symmetric benchmark for the comparison is defined by convolutions of distributions. A general analysis seems to be challenging, and our study on this issue is limited to a simple example that points the way for further analysis in the future.

The rest of the paper is organized as follows. In Section 2, we describe the common value model and state two conditions regarding the common value function and the distribution functions. In Section 3, we present equilibrium bids and revenues for the first price and second price auctions, and discuss the equilibrium selection issue in the second-price auction. The effect of asymmetry on auction revenues are discussed in Section 3.3. We provide some intuitive explanations for and formal statements of our main results on revenue ranking in Section 4. Examples are provided to illustrate the necessity of the conditions for the revenue ranking. In Section 5, after a description of the IPV auctions with resale, we establish the strategic equivalence of the common value auctions and the IPV auctions with resale. We apply our ranking results to the auctions with resale in Section 5.3. In Section 5.4, we give an example to show the opposite ranking when the monopolist has weakened bargaining power, and in section 5.5, we show the implications of the Coase conjecture in our ranking problem. Section 6 contains all the proofs.

[4] A different selection of equilibrium has been adopted by Parreiras (2006) in an environment with affiliated signals. Mares (2006) provides another equilibrium selection that maximizes the revenue for the seller among all equilibria.

2 The Common Value Model

We consider the following pure common value auction model. There are two risk neutral bidders in an auction for a single object. There is a common valuation for the object, and each bidder only receives partial information about the common value. Let $s_i, i = 1, 2$ be the private signal received by bidder i. We assume that s_1, s_2 are independently distributed with cumulative distribution function $F_i(s_i)$ and support $[0, a_i]$ for signal s_i. We assume that $F_i(s_i)$ is strictly increasing and continuously differentiable[5] with the density function $f_i > 0$ everywhere. The common value is given by $V = w(s_1, s_2)$. Assume that w is strictly increasing in each s_i and continuously differentiable[6].

We now relabel the signals by $t_i = F_i(s_i)$. Let $v_i(t_i) = F_i^{-1}(t_i)$. The common value function can be written as $V = w(v_1(t_1), v_2(t_2))$. Signal t_i is uniformly distributed over $[0, 1]$. Note that v_i is also strictly increasing and continuously differentiable. We have $v_1(0) = v_2(0) = 0$, and $v_1(1) = a_1, v_2(1) = a_2$, and we let $a = \max(a_1, a_2)$. The range of the function V is $[0, w(a_1, a_2)]$. In some of our discussions in this paper, we will consider a weak-strong pair of bidders in the sense that bidder 2 is a stronger bidder than bidder 1 if $v_1(t) \leq v_2(t)$ for all t.[7]

The common value function w is symmetric if $w(x, y) = w(y, x)$ for all x and y. The symmetry means that the common valuation does not depend on who receives which signal as long as the collection of individual beliefs are the same. There is a universal way of updating the information. No personal element is involved in the updating and re-valuation. This is in the spirit of the common value model, as the valuation should depend on the collection of the signals alone, and differences in valuation are only due to the differences in the information received. In this situation, we may think of the symmetry as part of the property of the common value model. However, in later applications to the auctions with resale, the common value defined need not be symmetric. Therefore we will not assume symmetry in the following presentation. In many places, symmetry does make the discussion easier to understand.

Function F_i is called regular if the following virtual value function is strictly increasing in s :

$$s - \frac{1 - F_i(s)}{f_i(s)},$$

which implies that for any $y \in (0, a_i)$, the following conditional virtual value is

[5] Allowing the distributions F_i to have kinks would not invalidate the revenue formulas and the ranking results of the paper. We also allow F_i to have infinite derivatives at 0 (such as power functions) in some of our examples.

[6] In some of discussions below, we allow $w(x, y)$ to be non-differentiable to include two interesting cases: $w(x, y) = \max(x, y)$ and $w(x, y) = \min(x, y)$. There is enough differentiability to make the results of this paper go through for the two cases.

[7] Here we only require that F_2 is dominated by F_1 in the sense of the first order stochastic dominance. Note that this concept is weaker than that of Maskin and Riley (2000a), in which conditional stochastic dominance is imposed.

strictly increasing in s :

$$s - \frac{F_i(y) - F_i(s)}{f_i(s)}.$$

The regularity condition can also be stated in terms of $v_i(t)$. The virtual value is given by

$$J(t) = v_i(t) - (1-t)v_i'(t).$$

Hence the regularity condition is simply the increasing property of $J(t)$. It is equivalent to the concavity of $(1-t)v_i(t)$ since

$$-\frac{d^2}{dt^2}[(1-t)v_i(t)] = \frac{d}{dt}[v(t) - (1-t)v'(t)] = J'(t) > 0.$$

For any $\tau \in (0,1)$, the conditional virtual value is given by

$$v_i(t) - (\tau - t)v_i'(t).$$

In our discussion below, we often use x to represent bidder i's private signal and y represent bidder j's signal. The common value function w is symmetric if $w(x,y) = w(y,x)$ for all x and y. The symmetry means that the common valuation does not depend on who receives which signal as long as the collection of individual beliefs are the same. There is a universal way of updating the information. No personal element is involved in the updating and re-valuation. This is in the spirit of the common value model, as the valuation should depend on the collection of the signals alone, and differences in valuation are only due to the differences in the information received. In this situation, we may think of the symmetry as part of the property of the common value model. However, in later applications to the auctions with resale, the common value defined need not be symmetric. Therefore we will not assume symmetry in the following presentation. In many places, symmetry does make the discussion easier to understand.

The common value function $w(x,y)$ is submodular if, for all (x,y) and (x',y'), $x \le x', y \le y'$, the following holds

$$w(x,y) + w(x',y') \le w(x,y') + w(x',y). \tag{1}$$

Given an increasing and concave function ϕ, $w(x,y) = \phi(x+y)$ is both symmetric and submodular.

One condition of w will be useful for our revenue ranking and can be stated as follows:

Condition (C): for all x, y, we have

$$w(x,y) \ge \frac{w(x,x) + w(y,y)}{2}. \tag{2}$$

Note that in (C), we do not necessarily impose symmetry. When w is symmetric, the submodular property implies (C). However, when w is not symmetric, condition (C) does not follow from submodularity. For example, $w(x,y) = \frac{2}{3}x + \frac{1}{3}y$ is submodular but does not satisfy condition (C).

We also provide another condition on w along with one of the distribution function. Given a bidder j's signal s_j, define the following function of the bidder i's signal:

$$H^{s_j}(s_i) = \frac{2w_i(s_i,s_j)}{w_1(s_i,s_i) + w_2(s_i,s_i)} - \frac{1 - F_j(s_i)}{1 - F_j(s_j)},$$

where $w_i(s_i,s_j)$ is the partial derivative of w with respect to s_i. We refer to the following single crossing condition as condition (R). It is a condition on both w and the distribution function F_j of bidder j. For our ranking result, it will be sufficient if one of the bidder distribution satisfies the condition.

Condition (R):For all $i \neq j$, *we have* $H^{s_j}(s_i) > 0$ *if* $s_i > s_j$ *and* $H^{s_j}(s_i) < 0$ *if* $s_i < s_j$.

When w is symmetric and submodular, it can be shown (see the beginning of Section 4.3) that condition (R) holds for any F_j. Therefore a symmetric submodular common value function satisfies both conditions (C) and (R). It should be emphasized however that when w is not symmetric, the two conditions tend to be different from the submodularity property.

We will use the above notations for a common value model to express an asymmetric private value model. This is useful for later applications to the model of asymmetric private value auctions with resale. This representation is first proposed in Milgrom and Weber (1985) and discussed extensively in Milgrom (2004). In Section 4.2 of Milgrom (2004), he discusses two advantages: (i) it easily generates predictions about bid distributions for use in empirical work; (ii) it unifies analysis of models with discrete or continuous valuation distributions. We will add another point: with this representation, it is easier to make a simple connection between private auctions with resale and common value auctions.

In this representation, a bidder is now described by a strictly increasing valuation function $v_i(t_i) : [0,1] \to R$, with the interpretation that $v_i(t_i)$ is the private valuation of bidder i. The word "private" refers to the important property that bidder i's valuation is not affected by the signal t_j of other bidders, while in the common value model, this is not the case. The function F_i is now the distribution function of the private valuation of bidder i. It will be shown in Section 5.3 that if bidder j's valuation distribution is convex then the optimal (single) offer from bidder i to bidder j in the resale stage satisfies condition (C). Similarly, when F_j is regular then condition (R) is satisfied for F_j and the optimal (single) offer from bidder i to bidder j.

3 Equilibria and Revenues in Common Value

Auctions

We shall derive the equilibrium revenue formulas for the first-price and second-price auctions in Sections 3.1 and 3.2. Equilibrium selection issue is discussed in Section 3.2 and the effect of asymmetry on equilibrium revenues is presented in Section 3.3.

3.1 First-Price Auctions

The existence and uniqueness of the equilibrium in the first-price common value auctions have been studied in the literature[8]. In this subsection, we derive the equilibrium bid and revenue using the distributional approach.

Let $b_i(t_i)$ be the strictly increasing bidding strategy of bidder i in the first-price auction, and $\phi_i(b)$ be its inverse. The following first order condition is satisfied by the equilibrium bidding strategy

$$\frac{d\ln\phi_i(b)}{db} = \frac{1}{w(v_1(\phi_1(b), v_2(\phi_2(b))) - b} \text{ for } i = 1, 2. \tag{3}$$

with the boundary conditions $\phi_i(0) = 0$. The ordinary differential equation system with the boundary conditions determine the equilibrium inverse functions.

In the pure common value model, it is well-known that in equilibrium, the winning probabilities of the two bidders are the same when they bid the same amount.[9] The symmetric property of the winning probabilities is exactly the property that both bidders have identical bidding strategies (as functions of t). In other words, we have $b_1(t) = b_2(t)$. Note that there is asymmetry in the signals as v_1, v_2 are different, and bidding strategies as functions of v_i are not symmetric. However, bidding strategies in terms of t are symmetric.

When signals are independent, the symmetry property of the equilibrium bidding strategy gives us very simple formulas for the bidding strategy and the revenue. The following result for the equilibrium strategy in the first-price common value auction has been established in the literature (for instance Parreiras (2006)). For the case of independent signals, we give a simple statement and proof based on the symmetry.[10]

[8] The existence of a non-decreasing equilibrium in the common value model is established in Athey (2001). The existence of a strictly increasing equilibrium has been shown in Rodriguez (2000). The uniqueness of equilibrium of the first price auction of the common value model can be found in Lizzeri and Persico (1998) and Rodriguez (2000).

[9] This can be found in Engelbrecht-Wiggans, Milgrom, and Weber (1983) for the Wilson track model and more generally in Parreiras (2006) and Quint (2006). This property also holds in first-price auctions with resale in Hafalir and Krishna (2007).

[10] We want to thank Jeremy Bulow for pointing out that the bidding formula can also be obtained from the theorem in Milgrom and Weber (1982) by using symmetric signals but asymmetric common value functions.

Proposition 1 *The equilibrium bidding strategy in the first-price common value auction is symmetric and is given by*

$$b(t) = \frac{1}{t}\int_0^t w(v_1(z), v_2(z))dz$$

with the revenue given by

$$R^F = 2\int_0^1 (1-z)w(v_1(z), v_2(z))dz.$$

For the separable case, we have the following revenue formula for the first-price auctions. A discrete version of this result was given by Hörner and Jamison (2007, supplement).

Corollary 2 *If the common value of the object is separable*

$$V = v_1(t_1) + v_2(t_2)$$

in the two signals, then the revenue of the first-price auction is

$$R^{FPA} = \int_0^1 (1-t)^2 dv_1(t) + \int_0^1 (1-t)^2 dv_2(t).$$

3.2 Second-Price Auctions

It is well-known that in the second-price pure common value auction, there is a continuum of equilibria (see Milgrom (1981)). In fact, for any increasing function h, the following is an equilibrium in the second-price auction (see Milgrom (2004), Theorem 5.4.8).

$$B_1(s_1) = w(s_1, h^{-1}(s_1)), B_2(s_2) = w(h(s_2), s_2).$$

The equilibrium as a function of t can be expressed as

$$b_1(t_1) = w(v_1(t_1), h^{-1}(v_1(t_1))), b_2(t_2) = w(h(v_2(t_2)), v_2(t_2)).$$

When we rank the revenues of the first-price and second-price auctions, we need to specify which equilibrium in the second-price auction is selected for the comparison.

We select the equilibrium with $h(s) = s$, that is,

$$B_i(s_i) = w(s_i, s_i), i = 1, 2$$

or

$$b_i(t_i) = w(v_i(t_i), v_i(t_i)), i = 1, 2. \tag{4}$$

Note that the selected equilibrium as functions of signals s_i is symmetric across the two bidders. The revenue from the second price auction for the selected equilibrium can be derived as follows.

Proposition 3 *The revenue of the selected second-price auction equilibrium (4) is*

$$R^S = \int_0^a (1 - F_1(x))(1 - F_2(x))dw(x, x),$$

where $a = \max(a_1, a_2)$.

Note that there is an important property associated with the selected equilibrium and revenue in the second price auction. That is, the selected equilibrium and revenue depend on $w(x, y)$ only through the diagonal $y = x$ and are not affected by the value of w off diagonal $y \neq x$. In particular, suppose $w(x, x) = x$, then the selected equilibrium bid is just $b_i(t_i) = v_i(t_i)$ and the revenue is given by

$$R^S = \int_0^a (1 - F_1(x))(1 - F_2(x))dx.$$

This is identical to the equilibrium revenue of the second price auction in an independent private value model. This property turns out to be very useful for applying our common value model to IPV auctions with resale. We will discuss this in Section 5.

In addition to the purpose of applications of our results to auctions with resale, there is another justification for the selected equilibrium above. In practice, it is rare to have a pure common value model. Instead, there might be a small private component in the valuation of the bidders. Assume that both bidders have the same small portion of the value derived from private value considerations, while the major portion of the valuation is common. We show that in the limit the unique second-price auction equilibrium converges to the selected equilibrium above.

To formalize this idea, assume that a small part of v_1 is a private component, meaning that when bidder 1 knows t_2, the updated valuation is given by

$$\varepsilon v_1(t_1) + (1 - \varepsilon)w(v_1(t_1), v_2(t_2)).$$

Similarly, when bidder 2 updates the valuation, it is given by

$$\varepsilon v_2(t_2) + (1 - \varepsilon)w(v_1(t_1), v_2(t_2)).$$

We call this an almost common value model. We have the following result on equilibrium refinement.

Proposition 4 *In a model of the almost common value with a small (ε) private value component, the equilibrium in the second-price auction is unique. As $\varepsilon \to 0$, the equilibrium converges to the selected equilibrium defined in (4).*[11]

We now compare our equilibrium selection with that of Parreiras (2006).[12] His selection is $h(s) = v_1(v_2^{-1}(s))$, or

$$b(t) = w(v_1(t), (v_2(t)).$$

This equilibrium as a function of t is symmetric across two bidders, while our equilibrium as a function of s is symmetric across two bidders. The two selections are identical when bidders are symmetric.

It can be shown that when the signals are independent, Parreiras (2006)'s selection has the same revenue as the first-price auction equilibrium.

Proposition 5 *The equilibrium selected by Parreiras (2006) in the second price auction is*

$$b(t) = w(v_1(t), v_2(t)),$$

yielding the revenue in the second price auction equal to that of the first-price auction.

In an affiliated common value model, Parreiras (2006) has shown that his selected second-price auction equilibrium revenue-dominates the first-price auction equilibrium. The Parreiras (2006) result implies that the ranking result of Milgrom and Weber (1982) is extended to the case when bidders are asymmetric amd that the effect of affiliation still favors the second price auction over the first price auction. In this paper, we focus on the effect of asymmetry on the ranking of the two auctions in absence of affiliation.

To see difference in the revenue ranking, consider the following example. Let $v_1(t_1) = t_1, v_2(t_2) = \sqrt{t_2}, w(x, y) = \frac{x+y}{2}$. We have the equilibrium bidding strategy for each bidder in the first-price auction

$$b(t) = \frac{1}{t} \int_0^t \frac{s + \sqrt{s}}{2} ds = \frac{t}{4} + \frac{\sqrt{t}}{3}.$$

[11] In this result, we use the same size ε for both bidders. If we allow $\varepsilon_1, \varepsilon_2$ to be different, the result remains true if the ration goes to 1. If the ratio does not go to one, we may get other equilibria in the limit. In this sense, the refinement concept has some limitations.

[12] By comparison, Parreiras (2006) selected an equilibrium based on a refinement concept through hybrid auctions. The second price auction equilibrium he selected is based on the limit of the hybrid auction when the weight on the first price is close to 0 (corresponding to the second price auction in the limit). It is a refinement idea through the perturbation in auction formats. Our refinement idea is through the perturbation in auction environments (the small private value components).

The revenue of the first-price auction is given by

$$R^{FPA} = 2\int_0^1 (1-t)\frac{t+\sqrt{t}}{2}dt = 0.433\,33.$$

The second-price auction equilibrium selected by Parreiras (2006) is the symmetric bidding strategy

$$b^*(t) = w(v_1(t), v_2(t)) = \frac{t+\sqrt{t}}{2},$$

with the revenue

$$2\int_0^1 \left(\int_0^s b(t)dt\right) ds = 2\int_0^1 \left(\frac{s^2}{4} + \frac{s^{1.5}}{3}\right) ds = 0.433\,33.$$

Hence the two auctions have the same revenue. The equilibrium we selected is this paper, expressed in signals, is

$$b_1(t_1) = t_1, b_2(t_2) = \sqrt{t_2},$$

yielding a revenue

$$\int_0^1 (1-v)(1-v^2)dv = 0.416\,67,$$

which is lower than the revenue in the first-price auction.

3.3 Revenue Effect of Asymmetry

In private value first-price auctions, Cantillon (2006) conjectures that the seller benefits from symmetry of the bidders. The intuition is that the competition is more intense when the bidders have similar valuations. This conjecture is shown to be true when the distributions of the valuation are power functions. Furthermore, for this class of asymmetric auctions, the seller's revenue is lower when there is more asymmetry and the total surplus is kept constant.

We shall ask a similar question in the common value auctions. We first consider a special case of the common value model with $w(x,y) = \max\{x,y\}$. The symmetric benchmark of this model has a valuation distribution given by geometric mean of the valuation distributions of the two signals in the asymmetric model. Let F_1, F_2 be the valuation distributions of the signals. The distribution of the common valuation is given by $F_1(x)F_2(x)$. There is a unique symmetric common value model to be compared with this asymmetric model. It is described by the valuation function $v_0(s)$ for both bidders such that the model has the same distribution of the common valuation. In this symmetric benchmark, let $F(x) = v_0^{-1}(x)$ be the valuation distribution of both bidders. To have the same distribution of the common value, we must have $F^2(x) = F_1(x)F_2(x)$, or $F(x) = \sqrt{F_1(x)F_2(x)}$.

For the first-price auction in the common value model, the opposite of Cantillon's result is true. The seller in this case benefits from asymmetry. When the valuation distributions are power functions, we also show that the seller's revenue is an increasing function of the degree of asymmetry of the model. This result suggests that in the common value model, the competition seems to be more intense when the bidders have access to different information signals.

Theorem 6 *If $w(x, y) = \max\{x, y\}$, then the revenue from the symmetric benchmark first-price auction is lower than that of the asymmetric first-price auction.*

The following shows that the seller's revenue is an increasing function of the degree of asymmetry in the first-price auction of the common value model $w(x, y) = \max\{x, y\}$ when the valuation distributions are power functions.

Proposition 7 *Let $w(x, y) = \max\{x, y\}$, and $F_i(x)$ are power function distributions of the form x^k over $[0, 1]$. The seller has higher FPA revenue when there is more asymmetry and the symmetric benchmark is fixed.*

We now consider the case when the common value is defined by $w(x, y) = x + y$. This is the wallet game studied in Klemperer (1998). In this case, the symmetric benchmark model has a valuation distribution $F(x)$ such that the convolution of F with itself is the same as the convolution of F_1 and F_2. Because of the complexity of convolutions, we only look at an example derived from exponential distributions. In this example, the seller also benefits from asymmetry in the first-price auction. Whether this result extends to other common value models is however an open question.

Let F_1 be the valuation distribution

$$F_1(x) = 1 - e^{-x}, x \geq 0,$$

and F_2 be given by

$$F_2(x) = 1 - e^{-x}(1 + x + 0.5x^2), x \geq 0.$$

The symmetric benchmark model has the following valuation distribution (see Billingsley (1986), page 273, example 20.5)

$$F(x) = 1 - e^{-x}(1 + x).$$

The seller's revenue in the symmetric model is

$$2\int_0^\infty e^{-2x}(1 + x)^2 dx = 2.5.$$

The seller's FPA revenue of the asymmetric model is

$$R^F = 2\int_0^1 (1-t)(v_1(t)+v_2(t))dt = 2\int_0^\infty (1-F_2(x))(h(x)+x)dF_2(x),$$

where

$$h(x) = F_1^{-1}(F_2(x)) = -\ln(1-F_2(x)).$$

We have

$$x + h(x) = 2x - \ln(1 + x + 0.5x^2),$$

hence

$$\begin{aligned} R^F &= 2\int_0^\infty e^{-x}(1+x+0.5x^2)(2x-\ln(1+x+0.5x^2))0.5x^2e^{-x}dx \\ &= \int_0^\infty x^2e^{-2x}(1+x+0.5x^2)(2x-\ln(1+x+0.5x^2))dx = 2.562\,5. \end{aligned}$$

In this example, the seller's FPA revenue is higher in the asymmetric model.

For the second-price auction, Cantillon's conjecture seems to be true. In the example above for the wallet game, the seller's SPA revenue is

$$R^S = 2\int_0^\infty (1-F_1(x))(1-F_2(x))dx$$
$$= 2\int_0^\infty e^{-2x}(1+x+0.5x^2)dx = 1.75,$$

which is lower than that of the symmetric benchmark. Thus the seller benefits from asymmetry in first-price auction, but loses from asymmetry in the second-price auction.

For the common value model given by $w(x,y) = \max\{x,y\}$, the Cantillon conjecture is indeed true for all possible valuation distributions of the signals. This is stated in the following theorem.

Theorem 8 *If the common value is given by $w(x,y) = \max\{x,y\}$, then the SPA revenue of the symmetric benchmark auction is higher than that of the asymmetric auction.*

4 Revenue Ranking

We give a simple proof of the ranking result when w is symmetric, and separable (and submodular), and explain the intuitive reasons why the general ranking results are possible for the common value auctions in section 4.1. We also give an intuitive explanation of the conditions needed for our results. In section 4.2, we present our main ranking results.

4.1 Intuitive Explanations

From now, on we shall study the revenue ranking problem with the equilibrium selection described in the last section. We are interested in ranking the revenues from two commonly used auctions: first-price and second-price auctions. Let R^F, R^S denote the revenue of the first-price and second-price auction respectively.

It is useful to give a simple proof of the ranking result when the common value function is separable of the form $w(x,y) = \frac{x+y}{2}$. As we have explained before, this function is submodular. For simplicity, assume that the support of F_i is $[0,1]$. By Corollary 2, we have

$$R^F = \frac{1}{2}\int_0^1 (1-t)^2 dv_1 + \frac{1}{2}\int_0^1 (1-t)^2 dv_2$$

$$= \frac{1}{2}\int_0^1 (1-F_1(x))^2 dx + \frac{1}{2}\int_0^1 (1-F_2(x))^2 dx$$

$$> \int_0^1 (1-F_1(x))(1-F_2(x))dx = R^S,$$

where the strict inequality holds as long as $F_1(x) \neq F_2(x)$ for a subset of $[0,1]$ with non-zero measure. Therefore, in this case the first-price auction generates higher revenue than the second-price auction. Note that the ranking result is a simple consequence of the revenue formulas and the inequality $A^2 + B^2 \geq 2AB$.

A generalization of this result is our first ranking theorem in the next section. The proof is not too different from the arguments shown in the last paragraph. The result relies on condition (C). It is somewhat easier to motivate the conditions when w is symmetric or satisfies $w(x,x) = x$ (this is always the case in the resale context) or both. In this case, condition (C) says that the common value is above the average of the two valuations x, y. Assume that $x < y$. We can think of the two common values $\max\{x,y\} = y, \min\{x,y\} = x$ as two extreme cases. When the common value $w(x,y) = (1-a)x + ay$ is close to $\min\{x,y\}$, the ranking is the opposite of that of $\max\{x,y\}$. To see this, let $v_1(t) = t, v_2(t) = Kt, K \geq 1$. When $K = 1$, we have equivalence between the two auction revenues. As K increases, the revenue $R^F_{\min}$ corresponding to the common value $\min\{x,y\}$ stays the same, while both $R^F_{\max}$ for $\max\{x,y\}$, and R^S increase. We have

$$R^F_{\max} = \int_0^1 2(1-t)Kt dt = \frac{K}{3},$$

and

$$R^S = \int_0^1 (1-x)(1-\frac{x}{K})dx = \frac{1}{2} - \frac{1}{6K}.$$

Hence $R^F_{\max} > R^S$ for all $K > 1$, and

$$R^F_{\max} > R^S > R^F_{\min}.$$

To have a ranking result, $w(x, y)$ must be closer to $\max\{x, y\}$ than $\min\{x, y\}$. It turns out that $a \geq 0.5$ is sufficiently close in general.

Condition (C) is particularly attractive because it requires no assumptions on the underlying distributions $F_i, i = 1, 2$. Therefore the ranking result applies to all specifications on the individual signals. However, when applied to the auctions with resale, the optimal pricing function need not satisfy this condition. To explain the meaning of condition (R), it is useful to rewrite condition (R) when w is symmetric. With symmetric w, we have

$$w_1(x, x) = w_2(x, x) = \frac{1}{2}.$$

Condition (R) for bidder j says that

$$w_i(x_i, x_j) > \frac{1}{2}\frac{1 - F_j(x_i)}{1 - F_j(x_j)} \text{ when } x_i > x_j.$$

This means that when the valuation x_i increases by one unit, the marginal increase of the common value has to be at least $\frac{1}{2}\frac{1-F_j(x_i)}{1-F_j(x_j)}$. It is a joint condition on w and F_j. Both conditions (C) and (R) express the tendency of the common value to be relatively high. From (3), a higher common value yields a lower solution for the differential equation, which means that the bidders are more aggressive, hence the revenue is relatively high.

Note that the common value function $w(x, y) = \max\{x, y\}$ satisfies both conditions as it is a symmetric submodular function. The common value function $w(x, y) = \min\{x, y\}$ is supermodular and does not satisfy either of the two conditions. One interesting case that should be mentioned is the Wilson (1968) drainage track model. In this model, one bidder observes the true value of the object, while the other bidder is uninformed or observes signals that are not informative, in the sense that the true value of the object only depends on the observed value of the informed bidder. In the Wilson drainage track model conditions (C), (R) all fail. Therefore our revenue ranking result cannot be applied to the Wilson drainage track model[13].

The following lemma clarifies the relationship between the submodular property and condition (R).

Lemma 9 *If w is symmetric, then condition (R) is satisfied for all F_j when w is submodular.*

It is useful to give some intuition as to why the symmetry property of the equilibrium bidding strategy in Proposition 1 has strong implications for revenue

[13] For the second price auction in the Wilson drainage track model, the uninformed bidder always gets 0 payoff no matter how he bids or how the informed bidder bids. However, the seller's revenue is affected by his bidding behavior. Hence there is no natural choice of an equilibirum selection to compare.

comparisons. In private value auctions, it is well-known (see Maskin and Riley (2000a)) that the weak bidder contributes more revenue to the seller in the first-price auction than in the second-price auction. For the strong bidder, it is just the reverse. This reversion is the source of the ambiguity in ranking the first-price and second-price private value auctions. When the strong bidder uses "lowball" strategies, the revenue of the second-price auction can be higher than that of the first-price auction. For common value auctions, the symmetry in the bidding strategy means that the weak and strong bidders contribute the same revenue to the seller. If we can show that the strong bidder bids sufficiently higher than before, then we have a general ranking result. In other words, our conditions combined with the symmetry property will prevent such lowball strategies from occurring.

Take the common value function $w(x,y) = \frac{x+y}{2}$. As we have shown earlier, for this common value function, the first-price auction has higher revenue than the second-price auction. To give more concrete numbers, take the following example of asymmetric private value auction in Cheng (2006), section 2.3. There is one weak buyer and one strong buyer with $F_w(v) = \frac{4}{3}v$ over $[0, \frac{3}{4}]$, and $F_s(v) = v^2$ over $[0,1]$. We have the following equilibrium bidding strategies in the private value auction

$$b_w(v) = \frac{2}{3}v, b_s(v) = \frac{1}{2}v.$$

The revenues from the weak bidder in the first, second-price auction are 0.125, 0.070313 respectively. The revenues from the strong bidder in the two auctions are 0.269 53, 0.25 respectively. The corresponding common value model is $v_1(t) = \frac{3}{4}t, v_2(t) = \sqrt{t}$. The equilibrium bidding strategy of either bidder is

$$\begin{aligned} b(t) &= \frac{1}{t}\int_0^t w(\frac{3}{4}s, \sqrt{s})ds = \frac{1}{t}\int_0^t \frac{\frac{3}{4}s + \sqrt{s}}{2}ds \\ &= \frac{1}{2t}(\frac{3}{8}t^2 + \frac{2}{3}t^{1.5}) = \frac{3}{16}t + \frac{1}{3}\sqrt{t}. \end{aligned}$$

We now show that in the common value model, the strong bidder in fact bids more aggressively, i.e.

$$b(t) \geq \frac{3}{16}t + \frac{1}{3}\sqrt{t} \geq \frac{1}{2}t = b_s(t). \tag{5}$$

The first-price auction revenue in this common value model is

$$2\int_0^1 (1-t)w(\frac{3}{4}t, \sqrt{t})dt = 2\int_0^1 (1-t)\frac{\frac{3}{4}t + \sqrt{t}}{2}dt = 0.391\,67,$$

which is higher than the second-price auction revenue

$$\int_0^1 (1 - \frac{4}{3}x)(1-x^2)dx = 0.33.$$

4.2 Main Ranking Results

The first result we offer is based on condition (C) of the common value function w. When this condition holds, the ranking holds without detailed knowledge of the valuation distributions $F_i, i = 1, 2$.

Theorem 10 *Suppose w satisfies condition (C), and $v_1(t)) \neq v_2(t)$ for a subset of $[0,1]$ of non-zero measure. Then $R^F > R^S$.*

The common value function $w(x, y) = \max\{x, y\}$ satisfies condition (C). To see this, we have

$$w(x, x) + w(y, y) = x + y \leq \max\{x, y\} + \max(y, x\} = 2w(x, y),$$

therefore condition (C) holds, and the ranking result always applies. When $w(x, y) = \min\{x, y\}$, the ranking is always reversed. Before we state this result, we want to note that the revenue equivalence holds when bidders are symmetric ($v_1(t) = v_2(t) = v(t)$ for all t). This is known in the literature, and can be proved easily by our revenue formulas. We have

$$R^F = \int_0^1 2(1-t)w(v(t), v(t))dt = \int_0^1 (1-t)^2 dw(v(t), v(t))$$
$$= \int_0^a (1 - F(x))^2 dw(x, x) = R^S.$$

We state this as a proposition.

Proposition 11 *Assume that $v_1(t)) = v_2(t)$ for all t, then we have $R^F = R^S$.*

In view of the importance of the maximum and minimum value function, we have the following simple result.

Proposition 12 *Assume that $v_1(t)) \neq v_2(t)$ for a subset of $[0,1]$ of non-zero measure. (i) If $w(x, y) = \max\{x, y\}$, then $R^F > R^S$; (ii) If $w(x, y) = \min\{x, y\}$, then $R^F < R^S$.*

Our second result is based on condition (R) which uses properties of one of the valuation distributions. It has a very different proof.

Theorem 13 *Assume that condition (R) holds for w and some bidder F_j, and $v_1(t) \neq v_2(t)$ with strict inequality for a subset of $[0,1]$ of non-zero measure. Then $R^F > R^S$.*

For the above results, the conditions (C), (R) cannot be dispensed with completely. Without them, the result may fail. We given two examples here. The first one involves a typical supermodular common value function, and the ranking fails. The second example illustrates a necessary condition for a general ranking result.

A typical example of a supermodular common value function is of the form $w(x, y) = (x+y)^n, n > 1$. For instance, let $w(x, y) = (x+y)^4$. Let the two bidders be $v_1(t_1) = \sqrt{t_1}, v_2(t_2) = t_2$, for t_1, t_2 in $[0, 1]$. We have $F_1(x) = x^2, F_2(x) = x$. The revenue of the first-price auction is

$$R^F = 2\int_0^1 (1-t)(t+t^{0.5})^4 dt = 1.6645,$$

and

$$R^S = 64\int_0^1 (1-x)(1-x^2)x^3 dx = 1.6762.$$

We have a reversal of the inequality. Since w is supermodular, condition (C) fails.

Note that in condition (R), there is no restriction on the other bidder's distribution F_i. We now use an example to illustrate the idea that condition (R) is essentially necessary along the diagonal if we want the ranking result to hold for all F_i. In this example, the two distributions F_1, F_2 differ only in some small interval $[0, \delta]$. When x_i is in this interval, condition (R) is violated. The ranking is reversed.

Let the two bidders be given by

$$\begin{aligned} v_1(t) &= 0.9t + t^2 \text{ for } t \le 0.1 \\ &= t \text{ for } t \ge 0.1, \end{aligned}$$

and $v_2(t) = t$ for all t. The two bidders have the same valuation distribution above $t \ge 0.1$, but for $t \le 0.1$, bidder two is slightly stronger. The common value is given by $w(x, y) = (\frac{\sqrt{x}+\sqrt{y}}{2})^2$. To find F_1, solve $x = 0.9t + t^2$, and we have

$$\begin{aligned} F_1(x) &= \frac{-0.9 + \sqrt{0.9^2 + 4x}}{2} \text{ for } x \le 0.1 \\ &= x \text{ for } x \in [0.1, 1]. \end{aligned}$$

We have the following revenues

$$\begin{aligned} R^F &= 2\int_0^{0.1} (1-t)(\frac{\sqrt{t} + \sqrt{0.9t + t^2}}{2})^2 dt + 2\int_{0.1}^1 (1-t)t dt \\ &= 0.33317397, \end{aligned}$$

and

$$R^S = \int_0^{0.1000} (1-x)(1 - \frac{-0.9 + \sqrt{0.9^2 + 4x}}{2}) dx$$

$$+\int_{0.1000}^{1}(1-x)^2dx = 0.333\,174\,83 > R^F.$$

Note that in this example, we have the partial derivative $w_2 = \frac{1}{4}(1+\sqrt{\frac{x}{y}})$. Since w_2 is increasing in x, it is not submodular. We also have $w(x,x) = x$, and w does not satisfy condition (C). Next we want to show that w does not satisfy condition (R). For condition (R) to hold, it must be the case that for all $x < y$,

$$w_2 = \frac{1}{4}(1+\sqrt{\frac{x}{y}}) > \frac{1}{2}\frac{1-F_2(y)}{1-F_2(x)} = \frac{1}{2}\frac{1-y}{1-x}. \tag{6}$$

We claim that (6) is false around some neighborhood of $(x,x), x < 0.2$. To see this, it is sufficient to show that the second partial derivative of the left-hand side of (6) is smaller, when we evaluate at $(x,x), x < 0.2$, i.e.

$$w_{22} = \frac{-1}{8x} < \frac{-1}{2(1-x)},$$

which is exactly the condition $x < 0.2$. We conclude that condition (R) is violated around the point $(x,x), x < 0.2$. To confirm our arguments, let $x = 0.1, y = 0.12$, then

$$w_2 = \frac{1}{4}(1+\sqrt{\frac{0.1}{0.12}}) = 0.478\,22 < 0.488\,89 = \frac{1}{2}\frac{1-0.12}{1-0.1},$$

which reverses the inequality in (6).

The idea in the above example can be generalized to the following necessary condition for the ranking result. It simply says that the function H^{x_j} in condition (R) has a non-negative derivative at (x_j, x_j).

Theorem 14 *Fix F_j, w. Assume that w is symmetric and continuously differentiable up to the second order. If $R^F \geq R^S$ for all F_i, then we must have*

$$w_{ii}(x,x) + \frac{1}{2}\frac{f_j(x)}{1-F_j(x)}\frac{dw(x,x)}{dx} - \frac{1}{2}\frac{d^2w(x,x)}{dx^2} \geq 0 \text{ for all } x \text{ in } [0,a_j].$$

When $w(x,x) = x$, the condition becomes

$$w_{ii}(x,x) + \frac{1}{2}\frac{f_j(x)}{1-F_j(x)} \geq 0 \text{ for all } x \text{ in } [0,a_j].$$

The necessary condition by itself is not sufficient for the ranking result. For example, let $w(x,y) = \min\{x,y\}$. Let $v_1(t) = t, t \in [0,1], v_2(t) = 2t, t \in [0,1]$, hence $F_1(x) = x, F_2(x) = 0.5x$. We have

$$R^F = 2\int_0^1 (1-t)\min(t,2t)dt = 2\int_0^1(1-t)tdt = \frac{1}{3},$$

and

$$R^S = \int_0^1 (1-x)(1-0.5x)dx = 0.416\,67 > R^F.$$

which reverses the ranking. Hence the necessary condition is not sufficient. Note also that when w is linear and $w(x,x) = x$, the necessary condition has no bite.

5 Application to Auctions with Resale

We shall describe the auctions with resale model in Section 5.1, and prove the strategic equivalence between first-price auctions with resale and common value auctions in Section 5.2. Section 5.3 contains the main ranking results for the single-offer case. Section 5.4 analyzes the relationship between bargaining power and the ranking property for the case of multiple offers. Section 5.5 deals with the implications of the Coase conjecture. In this section, we assume that the valuations are private, so that $F_i(v_i)$ is the c.d.f. of the private valuation of bidder i.

5.1 Auctions with Resale

The first-price auction with resale is a two-stage game. The bidders first participate in a standard sealed-bid first-price auction. In the second stage, either the winner or the loser of the auction may offer to sell or buy the object from the other bidder. At the end of the auction and before the resale stage, some information about the submitted bids may be available. We find it useful to assume that there is no disclosure of bid information in our treatment. If the winning bid is announced, while the lower bid is not (as is often the case in real-world auctions), and the winning bidder makes the offer in the resale stage, the new information has no impact in the equilibrium behavior (in the single-offer case). Therefore we might as well assume that no bid is announced. If the loser of the auction makes the offers instead, the winning bidder's valuation will be revealed in a monotone equilibrium, and this gives advantage to offer-maker in extracting the surplus from the offer-receiver. It can be shown that in this case, there is no monotone equilibrium. To have a unified and general treatment (through a mechanism design approach in which the mechanism does not take such information revelation into account), the best approach is the no disclosure assumption. We will do this when we assume that there is only one (take-it-or-leave-it) offer. In the case of the auction loser making the offer, this assumption is the same as saying that the payment price is announced, but the highest bid is not. Despite the no disclosure assumption, the bidders do update their beliefs, as the identity of the winner is common knowledge, and this information alone will lead to updated valuation distributions regarding the highest valuation of the buyer and the lowest valuation of the seller[14]. In section 5.4 when we consider the resale mechanism with multiple offers, the announcement of the bid of the offer-maker simplifies the analysis, as there is only one uninformed party in the resale bargaining stage.

If the winner of the auction makes a take-it-or-leave-it offer to the loser, we call it the (single-offer) monopoly resale mechanism. If the loser of the auction makes a take-it-or-leave-it offer to the winner, we call it the (single-offer) monopsony resale mechanism. The offer-maker can be fixed before the auction, or contingent on winning or losing the auction. More generally, there can be repeated offers with delay costs, and section 5.2 gives a general trade mechanism of the resale stage. In making bids in the first stage, the bidders take into account the resale opportunities in the second stage.

In the second-price auction with resale, the game differs only in the first stage, in which the first-price auction is replaced by the second-price auction. In a second-price auction with resale, the winner of the auction knows the losing bid, as this is the price he pays in the auction. Therefore, this puts the winner at some advantage in making offers. There is in fact a continuum of equilibria (see Blume and Heidhues (2004) in the second-price auction with resale. It is an equilibrium for both bidders to bid their valuation (see Proposition 2 in Hafalir and Krishna (2007)), and this is an efficient equilibrium. The efficiency means that there is no need for resale after the auction, so that the revenue is the same with or without resale. When there is no resale, the "bid-your-value" strategies constitute a weakly dominant equilibrium strategy. With resale, it is no longer weakly dominant. However it is robust in the sense of Borgers and McQuade (2007), and is the only robust equilibrium (see the supplement to Hafalir and Krishna (2007)). This is the equilibrium used in the revenue ranking in the auctions with resale, as well as in common value auctions. Since there is no resale, and $w(x,x) = x$ in the bilateral trade mechanisms, the second-price auction revenue does not depend on the different trade mechanisms in the second stage.

Gupta and Lebrun (1999) assume that there is a resale stage after the auction, and all private information is disclosed at the resale stage before resale. In this case, the private value auction model with resale becomes a common value model with the common value given by

$$w(x,y) = \max\{x,y\}.$$

This common value function satisfies condition (C). Therefore the revenue ranking applies to this common value model, and the strategic equivalence of the next section implies that the ranking applies to the auctions with resale as well.

We now consider the more realistic case in which there is incomplete information at the resale stage. The auction with resale is not a common value auction due to incomplete information. Let $b_i(t)$ be the equilibrium bidding

[14] For more details about the information asumption, and implications of the full disclosure of bids, see Remark 1 of Hafalir and Krishna (2006) and Lebrun (2007).

strategy of bidder i, and $\phi_i(b)$ its inverse function (mapping bids to probabilities) in the first-price auction with resale. Let x_i be the valuation of the winner of the auction bidding b. Bidder i will make offers to sell to bidder j only if $x_j = v_j(\phi_j(b)) > x_i$. Assume that this is the case, and bidder j has a regular valuation distribution F_j, then the optimal monopoly price $p(x_i, x_j)$ is the unique solution of the following equation in p determined by the first order condition in maximization:

$$p - \frac{F_j(x_j) - F_j(p)}{f_j(p)} = x_i. \tag{7}$$

We have $p(x, x) = x$, and $x_j > p(x_i, x_j) > x_i$ when $x_i < x_j$.

Hafalir and Krishna (2007) have shown that the following system of differential equations must be satisfied (translated into our notations) by the equilibrium bidding strategy when the resale market is the monopoly resale market:

$$\frac{d\ln(\phi_i(b))}{db} = \frac{1}{p(v_i(\phi_i(b)), v_j(\phi_j(b))) - b}. \tag{8}$$

This is also the system of differential equations (3) satisfied by the equilibrium inverse bidding functions of the first-price common value auctions shown in Milgrom and Weber (1982) when the common value is given by $p(x, y)$. Since the equilibrium is uniquely determined by the system of equations with the boundary conditions $\phi_2\phi_1^{-1}(1) = 1, \phi_1(0) = 0 = \phi_2(0)$. The two auctions must have the same equilibrium bidding strategies. We call this property strategic equivalence. For weak-strong pairs, we can easily extend the definition of $p(x, y)$ to pairs $x > y$ by inflection, i.e. let $w(x, y) = p(x, y)$ (or $r(x, y)$) when $x \leq y$, and $w(x, y) = p(y, x)$ when $x > y$. Since p has both partial derivatives equal to $\frac{1}{2}$ when $x = y$, the function $w(x, y)$ so defined is symmetric and continuously differentiable.

We shall use the notation i for the bidder who makes offers, and j for the bidder who accepts or rejects the offers. For a weak-strong pair, i can either be the weak bidder or the strong bidder. For our analysis , it does not matter who is the strong or weak bidder, but it does matter who makes offers. Let $w_i(x_i, x_j)$ be the partial derivative of the common value function with respect to the valuation of offer-making bidder i. Bidder i could be either the winning bidder who makes a monopoly offer or a losing bidder who makes a monopsony offer.

In the monopsony resale mechanisms after the auction, let x_i be the valuation of the loser of the auction bidding b. Bidder i will make offers to buy from bidder j only if $x_j = v_j(\phi_j(b)) < x_i$. Assume that this is the case, and bidder j has a regular valuation distribution F_j. The optimal monopsony price r maximizes

$$(F_j(r) - F_j(x_j))(x_i - r),$$

with the first order condition given by

$$r - \frac{F_j(x_j) - F_j(r)}{f_j(r)} = x_i. \tag{9}$$

Note that (9) is exactly the same as (7). We can in fact have a unified treatment if we think of bidder i as the offer-maker and bidder j as the offer-receiver. There is a unique solution to this equation when $x_j \leq x_i$, and let $r(x_j, x_i)$ be the optimal offer satisfying (9). We can extend the definition to the region $x_j > x_i$, just as for the function p. We have $r(x,x) = x, x_j < r(x_j, x_i) < x_i$ when $x_j < x_i$. The system of differential equations (8) is now satisfied when p is replaced by r. Hence we have strategic equivalence between the first-price auction with the monopsony resale mechanism and the first-price common value auction with the common value defined by $r(x,y)$.

For weak-strong pairs, the weak bidder always finds it desirable to make selling-offers to the strong bidder after winning the auction, but has no reason to make buying-offers after losing the auction. For the strong bidder, it is just the opposite. When it is not a weak-strong pair, a bidder may not want to make selling offers after winning the auction, but may want to make buying-offers after losing the auction. If we allow a bidder i to make offers whether he or she is a winner or not, we give the bargaining power to bidder i. If, on the other hand, we only allow the winner of the auction to make selling-offers, we call this contingent bargaining power, as the bargaining power depends on winning the auction. Either kind of bargaining power arrangement will be allowed, and the ranking result does not depend on this particular allocation. We can also imagine a (commonly known) random process of assigning an offer-maker with or without contingency on winning the object. For instance, Hafalir and Krishna (2007) consider a resale mechanism in which an independent exogenous random process determines who makes the offer: with probability π, the winner of the auction makes a take-it-or-leave-it offer to the loser, and with probability $1-\pi$, the loser of the auction makes a take-it-or-leave-it offer to the winner. For a weak-strong pair, this corresponds to the common value function

$$w(x,y) = \pi p(x,y) + (1-\pi) r(x,y) \text{ for } x < y.$$

The strategic equivalence in this case is more difficult to explain. However, the strategic equivalence holds for very general bilateral trade mechanisms, as shown in the following section.

5.2 Strategic Equivalence

The idea that resale opportunities generate elements of common value in an auction is quite intuitive. In this section, we will show that for a general bilateral trade mechanism satisfying a weak property, a first-price auction with this resale mechanism is strategically equivalent to a first-price common value auction derived from the mechanism. The auctions with resale is a two-stage game, while the common value auction is a one-stage game. When we say that the two auctions are strategically equivalent, we only refer to the bidding strategy in the first stage. Strategic equivalence means that the equilibrium bidding strategy profile is the same for both auctions. The auctioneer cannot tell the difference between the auction with resale and the common value auction from the bidding behavior, and the expected revenue from the two auctions are identical. The two auctions are obviously quite different, but when we compare the bidding strategies in the first stage, there is no difference in the way the bidders behave.

The weak property needed for this result is a general version of the sure-trade property first proposed in Hafalir and Krishna (2007). It says that if the difference in the seller's value x and the buyer's value y is the highest possible, then trade takes place with probability 1 at the sure-trade price $p(x, y)$. The common value auction derived from the trade mechanism is defined by the sure-trade price, i.e. the common value function is $w(x, y) = p(x, y)$. This property is satisfied by the monopoly or monopsony resale mechanisms with repeated offers from one side of the transaction. In a dynamic bargaining with repeated offers and delay costs, the sure-trade price is the first offer price, and later offers are not involved in the definition of the common value.

In a general bilateral trade mechanism with two-sided asymmetric information, we have a seller who has an object to sell, and a buyer who in interested in buying it from the seller. Each side has independent private valuation for the object. Assume that the seller's valuation distribution $F_i(v_i)$ has the support $[x, a]$, and the buyer's valuation distribution $F_j(v_j)$ has the support $[0, y]$. We will fix a, and consider a general bilateral trade mechanism with varying parameters x, y. By the revelation principle, we can focus our attention on the direct trade mechanisms.

A (direct) trading mechanism consists of a vector of prices

$$p_1(v_i, v_j), p_2(v_i, v_j), .., p_n(v_i, v_j)$$

with corresponding probabilities

$$q_1(v_i, v_j), q_2(v_i, v_j), ..., q_n(v_i, v_j),$$

satisfying

$$\sum_{k=1}^{n} q_k(v_i, v_j) \leq 1.$$

The interpretation is that when the reported valuations are (v_i, v_j), seller i sells the object to buyer j at the price $p_k(v_i, v_j)$ with probability $q_k(v_i, v_j)$. There is no trade with probability $1 - \sum_{k=1}^{n} q_k(v_i, v_j)$. We assume that the mechanism is incentive compatible and individually rational. This implies that $q_k(v_i, v_j) = 0$ for all k whenever $v_i \geq v_j$. If $x > y$, any incentive compatible and individually rational mechanism will result in no trade. By convention, there is no trade when $x = y$. Therefore trade can only take place in equilibrium when $x < y$. We say that the trade mechanism satisfies the sure-trade property if $q_1(x, y) = 1$, and we refer to $p_1(x, y)$ as the sure-trade price. A common value function $w(x, y)$ is defined by the sure-trade price $w(x, y) = p_1(x, y)$.

A first-price auction is conducted and then resale takes place via the mechanism. This is a two-stage game, and the bidders propose their bids in anticipation of the resale outcome after the auction. Consider a perfect Bayesian equilibrium bidding strategy profile $b_i(t), i = 1, 2$. Assume that both are strictly increasing and continuous. Define $\phi_i(b) = v_i(b_i^{-1}(b)), i = 1, 2$. In this equilibrium, both players report their true valuation after the auction. Consider an auction winner i with bid b. He knows that the valuation of bidder j lies in $[0, \phi_j(b)]$, and bidder j knows that bidder i's valuation lies in $[\phi_i(b), a]$. The sure-trade property says that

$$q_1(\phi_i(b), \phi_j(b)) = 1, \text{ and } q_k(\phi_i(b), \phi_j(b)) = 0 \text{ for all } k > 1 \tag{10}$$

Theorem 15 *Assume that the resale mechanism satisfies the sure-trade property. A first-price auction with resale is strategically equivalent to the first-price common value auction with the common value defined by the sure-trade price.*

In a monopoly resale mechanism with a take-it-or-leave-it offer, the seller chooses an optimal monopoly price lower than the highest possible valuation of the buyer. The offer is accepted when the buyer has that highest valuation, hence the sure-trade property holds, and the sure-trade price is the optimal monopoly price. In a monopsony resale mechanism with a take-it-or-leave-it offer, the buyer chooses an optimal monopsony price higher than the lowest possible valuation of the seller. The offer is accepted when the seller has the lowest valuation, hence the sure-trade property also holds, and the sure-trade price is the optimal monopsony price.

The sure-trade property holds in a monopoly resale mechanism with many rounds of offers from the seller, because the equilibrium first offer is lower than the highest valuation of the buyer, and the sure-trade price is the optimal first offer by the seller.

A trade mechanism need not satisfy the sure-trade property. For example, suppose that trader one is the buyer, and trade two is the seller. Both the seller and the buyer's valuation distribution are uniform over $[0, 1]$ An incentive efficient trade (Myerson and Satterthwaite (1983)) is to have $q(v_1, v_2) \geq 1$ for $v_1 - v_2 \geq 0.25$, and $q(v_1, v_2) = 0$ when $v_1 - v_2 < 0.25$. The transaction price is denoted by $p(v_1, v_2) \in [0, 1]$. The incentive efficient trade mechanism certainly satisfies the sure-trade property. If we redefine the trade probabilities by

$$\begin{aligned} \hat{q}(v_1, v_2) &= 0.5 \text{ whenever } v_1 - v_2 \geq 0.25 \\ &= 0 \text{ whenever } v_1 - v_2 < 0.25 \end{aligned}$$

keeping $p(v_1, v_2)$ unchanged. Then all the incentive and participation constraints are satisfied. The new trade mechanism does not satisfy the sure-trade property. It can be shown that any incentive efficient bilateral trade satisfies the sure-trade property as stated in the following proposition.

Proposition 16 *If (i) there is a positive probability of gains from trade, (ii) the trade mechanism is incentive efficient, and (iii) the valuation distributions are regular, then the sure-trade property is satisfied.*

In our paper, we focus on repeated offers in sequential bargaining between the two bidders in the resale stage. One may ask to what extent any incentive compatible and individually rational direct mechanism can be implemented by such sequential offers. This question has been studied in Ausubel and Denechere (1989b, 1993).

To illustrate, how the repeated-offer bargaining can be represented by the direct trade mechanism, consider bargaining with two rounds of offers by the seller. The seller makes an offer p_1 in the first period. This offer is either accepted or rejected, with the threshold of acceptance represented by z, *i.e.* a buyer accepts the first offer if and only if his or her valuation is above z. If the first offer is accepted, the game ends. If it is not accepted, the seller makes a second offer p_2 which is a take-it-or-leave-it offer. An equilibrium analysis of this model is provided in section 5.4. Let $p_1(x,y), p_2(x,y), z(x,y)$ denote the equilibrium in this bargaining problem. The direct trade mechanism is described as follows. Given the reported valuations (v_i, v_j), there is no trade if $v_j < p_2(x,y)$. Trade occurs with the transaction price $p_1(x,y)$ if $v_j \geq z(x,y)$, and the transaction price $\delta p_2(x,y)$ if $p_2(v_i,y) \leq v_j < z(v_i,y)$. The probability of transaction is defined by

$$\begin{aligned} q_1(v_i,v_j) &= 1 \text{ if } v_j \geq z(v_i,y) \\ &= 0 \text{ if } v_j < z(v_i,y), \end{aligned}$$

$$\begin{aligned} q_2(v_i,v_j) &= 1 \text{ if } p_2(v_i,y) \leq v_j < z(v_i,y) \\ &= 0 \text{ otherwise.} \end{aligned}$$

5.3 Revenue Ranking with a Single Offer

Condition (R) can be interpreted as a condition on the monopoly pricing behavior when the resale mechanism is a monopoly market. In the single-period monopoly-pricing problem, essentially we have provided an upper bound on how monopoly price varies with marginal cost. Assume that a monopolist with marginal cost c faces a demand curve $D(p)$. Suppose $p + \frac{k+D(p)}{D'(p)}$ is increasing in p for a parameter $k > 0$. Then

$$\frac{dp^*}{dc} \leq \frac{1}{2} + \frac{D(p^*)}{2k} \leq \frac{1}{2} + \frac{D(c)}{2k}.$$

This is essentially our condition (R) in the case of monopoly pricing. In our model, we let $k = 1 - F_j(x_j)$,and $D(p) = F_j(x_j) - F_j(p)$. The assumption on demand is the regularity condition.

Before we apply the ranking results for common value auctions to auctions with resale, it is useful to have some characterization of the common value function arising from resale. We do not have a sharp characterization yet. We do have some useful properties. We say that w is quasi-convex (quasi-concave) if the level curves are concave (convex) to the origin.

Lemma 17 *If the common value function $w(x, y)$ is derived from a (single-offer) monopoly or monopsony resale market, then*

$$w(x,x) = x, w_1(x,x) = w_2(x,x) = \frac{1}{2}. \tag{11}$$

Furthermore, w is quasi-convex (quasi-concave) if and only if the underlying valuation distribution function is convex (concave).

Note that the condition (11) rules out $w(x, y) = \frac{x+2y}{3}$ as a (single-offer) monopoly or monopsony pricing function.

We need to know whether conditions (C) or (R) are satisfied, if the common value function arises from resale. The following lemma says that if bidder i makes a take-it-or-leave-it offer to bidder j in the resale market, then the optimal offer price satisfies condition (C) if bidder j has convex valuation distributions. The optimal offer price is the optimal monopoly (monopsony) price when bidder i wins (loses) the object in the auction.

Lemma 18 *If the offer receiver has a convex valuation distribution F_j, then the optimal offer price function satisfies condition (C).*

Condition (C) does not necessarily hold when valuation distributions are regular. However, condition (R) holds for F_j and the optimal offer function as the common value function when F_j is regular. This is our next lemma. Regularity is somewhat weaker than convexity.

Lemma 19 *If the offer receiver has a regular valuation distribution F_j, then the optimal offer price function and F_j satisfies condition (R).*

We now state a general ranking result for auctions with resale. Unlike the weak-strong pairs of Hafalir and Krishna (2007), we prove the result more generally.

Theorem 20 *Assume that $v_i(t) \neq v_j(t)$ with strict inequality for a subset of $[0, 1]$ of non-zero measure. We have $R^F > R^S$ if one of the bidder is chosen to make offers in the resale market, and the other bidder has a regular valuation distribution. The choice of the offer-making bidder is fixed before the auction, or randomly determined independently of whether the bidder wins the auction or not.*

Another approach is to give contingent bargaining power to bidders, such as allowing a bidder to make offers only when he or she wins the auction. The ranking result holds if both bidders have regular valuation distributions.

Corollary 21 *Assume that* $v_i(t) \neq v_j(t)$ *with strict inequality for a subset of* $[0,1]$ *of non-zero measure. We have* $R^F > R^S$ *if both bidders have regular valuation distributions, and a bidder only makes offers contingent on winning (or losing) the object.*

We have the following necessary condition for the revenue ranking result in auctions with resale. It is a consequence of the necessary condition for the ranking result in common value auctions.

Theorem 22 *Let* w *be the optimal offer function of bidder* i *to bidder* j *with a valuation distribution function* F_j*. Fix* F_j, w*. If* $R^F \geq R^S$ *for all* F_i*, then*

$$4 + \frac{(1 - F_j(x))f_j'(x)}{f_j^2(x)} \geq 0.$$

Now we give an example of the reversal of revenue ranking when the distribution function of the offer-receiver is not regular. There is a weak-strong pair, and the resale market is the monopoly market. Let the valuation distribution of the strong bidder be $F_s(x) = x^{\frac{1}{2}}$ with the support $[0,1]$. For $n > 2$, let the weak bidder be defined by[15]

$$\begin{aligned} F_w(x) &= 0.02^{\frac{1}{2} - \frac{1}{n}} x^{\frac{1}{n}}, x \leq 0.02; \\ &= x^{0.5}, 0.02 \leq x \leq 1. \end{aligned}$$

We have $v_s(t) = t^2$, and

$$\begin{aligned} v_w(t) &= 0.02^{1 - \frac{n}{2}} t^n, t \leq 0.02^{0.5}; \\ &= t^2, 0.02^{0.5} \leq t \leq 1. \end{aligned}$$

The resale market is a monopoly. The virtual value of F_s is

$$J(x) = x - \frac{1 - x^{0.5}}{0.5x^{-0.5}} = 3x - 2x^{0.5},$$

which is not increasing in x as

$$J'(x) = 3 - x^{-0.5} < -4 \text{ when } x < 0.02.$$

[15] Although the density function of F_i has infinite derivative at 0, and there is a kink in F_w at $x = 0.02$, the example can be slightly modified to produce an example satisfying all the smooth conditions we assume for F_w, F_s. and the ranking is still reversed.

Therefore the regularity condition is not satisfied. However we shall see that the optimal monopoly price is uniquely determined. Given $v_w = x$, and the maximum valuation $v_s = y > v_w$ of the strong bidder, the optimal resale price maximizes

$$R(p) = (F_s(y) - F_s(p))p + F_s(p)x = y^{0.5}p - p^{1.5} + p^{0.5}x.$$

The objective function is strictly concave in p. Hence there is a unique optimal price given by the solution of the first order condition

$$y^{0.5} - 1.5p^{0.5} + 0.5p^{-0.5}x = 0.$$

The unique solution is given by

$$p(x, y) = (\frac{\sqrt{y} + \sqrt{y + 3x}}{3})^2.$$

We have the first-price auction revenue

$$\begin{aligned} R^F &= 2\int_0^1 (1-t)p(v_w(t), v_s(t))dt \\ &= 2\int_0^{\sqrt{0.02}} (1-t)p(0.02^{1-\frac{n}{2}}t^n, t^2)dt + 2\int_{\sqrt{0.02}}^1 (1-t)t^2 dt. \end{aligned}$$

When $n = 4$, we have

$$\begin{aligned} R^F &= 2\int_0^{\sqrt{0.02}} (1-t)(\frac{t + \sqrt{t^2 + 150t^4}}{3})^2 dt + 2\int_{\sqrt{0.02}}^1 (1-t)t^2 dt \\ &= 0.166\,305\,4, \end{aligned}$$

and

$$\begin{aligned} R^S &= \int_0^1 (1 - F_w(x))(1 - F_s(x))dx \\ &= \int_0^{0.02} (1 - 0.02^{0.25}x^{0.25})(1 - x^{0.5})dx + \int_{0.02}^1 (1 - x^{0.5})^2 dx \\ &= 0.166\,318\,11 > R^F, \end{aligned}$$

hence the ranking is reversed. When $n = 6$, we have

$$\begin{aligned} R^F &= 2\int_0^{\sqrt{0.02}} (1-t)(\frac{t + \sqrt{t^2 + 3(2500t^6)}}{3})^2 dt + 2\int_{\sqrt{0.02}}^1 (1-t)t^2 dt \\ &= 0.166\,143\,44, \end{aligned}$$

and

$$\begin{aligned} R^S &= \int_0^{0.02} (1 - 0.02^{\frac{1}{3}}x^{\frac{1}{6}})(1 - x^{0.5})dx + \int_{0.02}^1 (1 - x^{0.5})^2 dx \\ &= 0.166\,167\,92 > R^F. \end{aligned}$$

The revenue ranking is reversed with an even greater difference. In the limit, the difference is the greatest, with

$$R^F = 2\int_0^{\sqrt{0.02}} (1-t)\frac{4}{9}t^2 dt + 2\int_{\sqrt{0.02}}^1 (1-t)t^2 dt = 0.165\,73,$$

and

$$\begin{aligned} R^S &= \int_0^{0.02} (1-0.02)(1-x^{0.5})dx + \int_{0.02}^1 (1-x^{0.5})^2 dx \\ &= 0.167\,99 > R^F. \end{aligned}$$

5.4 Repeated Offers and Bargaining Power

When there is only one offer (which is equivalent to a commitment equilibrium in the bargaining literature) in the resale mechanism, the regularity assumption insures that the bidders derive sufficient benefits from resale so that the general ranking is possible. If we allow repeated offers with no commitment, it is well-known (Sobel and Takahashi (1983), Fudenberg and Tirole (1983)) that high delay costs weaken the bargaining power of the monopolist. The weakened bargaining power may lead to low trade prices when the auction winner makes offers to the loser. We show by an example that the opposite ranking can occur when the bargaining power is substantially reduced in bargaining with repeated offers.

The bargaining problem with repeated offers from one-side to the other with delay costs is similar to that of Sobel and Takahashi (1983). In repeated offers, the equilibrium analysis is substantially simplified when the valuation of the offer-maker is common knowledge. Therefore, in this part of the analysis, we assume that the bid of the offer-maker is announced, while the bid of the offer-receiver is not. The delay costs are expressed by discount factors δ_1, δ_2 for bidder one, two respectively. Our example assumes that bidder one has low δ_1 (close to 0), and bidder two has high δ_2 (close to 1).

Consider the weak-strong pair of bidders $v_1(t) = t, v_2(t) = 1.5t$ over $[0,1]$. There are only two rounds of offers. For the example, we adopt the notations x, y for x_i, x_j respectively. We have $F_1(x) = x, F_2(y) = \frac{2}{3}y$. Given the first price offer p_1, bidder two has a threshold of acceptance z. The offer will be accepted if and only if bidder two's valuation is higher than z. When bidder two rejects the offer, the equilibrium period two offer is given by $p_2(x,z) = \frac{x+z}{2}$. The following equation determines the equilibrium z

$$z - p_1 = \delta_2(z - \frac{z+x}{2}),$$

and we have

$$z = \frac{p_1 - 0.5\delta_2 x}{1 - 0.5\delta_2}.$$

The optimal first offer p_1 maximizes the profit function

$$(y-z)(p_1-x) + \delta_1(z-p_2)(p_2-x) = (y-z)(p_1-x) + \frac{\delta_1}{4}(z-x)^2$$

$$= (y - \frac{p_1 - 0.5\delta_2 x}{1-0.5\delta_2})(p_1 - x) + \frac{\delta_1}{4}(\frac{p_1 - x}{1-0.5\delta_2})^2.$$

The first order condition for p_1 is

$$y - \frac{2p_1 - (1+0.5\delta_2)x}{1-0.5\delta_2} + \frac{\delta_1}{2(1-0.5\delta_2)^2}(p_1 - x) = 0,$$

and we get the optimal first period offer

$$p_1(x, y, \delta_1, \delta_2) = \frac{(1-0.5\delta_2)^2}{2-\delta_2-0.5\delta_1}y + \frac{1-0.5\delta_1-0.25\delta_2^2}{2-\delta_2-0.5\delta_1}x.$$

Since the first price auction revenue R^F with resale is increasing in p_1, and p_1 is increasing in δ_1, and decreasing in δ_2, we know that R^F is increasing in δ_1 and decreasing in δ_2. Therefore we know that a higher delay cost (or lower bargaining power) for bidder one hurts the revenue in the first price auction, while the opposite is true for bidder two. When $\delta_1 = 0, \delta_2 = 1$, we have the lowest revenue in the first price auction. In this case, we have $w(x, y) = \frac{1}{4}y + \frac{3}{4}x = 1.125x$, hence

$$R^F = \int_0^1 2(1-t)1.125tdt = 0.375,$$

which is lower than the revenue from the second price auction

$$\int_0^1 (1-x)(1-\frac{2}{3}x)dx = 0.388\,89.$$

Thus we have an example in which the opposite ranking holds when the monopolist has low bargaining power due to a high delay cost while the buyer has no delay cost.

5.5 The implication of Coase Theorems

When both bidders are very patient, the opposite ranking can also occur. The Coase (1972) conjecture in fact says that the monopolist may lose all bargaining power if the buyer anticipates lower prices in future offers. This has been formalized in Gul, Sonnenschein and Wilson (1986). In their model, the monopolist makes offers in increasingly short intervals. Assuming stationarity in the equilibrium, they show that all prices including the first offer goes to the marginal cost of the monopolist. In our model, the marginal cost of the monopolist is his or her own valuation for the object. This means that the first offer price, which is the common value function in the associated trade mechanism, will converge to $\min\{x, y\}$. By Proposition 12, the ranking must be reversed when the Coase conjecture holds.

In Gul, Sonnenschein and Wilson (1986), only the monopolist makes offers to the buyer. When alternating offers are allowed, Ausubel and Deneckere (1992) show that the Coase conjecture also holds under the same conditions. The reason is that when the informed party makes offers, only non-serious offers will be made. In fact, the informed party prefers to reveal information only passively by accepting or rejecting offers. This is called the Silence Theorem. The Silence Theorem gives a justification to the model of repeated offers from the uninformed party to the informed party. Again we have the opposite ranking in the model of alternating offers when Coase conjecture holds.

In the literature on Coase conjecture, the seller's cost is usually fixed, and equal to 0. In our resale model, the seller's cost can be any number within the range. To show how the Coase Theorem can be adapted for any cost of the seller, we illustrate with the finite horizon model of Sobel and Takahashi (1983). We show that for any given discount factor $\delta_1 < 1$ of the seller, the Coase conjecture holds as $\delta_2 \to 1$, and the number of periods goes to infinity. We focus on the linear case of Sobel and Takahashi (1983).

Assume that bidder one and two have uniform IPV distributions over the intervals $[0, a_1], [0, a_2]$ respectively and $a_1 \leq a_2$. After the first-price auction in stage one, the winning bid is announced. In stage two, the winner of the auction makes no commitment offers (except the last one which is a take-it-or-leave-it offer) to the loser for n periods. In this case, only bidder one will make offers after winning the auction. First we derive the unique perfect Baysian equilibrium of this finite-offer game and show that the revenue ranking is reversed. Let the seller has the valuation x (which is common knowledge) and the buyer's valuation is uniformly distributed over $[0, y], y \geq x$. We deonte this bargaining game by $L_n(x, y)$.

Proposition 23 *The first period offer of the bargaining game $L_n(x, y)$ in the resale stage with n periods of offers is given by*

$$p = c_n y + (1 - c_n)x$$

where c_n is defined recursively by

$$c_1 = \frac{1}{2}, c_k = \frac{(1 - \delta_2 + \delta_2 c_{k-1})^2}{2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}}.$$

Fix $\delta_1 < 1$, and let $\delta_2 \to 1$, we have

$$c_k \to \frac{c_{k-1}}{2 - \delta_1} \text{ for all } k.$$

Since $c_1 = \frac{1}{2}$, we have $c_n = \frac{1}{2(2-\delta_1)^{n-1}} \to 0$, as $n \to \infty$. Therefore the first period offer p converges to $x = \min\{x, y\}$ as $n \to \infty$. By Proposition 12, the revenue ranking is reversed if $\delta_1 < 1$ is fixed, δ_2 is close to 1, and the number of offer periods n is sufficiently large. In this result, Coase Theorem holds as long as the buyer is sufficiently patient, and the number of bargaining period is sufficiently large.

6 Proofs

Proof of Proposition 1:

Let $V_i = \frac{\partial V}{\partial s_i}$ be the partial derivative with respect to s_i. The equilibrium payoff of bidder one with the signal s_1 is

$$U(s_1) = \max_b \int_0^{s_1} (V(s_1, t) - b)dt.$$

By the envelop theorem, the marginal utility is

$$U'(s_1) = \int_0^{s_1} V_1(s_1, t)dt,$$

hence we have

$$U(s_1) = \int_0^{s_1} \left(\int_0^{s} V_1(s, t)dt \right) ds = \int_0^{s_1} \left(\int_t^{s_1} V_1(s, t)ds \right) dt \tag{12}$$

$$= \int_0^{s_1} \left(V(s_1, t) - V(t, t)\right) dt = \int_0^{s_1} V(s_1, t)dt - \int_0^{s_1} V(t, t)dt.$$

We also have

$$U(s_1) = \int_0^{s_1} (V(s_1, t) - b(s_1))dt = -b(s_1)s_1 + \int_0^{s_1} V(s_1, t)dt. \tag{13}$$

Equating (12) and (13), we have

$$s_1 b(s_1) = \int_{s_0}^{s_1} V(t, t)dt,$$

and

$$b(s_1) = \frac{1}{s_1} \int_0^{s_1} V(t, t)dt = \frac{1}{s_1} \int_0^{s_1} w(v_1(t), v_2(t))dt.$$

The seller's revenue from each bidder is

$$A = \int_0^1 tb(t)dt = \int_0^1 \left(\int_0^s w(v_1(t), v_2(t))dt \right) ds.$$

Using integration by parts, we have

$$A = \int_0^1 (1 - t)w(v_1(t), v_2(t))dt.$$

Since the equilibrium bidding strategy is symmetric, the revenue from each bidder is the same. Hence the theorem is proved.

Proof of Corollary 2:

By Proposition 1, we have

$$R^{FPA} = 2\int_0^1 (1-t)w(v_1(t), v_2(t))dt = 2\int_0^1 (1-t)(v_1(t)+v_2(t))dt.$$

Using integration by parts, we have

$$2\int_0^1 (1-t)v_1(t)dt = \int_0^1 (1-t)^2 dv_1(t) = \int_0^1 (1-t)^2 dv_1(t).$$

Similarly,

$$2\int_0^1 (1-t)v_2(t)dt = \int_0^1 (1-t)^2 dv_2(t),$$

and the theorem is proved.

Proof of Proposition 3:
The selected equilibrium has the following bidding strategy

$$b_i(v) = w(v, v) \text{ for } i = 1, 2.$$

The expected revenue from the second-price auction is given by

$$\begin{aligned} R^{SPA} &= \int_0^a w(x,x)d[1-(1-F_1(x))(1-F_2(x))] \\ &= -\int_0^a w(s,s)d[(1-F_1(x))(1-F_2(x))] \end{aligned}$$

Using integration by parts, we have

$$R^{SPA} = \int_0^a (1-F_1(x))(1-F_2(x))dv(x,x),$$

and the proof is complete.

Proof of Proposition 4:
Let $\phi_1(b), \phi_2(b)$ be the inverse bidding functions (mapping bids to signals) of the two bidders in a second-price auction equilibrium with a small private value component. Let $F_i(x) = v_i^{-1}(x)$. Bidder one with signal t_1 chooses b to maximize

$$\int_0^{\phi_2(b)} [\varepsilon v_1(t_1) + (1-\varepsilon)w(v_1(t_1), v_2(t_2)) - b_2(t_2)]dt_2.$$

The first order condition is

$$[\varepsilon v_1(t_1) + (1-\varepsilon) w(v_1(t_1), v_2(\phi_2(b))) - b]\, \phi_2'(b) = 0.$$

Since $t_1 = \phi_1(b)$, we have

$$\varepsilon v_1(\phi_1(b)) + (1-\varepsilon) w(v_1(\phi_1(b)), v_2(\phi_2(b))) - b = 0. \tag{14}$$

A similarly argument for bidder two gives us

$$\varepsilon v_2(\phi_2(b)) + (1-\varepsilon) v(v_1(\phi_1(b)), v_2(\phi_2(b))) - b = 0. \tag{15}$$

Combine the two equations (14),(15), we get

$$v_1(\phi_1(b)) = v_2(\phi_2(b)).$$

From (14), we have

$$\varepsilon \phi_1(b) + (1-\varepsilon) w(v_1(\phi_1(b)), v_1(\phi_1(b))) - b = 0, \tag{16}$$

which can be rewritten as

$$b_1(t_1) = \varepsilon t_1 + (1-\varepsilon) w\left(v_1(t_1), v_1(t_1)\right).$$

Hence the equilibrium bidding strategy is unique. As $\varepsilon \to 0$, (16) in the limit we have

$$b_1(t_1) = w\left(v_1(t_1), v_1(t_1)\right), \tag{17}$$

and similarly

$$b_2(t_2) = w(v_2(t_2), v_2(t_2)).$$

Our proof is complete.

Proof of Proposition 5:

The revenue of the second-price auction equilibrium is

$$R^{SPA} = \int_0^s \int_0^t \min(b(s), b(t)) ds dt = 2\int_0^1 \left(\int_0^s b(t) dt \right) ds = 2\int_0^1 \left(\int_0^s w(v_1(t), v_2(t)) dt \right) ds.$$

Using integration by parts, we have

$$\begin{aligned} R^{SPA} &= 2\int_0^1 w(v_1(s), v_2(s)) ds - \int_0^1 s w(v_1(s), v_2(s)) ds \\ &= 2\int_0^1 (1-s) w(v_1(s), v_2(s)) ds, \end{aligned}$$

which is the same as the revenue in the first-price auction by Proposition 1.

We need the following simple lemma for the proof of Theorem 6.

Lemma 24 *Let F_1, F_2 be continuous c.d.f. defined over $[a,b], a \geq 0$. Assume that F_i are strictly monotone except possibly when $F_i(x) = 0$ or 1. Let $F(x) = \sqrt{F_1(x)F_2(x)}$ be the geometric mean, then*

$$F^{-1}(s) \leq \max\{F_1^{-1}(s), F_2^{-1}(s)\} \textit{ for all } s \in (0,1).$$

Proof. Let $s \in (0,1)$, and assume without loss of generality that $F_2^{-1}(s) = \max\{F_1^{-1}(s), F_2^{-1}(s)\}$. Let $x_i = F_i^{-1}(s)$ for $i = 1,2$, then $x_1 \leq x_2$. We need to show $F^{-1}(s) \leq F_2^{-1}(s) = x_2$ or equivalently $s \leq F(x_2)$. By monotonicity, $F_1(x_2) \geq F_1(x_1) = F_2(x_2) = s$. We have therefore $F(x_2) = \sqrt{F_1(x_2)F_2(x_2)} \geq F_2(x_2) = s$, and the proof is complete. ∎

Proof of Theorem 6:

For the first-price auction, we have the following revenue formula of the asymmetric auction,

$$R^{FPA} = 2\int_0^1 (1-s)w(F_1^{-1}(s), F_2^{-1}(s))ds.$$

By the lemma, we have

$$R^{FPA} = 2\int_0^1 (1-s)\max\{F_1^{-1}(s), F_2^{-1}(s)\}ds$$

$$\geq 2\int_0^1 (1-s)F^{-1}(s)ds = 2\int_0^1 (1-s)w(F^{-1}(s), F^{-1}(s))ds.$$

The last term is the revenue of the first-price auction in the symmetric benchmark model. Therefore the theorem is proved.

Proof of Proposition 7:

Let $F_1(x) = x^{r_1}, F_2(x) = x^{r_2}$ over $[0,1]$ and $r_1 < r_2$. Assume that $\frac{r_1+r_2}{2}$ remains fixed while r_2 becomes larger (and r_1 becomes smaller). This means that there is more asymmetry while the symmetric benchmark is fixed. The FPA revenue is

$$R^{FPA} = 2\int_0^1 (1-s)\max\{F_1^{-1}(s), F_2^{-1}(s)\}ds$$

$$= 2\int_0^1 (1-s)s^{\frac{1}{r_2}}ds,$$

which is increasing in r_2.

Proof of Theorem 8:

The revenue of the second-price auction in the asymmetric auction is

$$\begin{aligned} & \int_0^1 (1 - F_1(x))(1 - F_2(x))dw\,(x, x) \\ = & \int_0^1 (1 - F_1(x))(1 - F_2(x))dx. \end{aligned}$$

The revenue of the second-price auction in the symmetric auction is

$$\int_0^1 (1 - F(x))^2 dx.$$

We have

$$\begin{aligned} & \int_0^1 (1 - F(x))^2 dx - \int_0^1 (1 - F_1(x))(1 - F_2(x))dx \\ = & \int_0^1 (F_1(x) + F_2(x) - 2\sqrt{F_1(x)F_2(x)})dx \\ & = \int_0^1 (\sqrt{F_1(x)} - \sqrt{F_2(x)})^2 dx \geq 0, \end{aligned}$$

hence the theorem is proved.

Proof of Lemma 9:

The single crossing condition on H^{x_j} is equivalent to the single crossing condition of the following function in x_i :

$$\frac{w_i(x_i, x_j)}{w_i(x_i, x_i)} - \frac{1 - F_j(x_i)}{1 - F_j(x_j)}. \tag{18}$$

If we fix x_i, and let x_j vary, the single crossing condition of the function in (18) is equivalent to the single crossing condition of the following function in x_j

$$L^{x_i}(x_j) = \frac{1 - F_j(x_i)}{1 - F_j(x_j)} - \frac{w_i(x_i, x_j)}{w_i(x_i, x_i)}. \tag{19}$$

The first term in (19) is increasing in x_j, therefore the single crossing condition of $L^{x_i}(x_j)$ is an immediate consequence of the submodular property of w.

Proof of Theorem 10:

From Proposition 3, we have

$$R^{SPA} < \frac{1}{2} \int_0^a [(1 - F_1(x))^2 + (1 - F_2(x))^2] dw(x, x).$$

Using arguments similar to the proof of Corollary 2, we have

$$\frac{1}{2} \int_0^a [(1 - F_1(x))^2 + (1 - F_2(x))^2] dw(x, x)$$

$$= \int_0^{v_1(1)} (1 - F_1(x))w(x,x)dF_1(x) + \int_0^{v_2(1)} (1 - F_2(x))w(x,x)dF_2(x)$$

$$= \int_0^1 (1 - s_1)w(v_1(s_1), v_1(s_1))ds_1 + \int_0^1 (1 - s_2)w(v_2(s_2), v_2(s_2))ds_2$$

$$= \int_0^1 (1 - s)\left[w(v_1(s), v_1(s)) + w(v_2(s), v_2(s))\right] ds.$$

Condition (C) now implies that

$$R^{SPA} < 2\int_0^1 (1 - s)w(v_1(s), v_2(s))ds = R^{FPA},$$

and the theorem is proved.

Proof of Proposition 12:
Let F_i, F_j be the corresponding distributions. Let

$$F(x) = \max\{F_1(x), F_2(x)\}.$$

Let $v(t) = F^{-1}(t)$. Then we have $\min\{v_1(t), v_2(t)\} = v(t)$. By Proposition 11, we have

$$R^{FPA} = \int_0^1 2(1 - t)v(t)dt = \int_0^a (1 - F(x))^2 dx.$$

Hence

$$R^{FPA} = \int_0^a (1 - F(x))^2 dx < \int_0^a (1 - F_1(x))(1 - F_2(x)dx.$$

Proof of Theorem 13:

Let $t = F_i(v_i), h(x) = w(x,x)$. The difference of the revenue is

$$R^{FPA} - R^{SPA} = \int_0^1 2(1 - t)w(v_i(t), v_j(t))dt - \int_0^1 (1 - F_i(x))(1 - F_j(x))dh(x)$$

$$= \int_0^1 2(1 - t)w(v_i(t), v_j(t))dt - \int_0^1 (1 - t)(1 - F_j(v_i(t)))h'(v_i)v_i'(t)dt.$$

Using Integration by parts, we have

$$\begin{aligned} & \int_0^1 (1 - t)(1 - F_j(v_i(t)))]h'v_i'dt \\ = & \int_0^1 (1 - t)d\left[\int_0^{v_i(t)} (1 - F_j(v))h'(v)dv\right] \end{aligned}$$

$$= \int_0^1 \left[\int_0^{v_i(t)} (1 - F_j(v))h'(v)dv \right] dt. \quad (20)$$

Hence we have

$$R^{FPA} - R^{SPA} = \int_0^1 2(1-t)w(v_i(t), v_j(t))dt - \int_0^1 \left[\int_0^{v_i(t)} (1 - F_j(v))h'(v)dv \right] dt.$$

Let $p\,(k,t) = v_j(t) + k(v_i(t) - v_j(t)), 0 \leq k \leq 1$, and

$$D(k) = \int_0^1 2(1-t)w(p(k,t), v_j(t))dt - \int_0^1 \left[\int_0^{p(k,t)} (1 - F_j(v))h'(v)dv \right] dt.$$

We want to show that $D'(k) > 0$ on a set of non-zero measure. Since $D(0) = 0$, this proves that $D(1) = R^{FPA} - R^{SPA} > 0$. We have

$$D'(k) = \int_0^1 2(1-t)w_i(p(k,t), v_j(t))(v_i(t) - v_j(t))dt$$

$$- \int_0^1 ((1 - F_j(p(k,t)))h'(p(k,t))(v_i(t) - v_j(t))dt$$

$$= \int_0^1 (v_i(t) - v_j(t))[2(1-t)w_i(p(k,t), v_j(t)) - (1 - F_j(p(k,t)))h'(p(k,t))]dt.$$

Since $v_i(t) > v_j(t)$, if and only if $p(k,t) > v_j(t)$ for $k > 0$, if and only if

$$w_i(p(k,t), v_j(t)) > \frac{1}{2} \frac{1 - F_j(p(k,t))}{1 - F_j(x_j)} h'(p(k,t)) = \frac{1}{2} \frac{1 - F_j(p(k,t))}{1 - t} h'(p(k,t))$$

for $k > 0$, if and only if

$$2(1-t)w_i(p(k,t), v_j(t)) > (1 - F_j(p(k,t)))h'(p(k,t))$$

for $k > 0$. We conclude that $D'(k) > 0$, for $k > 0$ when $v_i(t) \neq v_j(t)$, and the proof is complete.

Proof Theorem 14:

When w is symmetric, we have $w_1 = w_2$ at (x,x). Let $h(x) = w(x,x)$, then we have $h'(x) = 2w_i(x,x)$. Let $K^{x_j}(x_i) = 2w_i(x_i, x_j) - \frac{1}{2}\frac{1-F_j(x_i)}{1-F_j(x_i)}h'(x_i)$. We have $K^{x_j}(x_j) = w_i(x_j, x_j) - \frac{1}{2}h'(x_j) = 0$. Taking the derivative of K^{x_j} at x_j, we get

$$\frac{\partial}{\partial x_i} K^{x_j}(x_j) = w_{ii}(x_j, x_j) + \frac{1}{2} \frac{f_j(x_j)h'(x_j)}{1 - F_j(x_j)} - \frac{1}{2}h''(x_j).$$

Assume that $w_{ii}(x,x) + \frac{1}{2}\frac{f_j(x)h'(x)}{1-F_j(x)} - \frac{1}{2}h''(x) < 0$ at some point $(x_0, x_0), x_0 \in (0, a_2)$. We have $\frac{\partial}{\partial x_i} K^{x_j}(x_i) < 0$ near x_0. Since $K^{x_j}(x_i) = 0$, we must have

$K^{x_j}(x_i) < 0$ for $x_i < x_j, x_i, y_j$ near x_0. This implies that there exists a neighborhood U around x_0 such that

$$w_i(x_i, x_j) < \frac{1}{2}\frac{1-F_j(x_j)}{1-F_j(x_j)}h'(x_j) \text{ for } (x_i, x_j) \in U, x_i > x_j.$$

Let $v_j(t_0) = x_0$. There exists a smooth function $k(t)$ such that $s(t) = 1$ outside a neighborhood I of t_0, and $1 + \varepsilon > s(t) > 1$ on I, such that the point $(s(t)v_j(t), v_j(t)) \in U$ for $t \in I$. Now define $v_i(t) = v_j(t)$ outside I, and $v_i(t) = s(t)v_j(t)$ in I. Define $p(k,t) = v_j(t) + k(v_j(t) - v_i(t)), k \in [0,1]$ as in the proof of Theorem 13. From the arguments in that proof, we know that $D(k)$ is a decreasing function of k. Since $D(0) = 0$, we have $D(1) < 0$, and we conclude that for the pair of bidders v_i, v_j so defined, we have $R^{FPA} < R^{SPA}$, violating the assumption of the theorem. This contradiction means that the theorem is proved.

Proof of Theorem 15:

We first establish the first order condition of the equilibrium bid. Suppose that in equilibrium the bidder i with valuation v_i bids b and wins the auction. This means bidder j has a valuation in the interval $[0, \phi_j(b)]$. The interesting case is when $v_i < \phi_j(b)$ and trade can take place. Let

$$R(v_i, v_j) = \sum_{k=1}^{n} q_k(v_i, v_j)p_k(v_i, v_j) + (1 - \sum_{k=1}^{n} q_k(v_i, v_j))h(v_i),$$

where $h(v_i)$ is the payoff if there is no trade[16]. If bidder i deviates by bidding $c \neq b$ but close to b, then the payoff from the deviation is

$$\int_0^{\phi_j(c)} U(v_i, v_j)f_j(v_j)dv_j - F_j(\phi_j(c))c.$$

The first order condition requires that the above derivative with respect to c be 0 at $c = b$, and we get

$$U(\phi_i(b), \phi_j(b))f_j(\phi_j(b))\phi_j'(b) - f_j(\phi_j(b))b\phi_j'(b) - F_j(\phi_j(b)) = 0.$$

By (10), we have $U(\phi_i(b), \phi_j(b)) = p_1(\phi_i(b), \phi_j(b))$, and

$$\left[p_1(\phi_i(b), \phi_j(b)) - b\right] f_j(\phi_j(b))\phi_j'(b) - F_j(\phi_j(b)) = 0,$$

hence

$$\frac{d}{db}\ln F_j(\phi_j(b)) = \frac{1}{p_1(\phi_i(b), \phi_j(b)) - b}.$$

[16]To make the proof applicable to a dynamic trade mechanism with a delay cost, we take his general expression. It is v_i if there is no delay cost, but it can be δv_i when there is no trade after two periods in the model with two rounds of offers by the seller.

For the bidder j with valuation $v_j = \phi_j(b)$, the payoff from bidding c is

$$(v_j - c)F_i(\phi_i(c)) + \int_{\phi_i(c)}^{a} \left[\sum_{k=1}^{n} q_k(v_i, v_j)(v_j - p_k(v_i, v_j)) \right] f_i(v_i) dv_i,$$

and the first order condition for equilibrium is

$$(v_j - b)f_i(\phi_i(b))\phi_i'(b) - F_i(\phi_i(b))$$
$$- \left[(v_j - p_1(\phi_i(b), \phi_j(b))) \right] f_i(\phi_i(b))\phi_i'(b) = 0$$

or

$$\left[p_1(\phi_i(b), \phi_j(b))) - b \right] f_i(\phi_i(b))\phi_i'(b) - F_i(\phi_i(b)) = 0.$$

Therefore

$$\frac{d}{db} \ln F_i(\phi_i(b)) = \frac{1}{p_1(\phi_i(b), \phi_j(b))) - b}.$$

The equilibrium in both auctions are uniquely defined by the two first order conditions and the boundary conditions, hence they have the same perfect Bayesian equilibrium bidding strategy.

Proof of Proposition 16:

By assumption $x < y$. Let trader j be the buyer and trade i be the seller. According to Myerson and Satterthwaite (1983), the incentive efficient mechanism has the property that $q(v_i, v_j) = 1$ if

$$v_j - \alpha \frac{1 - F_j(v_j)}{f_j(v_j)} > v_i + \alpha \frac{F_i(v_i)}{f_i(v_i)}, \tag{21}$$

where α is the Lagrangian of the participation constraint. When y is the highest valuation, and x is the lowest valuation, (21) becomes

$$y > x.$$

which is true by assumption.

Proof of Proposition 23:

Let the number of periods remaining be k, and denote the optimal offer by p_k. The updated belief of the highest valuation z_k of the buyer is the threshold of acceptance in the period before. By backward induction, p_k depends only on x, z_k, and we use the notation $p_k(x, z_k)$. Let $\pi_k(x, z_k)$ be the expected profit function when k periods are remaining. Again by backward induction, z_k depends only on x and z_{k+1}. Given p_k, p_{k-1}, bidder two has a threshold level of acceptance z_{k-1}. Bidder two will accept the offer p_k whenever his or her valuation is above z_{k-1}. Given p_k, p_{k-1}, we can determine z_{k-1} by the condition

$$z_{k-1} - p_k = \delta_2(z_{k-1} - p_{k-1})$$

Thus we have the equation

$$(1-\delta_2)z_{k-1} + \delta_2 p_{k-1} = p_k \tag{22}$$

If the offer p_k is rejected, the bidder i updates his belief of the valuation of bidder j, and the new highest (lowest) valuation of the buyer (seller) is now z_{k-1}. Let $p_{k-1}(x_i, z_{k-1})$ be the optimal offer with $k-1$ periods remaining with the updated z_{k-1}. We can rewrite (22) as

$$(1-\delta_2)z_{k-1} + \delta_2 p_{k-1}(x, z_{k-1}) = p_k \tag{23}$$

If the optimal offer p_{k-1} with $k-1$ periods remaining has been determined by backward induction and is increasing in z_{k-1}. The left-hand side of (23) is increasing in z_{k-1}, and there is a unique solution denoted by $z_{k-1}(x_i, p_{k-1})$. Thus we know how z_{k-1} is determined once p_k is chosen.

The choice of p_k is determined by the maximization of the profit function of the seller given by

$$\left[F_2(z_k) - F_2(z_{k-1}(x, p_k))\right](p_k - x) + \delta_1 \pi_{k-1}(x, z_{k-1}) \tag{24}$$

The first order condition for p_k is

$$F_2(z_k) - F_2(z_{k-1}) - f_2(z_{k-1})(p_k - x)\frac{\partial z_{k-1}}{\partial p_k} + \delta_1 \frac{\partial \pi_{k-1}}{\partial z_{k-1}} \frac{\partial z_{k-1}}{\partial p_k} = 0.$$

Take the implicit derivative of (22) with respect to p_k, we have

$$(1-\delta_2)\frac{\partial z_{k-1}}{\partial p_k} + \delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}} \frac{\partial z_{k-1}}{\partial p_k} = 1,$$

or

$$\frac{\partial z_{k-1}}{\partial p_k} = \frac{1}{(1-\delta_2) + \delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}}}. \tag{25}$$

Substitute (25) into the first order condition, we have

$$F_2(z_k) - F_2(z_{k-1}) - \frac{f_2(z_{k-1})(p_k - x) - \delta_1 \frac{\partial \pi_{k-1}}{\partial z_{k-1}}}{(1-\delta_2) + \delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}}} = 0.$$

For uniform distributions, we have $f_2 = 1$. Hence we have the first order condition

$$z_k - z_{k-1} - \frac{p_k - x - \delta_1 \frac{\partial \pi_{k-1}}{\partial z_{k-1}}}{(1-\delta_2) + \delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}}} = 0 \tag{26}$$

When $k = 1$, we have

$$p_1(x, y) = \frac{x + y}{2}, \pi_1(x, y) = (\frac{y - x}{4})^2$$

and $p_1(x,z_1) = \frac{x+z_1}{2}, \pi_1(x,z_1) = (\frac{z_1-x}{2})^2$. Hence

$$\frac{\partial p_1}{\partial z_1} = \frac{1}{2}, \frac{\partial \pi_1}{\partial z_1} = \frac{z_1 - x}{2}.$$

The theorem holds for $k = 1$ with $c_1 = \frac{1}{2}$. More generally, by mathematical induction, assume that the theorem holds for $k-1$, and we have

$$p_{k-1} = c_{k-1}z_{k-1} + (1-c_{k-1})x, \pi_{k-1} = 0.5c_{k-1}(z_{k-1}-x)^2$$

$$\frac{\partial p_{k-1}}{\partial z_{k-1}} = c_{k-1}, \frac{\partial \pi_{k-1}}{\partial z_{k-1}} = c_{k-1}(z_{k-1}-x).$$

The first order condition (26) for z_{k-1}, p_k is

$$y - z_{k-1} = \frac{(1-\delta_2)z_{k-1} + \delta_2(c_{k-1}z_{k-1} + (1-c_{k-1})x) - x - \delta_1 c_{k-1}(z_{k-1}-x)}{1-\delta_2+\delta_2 c_{k-1}}$$

or

$$y - z_{k-1} = \frac{(1-\delta_2+\delta_2 c_{k-1} - \delta_1 c_{k-1})(z_{k-1}-x)}{1-\delta_2+\delta_2 c_{k-1}}$$

or

$$\frac{2(1-\delta_2+\delta_2 c_{k-1}) - \delta_1 c_{k-1}}{1-\delta_2+\delta_2 c_{k-1}} z_{k-1} = y + \frac{1-\delta_2+\delta_2 c_{k-1} - \delta_1 c_{k-1}}{1-\delta_2+\delta_2 c_{k-1}} x$$

and we have

$$z_{k-1} = \frac{1-\delta_2+\delta_2 c_{k-1}}{2(1-\delta_2+\delta_2 c_{k-1}) - \delta_1 c_{k-1}} y + \frac{1-\delta_2+\delta_2 c_{k-1} - \delta_1 c_{k-1}}{2(1-\delta_2+\delta_2 c_{k-1}) - \delta_1 c_{k-1}} x.$$

Let

$$d_{k-1} = \frac{1-\delta_2+\delta_2 c_{k-1}}{2(1-\delta_2+\delta_2 c_{k-1}) - \delta_1 c_{k-1}},$$

then

$$z_{k-1} = d_{k-1}y + (1-d_{k-1})x.$$

We have

$$p_k = (1-\delta_2)z_{k-1} + \delta_2 p_{k-1} = (1-\delta_2)z_{k-1} + \delta_2(c_{k-1}z_{k-1} + (1-c_{k-1})x)$$

$$= (1-\delta_2+\delta_2 c_{k-1})z_{k-1} + \delta_2(1-c_{k-1})x = c_k y + (1-c_k)x$$

where

$$c_k = \frac{(1-\delta_2+\delta_2 c_{k-1})^2}{2(1-\delta_2+\delta_2 c_{k-1}) - \delta_1 c_{k-1}} = (1-\delta_2+\delta_2 c_{k-1})d_{k-1}.$$

The expected profit can be written as

$$\pi_k = (y - z_{k-1})(p_k - x) + \delta_1 \pi_{k-1}$$

$$= c_k(1-d_{k-1})(y-x)^2 + 0.5\delta_1 c_{k-1}(z_{k-1}-x)^2$$
$$= (y-x)^2(c_k - c_k d_{k-1} + 0.5\delta_1 c_{k-1} d_{k-1}^2)$$
$$= (y-x)^2(c_k - (1-\delta_2+\delta_2 c_{k-1})d_{k-1}^2 + 0.5\delta_1 c_{k-1} d_{k-1}^2)$$
$$= (y-x)^2(c_k - 0.5d_{k-1}^2(2(1-\delta_2+\delta_2 c_{k-1}) - \delta_1 c_{k-1}))$$
$$= (y-x)^2(c_k - 0.5\frac{(1-\delta_2+\delta_2 c_{k-1})^2}{2(1-\delta_2+\delta_2 c_{k-1}) - \delta_1 c_{k-1}})$$
$$= (y-x)^2(c_k - 0.5c_k) = 0.5c_k(y-x)^2.$$

By mathematical induction, the proof is complete.

Proof of Lemma 17:

Take the partial derivatives of both sides of the first order condition

$$p(x,y) - \frac{F(y) - F(p(x.y))}{f(p(x,y))} = x.$$

We get

$$2 + \frac{F(y) - F(p)}{f^2} f' = \frac{1}{p_1}, \tag{27}$$

and

$$2 - \frac{f(y)}{f(p)} \frac{1}{p_2} + \frac{F(y) - F(p)}{f^2} f' = 0. \tag{28}$$

From (27), (28), we have

$$\frac{p_1}{p_2} = \frac{f(p)}{f(y)}.$$

The slope of the level curve is given by

$$\left| \frac{dy}{dx} \right| = \frac{p_1}{p_2} = \frac{f(p)}{f(y)}.$$

When y increases (while x decreases) on the level curve keeping p constant, the slope becomes flatter. Hence the level curves of p is quasi-convex if and only if f is increasing. Similarly, for the monopsony pricing function, we have

$$\frac{r_1}{r_2} = \frac{f(x)}{f(r)},$$

and the same result holds for the quasi-convexity of r. We also have $p_1 = p_2, r_1 = r_2$ whenever $x = y$.

Proof of Lemma 18:

Assume that bidder i wins the object and wants to make offers to sell the object to bidder j. The optimal monopoly price $p(x,y)$ satisfies condition (C) if

$$p(x,y) \geq \frac{x+y}{2}.$$

Since $z = p(x, y)$ maximizes the following objective function in variable z

$$K(z) = [F_j(y) - F_j(z)](z - x),$$

it is sufficient to show that

$$K'(\frac{x+y}{2}) > 0,$$

or

$$F_j(y) - F_j(\frac{x+y}{2}) - F_j'(\frac{x+y}{2})(\frac{x+y}{2} - x) > 0.$$

Equivalently, we need to show that

$$\frac{F_j(y) - F_j(\frac{x+y}{2})}{\frac{y-x}{2}} > F_j'(\frac{x+y}{2}). \tag{29}$$

Note that the left-hand side (29) is the slope of the line through the two points $(\frac{x+y}{2}, F_j(\frac{x+y}{2})), (y, F_j(y))$, while the right-hand side is the slope of F_j at $\frac{x+y}{2}$. The convexity of F_j is sufficient for (29) to hold.

If bidder i loses the auction, and wants to make buying offers to bidder j, the arguments are very similar. Since $z = r(x, y)$ maximizes the following objective function in variable z

$$K(z) = (F_j(z) - F_j(x))(y - z),$$

it is sufficient to show that

$$K'(\frac{x+y}{2}) > 0,$$

or

$$F_j'(\frac{x+y}{2})(y - \frac{x+y}{2}) - F_j(\frac{x+y}{2}) + F_j(x) > 0.$$

Equivalently, we need to show that

$$F_j'(\frac{x+y}{2}) > \frac{F_j(\frac{x+y}{2}) - F_j(x)}{\frac{y-x}{2}}. \tag{30}$$

Note that the left-hand side (29) is the slope of the line through the two points $(x, F_j(x)), (\frac{x+y}{2}, F_j(\frac{x+y}{2}))$, while the right-hand side is the slope of F_j at $\frac{x+y}{2}$. The convexity of F_j is sufficient for (29) to hold. The proof is complete.

Proof of Lemma 19:

Let F_j be regular. Fix x_j, and let $p(x_i) = w(x_i, x_j)$. We have the first order condition for the optimal $p(x_i)$:

$$p(x_i) - x_i = \frac{F_j(x_j) - F_j(p(x_i))}{f_j(p(x_i))}, \tag{31}$$

or

$$(p(x_i) - x_i)f_j(p(x_i)) + F_j(p(x_i)) = F_j(x_j). \tag{32}$$

Taking the derivative of (32) with respect to x_i, we have

$$p'(x_i)[2f_j(p(x_i)) + (p(x_i) - x_i)f_j'(p(x_i)] = f_j(p(x_i)),$$

and

$$p'(x_i) = \frac{1}{2 + (p(x_i) - x_i)\frac{f_j'(p(x_i))}{f_j(p(x_i))}} > 0. \tag{33}$$

We need to show

$$p'(x_i) < (>)\frac{1}{2}\frac{1 - F_j(x_i)}{1 - F_j(x_j)} \text{ when } j = s(\text{or } w),$$

or

$$\frac{1}{2 + (p(x_i) - x_i)\frac{f_j'(p(x_i))}{f_j(p(x_i))}} < (>)\frac{1}{2}\frac{1 - F_j(x_i)}{1 - F_j(x_j)} \text{ when } j = s(\text{or } w).$$

Since $F_j(x_i) < (>)F_j(p(x_i))$ when $j = s(\text{or } w)$, it is sufficient to show

$$\frac{1}{2 + (p(x_i) - x_i)\frac{f_j'(p(x_i))}{f_j(p(x_i))}} < (>)\frac{1}{2}\frac{1 - F_j(p(x_i))}{1 - F_j(x_j)} \text{ when } j = s(\text{or } w),$$

or

$$2 + (p(x_i) - x_i)\frac{f_j'(p(x_i))}{f_j(p(x_i))} > (<)2\frac{1 - F_j(x_j)}{1 - F_j(p(x_i))},$$

which is equivalent to

$$2\frac{F_j(x_j) - F_j(p(x_i))}{1 - F_j(p(x_i))} + (p(x_i) - x_i)\frac{f_j'(p(x_i))}{f_j(p(x_i))} > (<)0.$$

Divide both sides by $F_j(x_j) - F_j(p(x_i)) > (<)0$, we need to show

$$\frac{2}{1 - F_j(p(x_i))} + \frac{p(x_i) - x_i}{F_j(x_j) - F_j(p(x_i))}\frac{f_j'(p(x_i))}{f_j(p(x_i))} > 0. \tag{34}$$

Using (31), we know (34) is equivalent to

$$\frac{2}{1 - F_j(p(x_i))} + \frac{f_j'(p(x_i))}{f_j(p(x_i))^2} > 0. \tag{35}$$

From the regularity of F_j, we have, for all p,

$$\frac{d}{dp}(p - \frac{1 - F_j(p)}{f_j(p)}) > 0, \tag{36}$$

hence

$$2 + \frac{1 - F_j(p)}{f_j(p)^2}f_j'(p) > 0,$$

which implies (35).

Proof of Theorem 20:

Define $w(x,y) = p(x,y)$, or $r(x,y)$ if $x \leq y$. The function w can be extended to a continuously differentiable strictly increasing function over all (x,y). The revenue of the auctioneer however depends on the definition of w on the pairs $(x,y), x \leq y$.

Apply 19, we know that condition (R) is satisfied for the optimal offer function w. By Theorem 13, the ranking result holds.

If we allow random assignment of the offer-maker, let π be the probability that bidder i makes the offer, and $1-\pi$ the probability that bidder j makes the offer. Let w^i, w^j be the corresponding pricing function. The common value is now

$$w(x,y) = \pi w^i(x,y) + (1-\pi) w^j(x,y) \text{ for } x < y.$$

Note that if condition (C) (or (R)) is satisfied by both w^i and w^j, then it is also satisfied by w. Since the revenue is linear in π, the revenue ranking property of this common value auction follows from those of w^i and w^j.

Proof of Corollary 21:

With contingent bargaining power, the definition of w depends on F_i in some region, and on F_j in others.

Because of Lemma 17, the function w can be extended to all pairs and remains continuously differentiable and strictly increasing. This is because all partial derivatives of p and r on the diagonal (x,x) are identical and equal to $\frac{1}{2}$ regardless of which $F_i, i = 1,2$ is used in the optimal pricing problem. If all distribution functions are regular, the condition (R) is always satisfied in each region. Hence Theorem 13 applies, and the ranking result holds.

Proof of Theorem 22:

From (33), we take the second derivative, and evaluate at (x_j, x_j), we have

$$p''(x_j) = \frac{-(\frac{1}{2}-1)\frac{f_j'(x_j)}{f_j(x_j)}}{4} = \frac{1}{8}\frac{f_j'(x_j)}{f_j(x_j)}.$$

According to Theorem 14, the necessary condition for w to have the ranking result for all F_j is

$$\frac{1}{8}\frac{f_j'(x_j)}{f_j(x_j)} + \frac{1}{2}\frac{f_j(x_j)}{1-F_j(x_j)} \geq 0,$$

or

$$\frac{(1-F_j(x_j))f_j'(x_j)}{f_j^2(x_j)} + 4 \geq 0,$$

and the proof is complete.

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