Heterogeneity in Nash Networks*

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Abstract

Heterogeneity in Nash networks with two-way flow can arise due to differences in the following four variables: (i) the value of information held by agents, (ii) the rate at which information decays or loses its value as it traverses the network, (iii) the probability with which a link transmits information, and (iv) the cost of forming a link. In this paper we show that heterogeneity plays an important role in determining the architectures of Nash and efficient networks: with heterogeneity all networks can be Nash or efficient. This sharply contrasts with the homogeneous cases. We also find that heterogeneity is crucial for non existence of Nash networks. Finally, we show that there is in general no relationship between the decay and probabilistic models of heterogeneity.

JEL Classification: C72, D85

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1 Introduction

"The weakness of the game-theoretic approach is that most of the explicit characterization of equilibrium networks are so stark that the predicted networks have overly simple structures." – Jackson (forthcoming [7]). This remarkably accurate observation implies that strategic models of network formation have limited applicability. One way to resolve this issue would be the introduction of heterogeneity in game-theoretic models of network formation.

We believe that incorporating heterogeneity in network models leads to formulations that are realistic, and therefore are also able to explain reality better. Indeed, in models that take heterogeneity into account, the predicted networks can have possibly complicated architectures. Since results about equilibrium (and efficient) networks are directly linked to the value of parameters, heterogeneity yields a richer set of results. Moreover, it acts as a robustness check for the Nash equilibrium and efficiency results obtained using homogeneous parameters.

The notion of Nash networks was introduced by Bala and Goyal in two papers (2000, [1], [2]).[1] In the first paper (2000, [1]), links never fail and always transmit all the information reliably. Given that link formation is costly, the authors find that Nash networks are always minimally connected. In the second paper (2000, [2]), each link is allowed to fail with some probability p. Although link formation is still a costly act, in this case they find that as the amount of information at stake increases, agents attempt to insure themselves against link failures by forming super-connected networks, i.e, links have back-ups. A key feature of both papers is that all utilized parameters are homogeneous. Subsequent research however shows that the homogeneity of the parameters plays a significant role in these

[1]In the following, we use Nash networks to refer to networks that satisfy Nash equilibrium as the stability concept instead of Jackson and Wolinsky's (1996, [8]) notion of pairwise stability.

two widely divergent results. Haller and Sarangi (2005, [6]) extend Bala and Goyal (2000, [2]) by allowing different links to have different success probabilities. Interestingly, they find that given *any* network $\boldsymbol{g}$, there exists a set of parameter values under which $\boldsymbol{g}$ is Nash – the model with heterogeneity can encompass the results of *both* Bala and Goyal papers.

In this paper we examine different possible heterogeneous Nash network scenarios using the popular "connections model" introduced by Jackson and Wolinksy (1996, [8]) and studied extensively by Bala and Goyal (2000, [1], [2]). In the typical model agents are endowed with some information which can be accessed by other agents forming links with them. Link formation is costly and the cost of establishing a link being incurred by the initiating agent. In these models heterogeneity manifests itself in the payoff function and can occur through four different variables: (i) the value of information held by agents, (ii) the rate at which information decays or loses value as it traverses the network, (iii) the probability with which a link transmits information, and (iv) the cost of forming a link.

We focus on the two-way flow models introduced by Bala and Goyal (2000, [1]).[2] The two-way flow model allows bi-directional flow of information through a link regardless of who establishes it. Here we examine Nash networks, efficient networks and the existence of equilibrium networks under different possible heterogeneous frameworks created by the above variables. Our main results can be summarized as follows:

- the introduction of heterogeneity alters the existence of Nash networks, and
- the nature of heterogeneity plays a crucial role in existence and characterization of Nash networks.

The paper is organized as follows. In Section 2, we present the model setup.

[2]For the role of heterogeneity in one-way flow models see Galeotti (2006, [3]).

Section 3 contains results about models with imperfect reliability. In Section 4, we study Nash networks and efficient networks in models which incorporate heterogeneity and decay. In Section 5 we discuss the relationship between these two types of model. Section 6 provides a summary and a discussion of our results.

2 Model Setup

In this section we define the formal elements of the strategic form network formation game. Let $N = \{1, \ldots, n\}$, $n \geq 3$, denote the set of players with generic elements i, j, k. The set N also constitutes the nodes set. For ordered pairs $(i, j) \in N \times N$, the shorthand notation $i\,j$ is used and for non-ordered pairs $\{i, j\} \subset N$ the shorthand $[i\,j]$ is used.

Strategies. For player i a pure strategy is a vector $\boldsymbol{g}_i = (g_{i,1}, \ldots, g_{i,i-1}, g_{i,i+1}, \ldots, g_{i,n}) \in \{0,1\}^{N \setminus \{i\}}$. Since our aim is to model network formation, $g_{i,j} = 1$ is interpreted to mean that there exists a direct link between i and j initiated by player i (link $i\,j$ is formed by i), whereas $g_{i,j} = 0$ means that i does not initiate the link ($i\,j$ is not formed). Regardless of what player i does, player j can always choose to initiate a link with i, or set $g_{j,i} = 0$, i.e., not initiate a link with i. We focus only on pure strategies. The set of all pure strategies of agent i is denoted by $\mathcal{G}_i$ and consists of 2^{n-1} elements. The joint strategy space is given by $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$. Note that there is a one-to-one correspondence between $\mathcal{G}$ and the set of all directed graphs or networks with vertex set N. Namely, to a strategy profile $\boldsymbol{g} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n) \in \mathcal{G}$ corresponds the graph $(N, E(\boldsymbol{g}))$ with edge set $E(\boldsymbol{g}) = \{(i,j) \in N \times N |\; i \neq j,\; g_{i,j} = 1\}$. In the sequel, we shall identify a joint strategy $\boldsymbol{g}$ by its corresponding graph and use the terminology directed graph or directed network $\boldsymbol{g}$.

Payoffs. Payoffs of player i are given by the difference between benefits $B_i(\boldsymbol{g})$ and costs $c_i(\boldsymbol{g})$. Hence the payoff of player i in network $\boldsymbol{g}$ is given by

$$u_i(\boldsymbol{g}) = B_i(\boldsymbol{g}) - c_i(\boldsymbol{g}). \tag{1}$$

Next we define different types of heterogeneity in networks by introducing different costs and benefits formulations.

2.1 Link Costs

Players incur costs only for the direct links they establish. We consider two possible kinds of costs.

1. Homogeneous costs. Each player i incurs a cost $c > 0$ when she initiates the direct link $i\,j$, i.e., if $g_{i,j} = 1$. Hence the total costs incurred by player i when network $\boldsymbol{g}$ is formed are given by:

$$c_i(\boldsymbol{g}) = c \cdot \sum_{j \neq i} g_{i,j}. \tag{2}$$

2. Heterogeneous costs. The cost of each link now depends on the specific pair of players who form the link. We can write the cost function of player i as follows:

$$c_i(\boldsymbol{g}) = \sum_{j \neq i} g_{i,j} \cdot c_{i,j}. \tag{3}$$

2.2 Link Benefits

In any given network, benefits depend on values held by agents, the reliability of links and the nature of information decay in the model. Note that reliability and decay models are treated as mutually exclusive scenarios in the networks literature and here we retain this distinction. In a model with perfect reliability and no decay, a player obtains the full value of information from all the agents she "observes" both through her direct and indirect links.[3] In a model with imperfect reliability, information obtained from a player through direct or indirect links has the same value. In other words, information obtained through indirect links does not diminish in value. However, links fail to transmit information with a certain probability. In order to ascertain the value of information agent i acquires from agent j, it is now necessary to take into account all possible pathways that link i with j. This captures the idea that when one channel of communication fails to deliver the information it may be obtainable through another path. In decay models however, links always manage to transmit information but have constraints on how much information they can convey. Since information loses value as it travels along a sequence of links it captures the idea that "it is always better to have the facts straight from the horse's mouth". In this scenario, instead of considering all possible paths between two agents we only consider the path that results in the least possible information decay.

2.2.1 Benefits with Perfectly Reliable Links and no Decay

In such a model, heterogeneity in benefits from links depend on the value parameter, $V_{i,j} > 0$. In principal it is possible for all agents to have the same value of information V which is usually normalized to one. Further, a link between agents i and j potentially allows for **two-way flow of information**. So the benefits from network $\boldsymbol{g}$ are derived from its closure $\overline{\boldsymbol{g}} \in \mathcal{G}$, defined by $\overline{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\}$ for $i \neq j$. Moreover, since information is acquired through direct and indirect links we say information flows from player j to player i, if i and j are linked by means of a path in $\overline{\boldsymbol{g}}$. A **path** of length m in $\boldsymbol{g} \in \mathcal{G}$ from player i to player $j \neq i$, is a finite sequence $i_0, i_1, \ldots, i_m$ of pairwise distinct players such that $i_0 = i$, $i_m = j$,

[3]We say an agent observes another agent directly if the geodesic distance between them is one, and observes her indirectly if the geodesic distance betwen them exceeds one.

and $g_{i_k,i_{k+1}} = 1$ for $k = 0, \ldots, m-1$. Let $\mathcal{C}_{i,j}(\boldsymbol{g})$ be the set of paths from j to i in the network $\boldsymbol{g}$, and let $C_{i,j}(\boldsymbol{g})$ be a typical element of $\mathcal{C}_{i,j}(\boldsymbol{g})$. Let us denote by

$$N_i(\boldsymbol{g}) = \{j \in N |\; j \neq i, \text{ there exists a path in } \overline{\boldsymbol{g}} \text{ between } i \text{ and } j\},$$

the set of other players whom player i can access or "observe" in the network $\boldsymbol{g}$. Information received from player j is worth $V_{i,j}$ to player i. Therefore, player i's benefits from a network $\boldsymbol{g}$ is given by:

$$B_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} V_{i,j}. \tag{4}$$

We now substitute this in (1) and use different cost formulations to obtain a class of models developed by Galeotti, Goyal, and Kamphorst (2006, [4]). Assuming homogeneous values and costs reduces this to one of the models examined in Bala and Goyal (2000, [1]).

Note that $\overline{\boldsymbol{g}}$ belongs to the set $\mathcal{H} = \{\boldsymbol{h} \in \mathcal{G} | h_{i,j} = h_{j,i} \text{ for } i \neq j\}$. There is a one-to-one correspondence between the elements of $\mathcal{H}$ and the non-directed networks (graphs) with node set N. Namely, for $\boldsymbol{h} \in \mathcal{H}$ and $i \neq j$, $[i\ j]$ is an edge of the corresponding non-directed network if and only if $h_{i,j} = h_{j,i} = 1$. In what follows, we identify $\boldsymbol{h}$ with the corresponding non-directed network. Hence, the notation $[i\ j] \in \boldsymbol{h}$ stands for "$[i\ j]$ is an edge of $\boldsymbol{h}$". Also, for $\boldsymbol{k} \in \mathcal{H}$, $\boldsymbol{k} \subset \boldsymbol{h}$ means that $\boldsymbol{k}$ is a subnetwork of $\boldsymbol{h}$.

2.2.2 Benefits with Imperfectly Reliable Links

In such a model heterogeneity in benefits from links depends on the value parameter, $V_{i,j} > 0$, and the probability of link success, $p_{i,j} > 0$. The closure of the network $\overline{\boldsymbol{g}}$ determines the possible flow of information in this setting. If $\overline{g}_{i,j} = 0$, then there is no (direct) information flow between i and j. If $\overline{g}_{i,j} = 1$, then the link succeeds (there is direct two-way information flow between i and j) with probability $p_{i,j} \in (0, 1)$ and fails (there is no direct information flow between i and j) with probability $1 - p_{i,j}$. Of course it is also possible that all links have the same uniform success probability $p > 0$. Further it is assumed that success probabilities across links are independent events.

Observe that the joint strategy $\boldsymbol{g}$ gives rise to a random network with possibly different realization probabilities for different edges. Formally, we treat $\boldsymbol{g}$ and the realizations of the random network as non-directed networks. Possible realizations of the random network consist of the non-directed networks $\boldsymbol{h}$ satisfying $\boldsymbol{h} \subset \overline{\boldsymbol{g}}$. Invoking the independence assumption, given $\boldsymbol{g}$, the probability of the network $\boldsymbol{h} \subset \overline{\boldsymbol{g}}$ being realized is:

$$\lambda(\boldsymbol{h} \mid \boldsymbol{g}) = \prod_{[i\ j] \in \boldsymbol{h}} p_{i,j} \prod_{[i\ j] \in \overline{\boldsymbol{g}} \setminus \boldsymbol{h}} (1 - p_{i,j}). \tag{5}$$

Note that this conditional probability can also be defined when all links have the same probability of success. Given a strategy profile $\boldsymbol{g}$, player i's expected benefits from the resulting random network is:

$$B_i(\boldsymbol{g}) = \sum_{\boldsymbol{h} \subset \boldsymbol{g}} \lambda(\boldsymbol{h} \mid \boldsymbol{g}) b_i(\boldsymbol{h}). \tag{6}$$

The realization of the network $\boldsymbol{h}$, which occurs with probability $\lambda(\boldsymbol{h} \mid \boldsymbol{g})$, gives player i benefits $b_i(\boldsymbol{h})$. Summing over all possible realizations $\boldsymbol{h} \subset \boldsymbol{g}$ yields expected benefits. Variations of the expected benefits formulation can be substituted in the payoff function to obtain different models.

The model of imperfect reliability where all values, costs and probabilities are identical across players and links was analyzed by Bala and Goyal (2000, [2]). We call this the **model of imperfect reliability with homogeneous parameters.** The payoff function where the link failure probability is different for each link but V and c are identical across all agents is called the **model of imperfect reliability with heterogeneous links**. Such a model was first analyzed by Haller and Sarangi (2005). In the latter paper, the authors introduce a model with identical link failure probabilities, but with values $V_{i,j}$ has been first introduced in this paper. We call this the **model of imperfect reliability with heterogeneous players**. Our paper also allows for heterogeneity in link formation costs.

2.2.3 Benefits with Decay of value through links

Decay models were introduced by Jackson and Wolinksy (1996, [8]) under the name of the "connections model". In the Nash networks setting they were analyzed by Bala and Goyal (2000, [1]) who assumed that the value of information, the costs of link formation, and the decay parameter were identical across all agents and links. In other words, they analyzed the case of homogeneous decay. We propose two different frameworks to capture heterogeneity in models with decay.

Decay with Heterogeneous Players. Here we utilize the homogeneous decay assumption in conjunction with the heterogeneous players framework of Galeotti, Goyal and Kamphorst (2006, [4]), i.e., we assume that there exists $(i,j) \neq (k,\ell)$ such that $V_{i,j} \neq V_{k,\ell}$. Then the payoff function can be written as:

$$u_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} \delta^{d_{i,j}(\boldsymbol{g})} V_{i,j} - \sum_{j \in N \setminus \{i\}} c_{i,j} \cdot g_{i,j}, \tag{7}$$

where δ is the decay parameter and $d_{i,j}(\boldsymbol{g})$ is the distance in the shortest path (or geodesic distance) between i and j in $\boldsymbol{g}$.[4] We use this label for the model regardless of whether link costs are homogeneous or heterogeneous.

[4]In Bala and Goyal (2000, [1]), it is also assumed that players can always access their own information.

Decay with Heterogeneous Links. Here we assume that decay associated with the link $[i\ j]$ is not the same as decay associated with the link $[\ell\ k]$ for $[\ell\ k] \neq [i\ j]$. This assumption captures the fact that allowing for decay, the quantity of information a link can convey is not the same across all links. In other words, some links or paths are "better" than others.

We measure the decay associated with a link $[i\ j]$ by the parameter $\delta_{i,j} \in (0,1)$. Given a network $\boldsymbol{g}$, it is assumed that if agent i has formed a link with agent j, then agent i receives information of value $\delta_{i,j}$ from j. For this model we retain the symmetry assumption, that is $\delta_{i,j} = \delta_{j,i}$. Without loss of generality we assume that the value of a link is $V = 1$. The payoff of agent i in the network $\boldsymbol{g}$ is then given by:

$$u_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} \left(\prod_{[\ell\ k] \in C^*_{i,j}(\boldsymbol{g})} \delta_{\ell,k} \right) - \sum_{j \in N \setminus \{i\}} c_{i,j} \cdot g_{i,j}, \tag{8}$$

where $C^*_{i,j}(\boldsymbol{g}) = \arg\max_{C_{i,j}(\boldsymbol{g}) \in \mathcal{C}_{i,j}(\boldsymbol{g})} \left\{ \prod_{[\ell\ k] \in C_{i,j}(\boldsymbol{g})} \delta_{\ell,k} \right\}$.

This expression fundamentally differs from the previous expression because it does not use the geodesic distance between agents to determine the value of information obtained. Also as before we retain this name for the model regardless of whether link costs are homogeneous or heterogeneous.

2.3 Network Definitions

Nash Networks. Given a network $\boldsymbol{g} \in \mathcal{G}$, let $\boldsymbol{g}_{-i}$ denote the network that remains when all of agent i's links have been removed. Clearly, $\boldsymbol{g} = \boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}$ where the symbol $\oplus$ indicates that $\boldsymbol{g}$ is composed of the union of links in $\boldsymbol{g}_i$ and $\boldsymbol{g}_{-i}$ (in similar way the symbol $\ominus$ is used to indicate that we remove a link). A strategy $\boldsymbol{g}_i$ is a **best response** of agent i to $\boldsymbol{g}_{-i}$ if

$$u_i(\boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}) \geq u_i(\boldsymbol{g}'_i \oplus \boldsymbol{g}_{-i}), \text{ for all } \boldsymbol{g}'_i \in \mathcal{G}_i.$$

Let $\mathcal{BR}_i(\boldsymbol{g}_{-i})$ denote the set of agent i's best responses to $\boldsymbol{g}_{-i}$. A network $\boldsymbol{g} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n)$ is said to be a **Nash network** if $\boldsymbol{g}_i \in \mathcal{BR}_i(\boldsymbol{g}_{-i})$ for each $i \in N$, that is if $\boldsymbol{g}$ is a Nash equilibrium of the strategic game with normal form $(N, (\mathcal{G}_i)_{i \in N}, (u_i)_{i \in N})$. A strict Nash network is a network where all players are playing a strict best response.

Efficient Networks. A network $\boldsymbol{g}$ is efficient if the total utility of players is maximum, that is $W(\boldsymbol{g}) = \sum_{i=1}^{n} u_i(\boldsymbol{g}) \geq \sum_{i=1}^{n} u_i(\boldsymbol{g}')$, for all $\boldsymbol{g}' \in \mathcal{G}$.

Graph-theoretic Concepts. We now introduce some graph-theoretic definitions. A network $\boldsymbol{g}$ is called a **star** if there is a vertex i_s, such that for all $j \neq i_s$, $\max\{g_{i_s,j}, g_{j,i_s}\} = 1$ and for all $k \notin \{i_s, j\}$, $\max\{g_{k,j}, g_{j,k}\} = 0$. Moreover if $g_{i_s,j} = 1$ for all $j \neq i_s$, then the star is a center-sponsored star, and if $g_{i_s,j} = 0$ for all $j \neq i_s$, then the star is a periphery-sponsored star, otherwise it is a mixed star. A network $\boldsymbol{g}$ is **connected** if there is a path in $\overline{\boldsymbol{g}}$ between all players $i, j \in N$. A network $\boldsymbol{g}$ is **minimally connected** if it is connected and for all $i\,j \in E(\boldsymbol{g})$, $\boldsymbol{g} \ominus i, j$ is unconnected. A network $\boldsymbol{g}$ is **superconnected** if there exists a link $i\,j \in E(\boldsymbol{g})$ such that $\boldsymbol{g} \ominus i, j$ is a connected network.

Finally, we introduce the notion of an essential network. A network $\boldsymbol{g} \in \mathcal{G}$ is **essential** if $g_{i,j} = 1$ implies $g_{j,i} = 0$. Note that if $\boldsymbol{g} \in \mathcal{G}$ is a Nash network, then it must be essential. This follows from the fact that for each link $i\,j$, $c_{i,j} > 0$ and the information flow is two-way and independent of which agent invests in forming the link.

3 Models with Imperfect Reliability

Based on the parameters three formulations of network reliability models are possible. First, is the model of Bala and Goyal (2000, [2]) which is homogeneous with respect to all parameters (the pV model). This was followed by the work of Haller and Sarangi (2005, [6]) who allow for heterogeneity in the links' failure probabilities (the $p_{i,j}V$ model). A third kind of heterogeneity in this framework concerns players with varying values of information. We call this model of imperfect reliability as the model with heterogeneous players (the $pV_{i,j}$ model) and analyze it next.

We begin by illustrating that Nash equilibrium under the heterogeneous players model need not coincide with Nash equilibrium under the heterogeneous links model. To allow for meaningful comparison we impose the restriction that the expected value of a direct link is the same under both formulations, i.e., $pV_{i,j} = p_{i,j}V$, for all $(i, j) \in N \times N$.

Example 1 Suppose $N = \{1, 2, 3\}$, $V = 1$, $p = 1/5$, $c = 1/5$, and

$$\begin{pmatrix} V_{1,1} & V_{1,2} & V_{1,3} \\ V_{2,1} & V_{2,2} & V_{2,3} \\ V_{3,1} & V_{3,2} & V_{3,3} \end{pmatrix} = \begin{pmatrix} 0 & 15/4 & 9/2 \\ 15/4 & 0 & 5/3 \\ 9/2 & 5/3 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ p_{2,1} & 0 & p_{2,3} \\ p_{3,1} & p_{3,1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3/4 & 9/10 \\ 3/4 & 0 & 1/3 \\ 9/10 & 1/3 & 0 \end{pmatrix}.$$

It is easy to check that $pV_{i,j} = p_{i,j}V$, for all $i, j \in N$. Straightforward calculations show that the network $\boldsymbol{g}$ with $E(\boldsymbol{g}) = \{1\,3, 3\,2, 2\,1\}$ is a Nash network in the $pV_{i,j}$ framework. But, in the $p_{i,j}V$ framework the difference between the expected utility of player 3 in $\boldsymbol{g}$ and $\boldsymbol{g}'$ with $E(\boldsymbol{g}') = \{1\,3, 2\,1\}$ is $-1/15 < 0$. Hence $\boldsymbol{g}$ is not a Nash network. Furthermore, it is easy to check that $\boldsymbol{g}'$ is a Nash network for the $p_{i,j}V$ game.

The intuition behind this diverging results is the following. Since we have $p_{i,j}V = pV_{i,j}$ in the example, the higher $p_{i,j}$ is , the higher is $V_{i,j}$ (for given V and p respectively). In the $pV_{i,j}$ framework, higher values provide an incentive to form links, while in $p_{i,j}V$ framework, higher probabilities reduce the marginal benefits of alternative paths. Thus player 3 is better off not forming the link 32.

We present two results for this model. First, instead of explicitly characterizing all the Nash equilibria, for the sake of brevity, we demonstrate that there exist conditions such that every essential network can be strict Nash. This is also shown to be true for efficient networks.

Theorem 1 *: (Imperfect reliability with heterogeneous players.) Let $\boldsymbol{g}$ be an essential network. Then, there exists a homogeneous link cost $c > 0$, a probability p, and an array $\boldsymbol{V} = [V_{i,j}]$ of values such that*

1. *$\boldsymbol{g}$ is a strict Nash network in the corresponding network formation game;*
2. *$\boldsymbol{g}$ is an efficient network in the corresponding network formation game.*

Proof Let $\boldsymbol{g}$ be an essential network.

1. Firstly, we prove that there is a link cost $c > 0$, a probability p, and an array $\boldsymbol{V}$ of values such that $\boldsymbol{g}$ is a strict Nash network. We assume that $V_{i,j} \in \{\varepsilon, V\}$. More precisely, if $g_{i,j} = 1$, then $V_{i,j} = V$, and if $g_{i,j} = 0$, then $V_{i,j} = \varepsilon < V$.

   Let $\boldsymbol{g} \oplus i, j$, with $V_{i,j} = \varepsilon$, be the network where i obtains the largest amount of resources thanks to the link $i\,j$. Then, the amount of resources that player i obtains through j is given by:

   $$\varepsilon p + \sum_{\ell=2}^{n(n-1)/2} p^{\ell}(a_\ell \varepsilon + a'_\ell V) \tag{9}$$

   where $a_\ell, a'_\ell \in \mathbb{Z}$ for each $\ell \in \{2, \dots, n(n-1)/2\}$, it is possible that some a_ℓ or some a'_ℓ are null. Let $A = \sum_{\ell=2}^{n(n-1)/2} |a_\ell| + |a'_\ell|$.

Let $\boldsymbol{g} \ominus i, j$ be the network where i obtains with the largest probability the resources of j, with $V_{i,j} = V$, given that there is no link between i and j in this network. Then, the amount of ressources of j that player i obtains, in $\boldsymbol{g} \ominus i, j$, is given by:

$$\sum_{\ell=2}^{(n-1)(n-2)/2} p^{\ell} b_{\ell} V$$

where $b_{\ell} \in \mathbb{Z}$ for each $\ell \in \{2, \dots, (n-1)(n-2)/2\}$. It is possible that b_{ℓ} is null. Let $B = \sum_{\ell=2}^{(n-1)(n-2)/2} |b_{\ell}|$.

Given A and B, there exists p such that

$$p < \frac{V - \varepsilon}{V(A+B)}, \; A \neq 0 \text{ or } B \neq 0,$$

and

$$p\varepsilon + p^2 AV < c < Vp - Bp^2 V.$$

Now, it is clear that a player i who did not form a link with j in $\boldsymbol{g}$ has no incentive to form this link. Indeed, if i forms a link with j, then her payoff function increases by an amount which is bounded above by $p\varepsilon + p^2 AV - c < 0$. Likewise, a player i has an incentive to preserve her link with player j if $g_{i,j} = 1$. Indeed, a player j brings to i an amount which is bounded below by $Vp - Bp^2 V - c > 0$. It follows that all players play a best response in $\boldsymbol{g}$. Hence $\boldsymbol{g}$ is a strict Nash network.

2. Secondly, we prove that there is a link cost $c > 0$, a probability p, and an array $\boldsymbol{V}$ of values such that $\boldsymbol{g}$ is an efficient network. We assume that if $g_{i,j} = 1$, then $V_{i,j} = V$, and if $g_{i,j} = 0$, then $V_{i,j} = \varepsilon < V/2n$. Let $\mathcal{F} = \{i\, j | g_{i,j} = g_{j,i} = 0\}$ be the set of links such that $V_{i,j} = \varepsilon$ and $\mathcal{D} \subset \mathcal{F}$. Let $\boldsymbol{g}'(\mathcal{D}) = \arg\max_{\boldsymbol{g}' \in \mathcal{G}} \{W(\boldsymbol{g}' \oplus \mathcal{D}) - W(\boldsymbol{g}')\}$ be the network where the n players obtain the largest amount of resources thanks to the links in $\mathcal{D}$.

Let $\mathcal{D}^* = \arg\max_{\mathcal{D}\subset\mathcal{F}}\{\boldsymbol{g}'(\mathcal{D})\}$ be the subset of $\mathcal{F}$ which brings the maximal increasing total utility. The latter is at most:

$$2|\mathcal{D}^*|\varepsilon p + \sum_{\ell=2}^{n(n-1)/2} p^{\ell}(e_\ell \epsilon + e'_\ell V) \qquad (10)$$

where $e_\ell, e'_\ell \in \mathbb{Z}$ for each $\ell \in \{1, \ldots, n(n-1)/2\}$. It is possible that some e_ℓ or some e'_ℓ are null. Let $E = \sum_{\ell=2}^{n(n-1)/2} |e_\ell| + |e'_\ell|$.

We can find p such that

$$p < \frac{V - 2n\varepsilon}{(E+B)V},\ E \neq 0 \text{ or } B \neq 0,$$

and

$$2np\varepsilon + p^2 EV < c < Vp - Bp^2 V.$$

Then, by using similar arguments as in the previous part, we conclude that $\boldsymbol{g}$ is an efficient network.

□

A few remarks are in order here. First, although Haller and Sarangi (2005, [6]) do not investigate this issue, it can be easily shown that a similar efficiency result holds for the heterogeneous links model.

Second, we observe that given any network, for this network to be Nash, it is enough to introduce a low level of value heterogeneity. Indeed, in the proof of Theorem 1 we only need two different values. Likewise, even if the difference between these two values is small we can always find parameters p and c such that the network is Nash. Lastly, Theorem 1 still holds if there is heterogeneity in link costs as well. A simple continuity argument can be used to verify this.

Next it is also worth asking what is the relationship between efficiency and stability in this model? Both Bala and Goyal (2000, [2] and Haller and Sarangi

(2005, [6]) claim that then costs are very high or very low, there is no conflict between stability and efficiency. We find that the same is true for the heterogeneous players model. It is possible to use a continuity argument to preserve the result of Bala and Goyal (2000, [2], pg. 223-224) concerning the conflict between Nash networks and efficient networks.

Finally, one can ask the question: Given Theorem 1 is there any role for explicit characterization of the Nash equilibrium networks? Although for the sake of brevity we do not pursue this, it can still be a meaningful exercise since the framework allows for the existence of multiple equilibria. Consider for instance Example 1 in Bala and Goyal (2000, [2]). This is the simplest possible imperfect reliability scenario since it only has 3 players with homogeneous values, costs and reliability. First, there exists a parameter range in which periphery-sponsored stars are Nash. Clearly, as there is no rule for selecting the central agent, the coordination problem for selecting this agent may still persist under heterogeneity. Moreover, the authors identify a parameter range where two kinds of architectures are Nash: the center-sponsored star and mixed stars. This outcome is possible even with parameter heterogeneity.

We now identify sufficient conditions for the simultaneous existence of two types of star networks in equilibrium in the heterogeneous players framework. Let $V^{\overline{m}} = \max_{(i,j)\in N\times N}\{V_{i,j}\}$ and $V^{\underline{m}} = \min_{(i,j)\in N\times N}\{V_{i,j}\}$.

Proposition 1 *: Consider the heterogeneous players model with homogeneous costs. If* $pV^{\underline{m}} > c$ *and* $(n-2)(1-p^2)V^{\overline{m}} + (1-p)V^{\overline{m}} < c$*, then center-sponsored stars and mixed stars are strict Nash networks.*

Proof The proof by contradiction is an adaptation of the proof of Bala and Goyal (2000, [2]). Let $i_s \in N$ be the central agent of the star. We focus on center-sponsored stars. Suppose that there is an agent j who is not linked with i_s ($g_{j,i_s} = g_{i_s,j} = 0$). Then the marginal payoff that i_s obtains from a player j is $pV_{i_s,j} \geq pV^{\underline{m}} > c$, a contradiction. Moreover, if player j forms $k \geq 1$ links, then her payoff is bounded above by

$$\sum_{i \in N \backslash \{j\}} V_{j,i} - kc. \quad (11)$$

Furthermore, the payoff that player j obtains in the center-sponsored star is:

$$pV_{j,i_s} + p^2 \sum_{i \in N \backslash \{j,i_s\}} V_{j,i}. \quad (12)$$

Subtracting (12) from (11) shows that j's maximum incremental payoff from one or more links is no larger than $(n-2)(1-p^2)V^{\overline{m}} + (1-p)V^{\overline{m}} - c$, which is negative by choice of p and c. Hence, j's best response is to form no link with a player $i \neq i_s$. The proof for the mixed stars can be made with similar arguments. □

Existence of Nash networks. We now show that there exist parameters p, c, $(V_{i,j})_{(i,j) \in N \times N \backslash \{i\}}$, such that there is no Nash network in the $pV_{i,j}$ framework. Note that this result is preserved when the costs of link formation are heterogeneous.

Example 2 : *(Non-existence of Nash networks.)* Let $N = \{1, 2, 3\}$ be the set of p layers. Let $pV_{i,j} < c$ for all links $i\ j$ except the link 1 3. Also, let $pV_{21} + p^2 V_{23} > \max\left\{c, p^2 V_{2,1} + pV_{2,3}\right\}$, $p(1-p)(pV_{1,2} + (1+p)V_{1,3}) < c$, $p(1-p)(pV_{2,1} + (1+p)V_{2,3}) < c$, $p(1-p)(pV_{3,1} + (1+p)V_{3,2,}) > c$.

1. The empty network is not a Nash network, since $pV_{1,3} > c$.
2. A network with one link cannot be a Nash network. Indeed, if this link is the link $i\ j \neq 1\ 3$, then player i can obtain a higher payoff by deleting this

link, since $pV_{i,j} < c$. If this link is the link $i\,j = 1\,3$, then player 2 has a higher payoff if she forms the link 2 1, since $pV_{2,1} + p^2V_{2,3} > c$.

3. Next, a network $\boldsymbol{g}$ with two links cannot be a Nash network. Given that Nash networks must be essential in such a network, there always exists a path in $\overline{\boldsymbol{g}}$ between the players.

   i. Since $pV_{i,j} < c$ for all links $i\,j$, except the link 1 3, no network where a link allows access to resources of only one other player can be Nash (except for the link 1 3). From this it follows that only networks with links $\{1\,3, 2\,3\}$, $\{1\,2, 3\,2\}$, $\{2\,1, 3\,1\}$ or $\{1\,3, 2\,1\}$ can be Nash.

   ii. We know that player 2 prefers the link 2 1 to the link 2 3 (since $pV_{2,1} + p^2V_{2,3} > p^2V_{2,1} + pV_{2,3}$). Thus, networks with links 1 3, 2 3, cannot be Nash.

   iii. We know that, every thing else being equal, player 1 prefers the link 1 3 to the link 1 2 (since $pV_{1,3} > c > pV_{1,2}$). Thus, network with links 1 2 and 3 2 cannot be Nash.

   iv. Since $p(1-p)(pV_{3,1} + (1+p)V_{3,2}) > c$, the networks with links $\{2\,1, 3\,1\}$ or $\{1\,3, 2\,1\}$ cannot be Nash. Indeed, in such a case, player 3 has an incentive to set a link with player 2.

   Hence, it follows that a network with two links cannot be a Nash network.

4. Finally, we show that a network $\boldsymbol{g}$ with three links cannot be a Nash network. Note that in this network there always exists a path in $\overline{\boldsymbol{g}}$ between the players. Since $p(1-p)(pV_{1,2} + (1+p)V_{1,3}) < c$ and $p(1-p)(pV_{2,1} + (1+p)V_{2,3}) < c$, then no network where player 1 or player 2 have formed links can be a Nash network. It follows that no network with three links can be a Nash network.

4 Models with Decay

In this section we focus on situations where links do not convey full information. Bala and Goyal (2000, [1]) analyze such networks in a homogeneous setting. Their main result consists in providing conditions which allow some architectures (complete and empty networks, and stars) to be Nash. They argue that it is difficult to provide a complete characterization of Nash or efficient networks in the presence of decay. We begin with situations where the parameters of decay are homogeneous and the values of players are heterogeneous.

4.1 Decay with Heterogeneous Agents

In this section we obtain two main results.[5] First, we demonstrate that all networks can be supported as strict Nash and efficient. Then, we show that there exist parameter values for which there is no Nash network.

Theorem 2 *: (Decay with heterogeneous agents.) Let* $\boldsymbol{g}$ *be an essential network. If the payoff function satisfies equation (7), then there exists a link cost* $c > 0$ *and an array* $\boldsymbol{V} = [V_{i,j}]$ *of values such that:*

1. $\boldsymbol{g}$ *is a strict Nash network in the corresponding network formation game;*
2. $\boldsymbol{g}$ *is an efficient network in the corresponding network formation game.*

Proof Let $\boldsymbol{g}$ be an essential network and $V^{\overline{m}} = \max_{(i,j)\in N\times N}\{V_{i,j}\}$. We successively prove both parts of the theorem.

1. For $g_{i,j} = 1$, let $V_{i,j}(\delta - \delta^2) > c$, and if $g_{i,j'} = 0$, then let $c > \delta V_{i,j'} + (n-2)\delta^2 V^{\overline{m}}$, with $V_{i,j} > V_{i,j'}$. These two conditions are compatible if δ is

[5] Galeotti, Goyal and Kamphorst (2006, [4]) consider the case of small levels of decay with heterogeneous agents in the context of the insider-outsider model. However, their results are not general since they only allow for two groups of players, which severely restricts the values of δ.

sufficiently close to zero. Now, it is obvious that player i has no incentive to form a link with j if $g_{i,j} = 0$, since the costs of establishing this link is greater than the maximum value that player j can provide to player i. Likewise, it is straightforward that if $g_{i,j} = 1$ player i has no incentive to delete the link with j, since i cannot obtain more than $\delta^2 V_{i,j}$ if she accesses the resources of j through an indirect link.

2. For $g_{i,j} = 1$, let $V_{i,j}(\delta - \delta^2) > c$, and if $g_{i,j'} = 0$, then let $c > \delta(V_{i,j'} + V_{j',i}) + \delta^2(n-2)nV^{\overline{m}}$ (we suppose that $g_{j',i} = 0$, otherwise i cannot form the link $i\ j'$ in an efficient network). These two conditions are compatible if δ is sufficiently close to zero. We can show that $\boldsymbol{g}$ is an efficient network by using the same arguments as in the previous part.

□

First, note that the above result concerning existence of Nash network is not true anymore if $V_{i,j} = V_i$ or $V_{i,j} = V_j$ for all players $i \in N$. For instance, let $N = \{1, 2, 3\}$ be the set of players, $V_{i,j} = V_i$ for all $i \in N$, and assume a network $\boldsymbol{g}$ where $g_{1,2} = 1$ and $g_{i,j} = 0$ for all $i, j \neq 1, 2$. It is obvious that player 1 either has an incentive to delete the link $1, 2$ or has an incentive to set the link $1, 3$. Henceforth $\boldsymbol{g}$ can not be a Nash network. Second, from the above result itself it is possible to deduce that Nash and efficient networks do not always coincide since the bounds on c in the proof are different. So, the introduction of decay does not allow a Nash network to be always an efficient network.

Existence of Nash networks. We begin by showing that the model of Bala and Goyal (2000, [1]), always has a Nash network. This result continues to hold if heterogeneity is not "too high", more precisely if $V_{i,j} = V_i$ for all $i \in N$.

Theorem 3 *: Suppose costs of setting links are homogeneous.*

1. *If the decay parameter and the values are homogeneous, then there always exists a Nash network.*

2. *If the payoff function satisfies equation (7) and, for all $i \in N$, $V_i = V_{i,j}$, for all $j \in N \setminus \{i\}$, then there always exists a Nash network.*

Proof We now prove both parts of the theorem.

1. If $\delta V \leq c$, then the empty network is a Nash network. If $\delta V > c$, then there are two possibilities: (i) if $(\delta - \delta^2)V \leq c$, then star networks are Nash, and (ii) if $(\delta - \delta^2)V > c$, then the complete network is Nash.

2. If $\delta V_i \leq c$ for all $i \in N$, then the empty network is a Nash network. If there exist players $i \in N$ such that $\delta V_i > c$, then we start with the empty network and we let one of these players, say i_0, form link with all other players. Then we obtain an center-sponsored star. If for all $i \in N \setminus \{i_0\}$, $(\delta - \delta^2)V_i \leq c$, then the previous network is Nash, otherwise, we let all players $i \in N \setminus \{i_0\}$ such that $(\delta - \delta^2)V_i > c$ be connected to all other players (obviously, in that case we choose i_0 such that $(\delta - \delta^2)V_{i_0} > c$), and we obtain a Nash network.

□

Notice that the above proof also identifies conditions under which we can obtain some simple architectures like stars as Nash networks. Our next result shows that this is not true for the decay model with heterogeneous players and identical costs of link formation. Note that it is also possible to modify the next example to incorporate heterogeneity in link formation costs and still achieve the same outcome.

Example 3 *(Non-existence of Nash networks.)* Let $N = \{1, \ldots, 5\}$ be the set of players, and assume that:

1. $V_{1,2}\left(\delta-\delta^4\right)+V_{1,3}(\delta^2-\delta^3) > c$, $\delta V_{1,3} < \delta V_{1,2} < c$, and for all $j \neq 2$, $\delta V_{1,j}+\delta^2\sum_{k\neq j} V_{1,k} < c$.

2. $V_{2,3}\left(\delta-\delta^4\right)+V_{2,4}(\delta^2-\delta^3) < c$, $\delta V_{2,3}+\delta^2 V_{2,4}+\delta^3 V_{2,5}+\delta^4 V_{2,1} > c$, and for all $j \neq 3$, $\delta V_{2,j}+\delta^2\sum_{k\neq j} V_{2,k} < c$.

3. $\delta V_{3,4} > c$ and $\delta \sum_{k\neq 4} V_{3,k}+\delta^2 V_{3,4} < c$.

4. $\delta V_{4,5} > c$ and $\delta \sum_{k\neq 5} V_{4,k}+\delta^2 V_{4,5} < c$.

5. $\delta V_{5,1} > c$ and $\delta \sum_{k\neq 1} V_{5,k}+\delta^2 V_{5,1} < c$.

These five points provide a list of the players with whom each of the others has no incentives to form links, as well as those who with whom they would like to form links. For example, the first point implies that player 1 will never form a link with players 3, 4 and 5. Moreover, a Nash network must contain the links 3 4, 4 5, 5 1. From all of this, it follows that there is four possible Nash networks: $E(\boldsymbol{g}^1) = (3\,4, 4\,5, 5\,1, 1\,2, 2\,3)$, $E(\boldsymbol{g}^2) = (3\,4, 4\,5, 5\,1, 1\,2)$, $E(\boldsymbol{g}^3) = (3\,4, 4\,5, 5\,1)$, $E(\boldsymbol{g}^4) = (3\,4, 4\,5, 5\,1, 2\,3)$. We know by point 2. that player 2 prefers the network $\boldsymbol{g}^2$ to the network $\boldsymbol{g}^1$, so $\boldsymbol{g}^1$ is not Nash. Likewise, player 1 prefers the network $\boldsymbol{g}^3$ to the network $\boldsymbol{g}^2$ by point 1, so $\boldsymbol{g}^2$ is not Nash. Player 2 prefers $\boldsymbol{g}^4$ to $\boldsymbol{g}^3$ by point 2, so $\boldsymbol{g}^3$ is not Nash. Finally, by point 1 player 1 prefers the network $\boldsymbol{g}^1$ to the network $\boldsymbol{g}^4$. Hence $\boldsymbol{g}^4$ is not Nash.

4.2 Decay with Heterogeneous Links

In this section we consider situations where players have homogeneous values while the decay of each link is different. We obtain the following result.

Theorem 4 *: (Decay with heterogeneous links.) Let* $\boldsymbol{g}$ *be an essential network. If the payoff function satisfies equation (8) and costs of setting links are homogeneous, then there exist* $c > 0$ *and an array* $\boldsymbol{\delta} = [\delta_{i,j}]$ *of decay such that:*

1. *$\boldsymbol{g}$ is a strict Nash network in the corresponding network formation game;*

2. *$\boldsymbol{g}$ is an efficient network in the corresponding network formation game.*

Proof We successively prove the two parts of the proposition.

1. Let $\boldsymbol{g}$ be an essential network. For $g_{i,j} = 1$, let $c < \left(\delta_{i,j} - (\delta^m)^2\right) V$, with $\delta^m = \max_{(i',j') \in N^2}\{\delta_{i',j'}\}$. Also, for $g_{i,j} = 0$, let $c > (\delta_{i,j} + \delta^m \delta_{i,j}(n-2))V$. It can be checked that these two conditions are compatible. We note that under these conditions a player i, who has formed a link with player j, has no incentive to remove it. Indeed, the situation where player i has the greatest incentive to delete a link with j occurs when she obtains the resources of j from a player k such that $\max\{g_{k,i}, g_{i,k}\} = \max\{g_{k,j}, g_{j,k}\} = 1$. The condition which allows player i to maintain her link with j is: $c < (\delta_{i,j} - \delta_{i,k}\delta_{j,k}) V$. This condition is always true if $c < \left(\delta_{i,j} - (\delta^m)^2\right) V$. Likewise, a player i who has not formed a link with a player j has no incentive to form a link with her. Indeed, a player j can provide at most information of value $(\delta_{i,j} + \delta^m \delta_{i,j}(n-2))V$ to player i.

2. The proof of the second part of the proposition is similar to the previous part, but now we assume that if $g_{i,j} = 0$, then $c > \delta_{i,j} + \delta_{j,i} + (\delta^m)^2 (n-2)nV$ (given that $g_{j,i} = 0$). Since the two conditions are again compatible, we can conclude.

□

Again based on therange of c in the proof it should be obvious that Nash and efficient networks does not always coincide. Moreover, if we assume that decay begins with indirect neighbors, then we can construct the following example where regardless of the value of the parameters, some essential networks are neither Nash nor efficient.

Example 4 Let $N = \{1, 2, 3, 4\}$ be the set of players, let $\boldsymbol{g}$ be a network such that $E(\boldsymbol{g}) = \{1\,2\}$. Then, $\boldsymbol{g}$ is not a Nash network. Indeed, if player 1 has an incentive to form a link with player 2, then $V < c$. In that case, player 3 has an incentive to form a link with player 1. Likewise $\boldsymbol{g}$ is not an efficient network.

Existence of Nash equilibrium. The question of existence of Nash equilibria in the models with decay and heterogeneous links remains an open question. However, if we add the heterogeneity of costs to the heterogeneity of links, then it is possible to adapt an argument from Haller, Kamphorst and Sarangi (2006, [5]) to show that Nash equilibria do not always exist. Indeed, they show in example 2 that there exist situations with $\delta = 1$ where links are perfectly reliable, values are homogeneous and costs are heterogeneous, such that there does not exist any Nash network. Hence, by continuity it is possible to construct a similar example with δ sufficiently close to 1 where Nash equilibria will not exist.

5 Relationship between Probabilistic and Decay Models

The probabilistic model uses all paths between two players to compute payoffs while the decay model only uses the shortest path between the two players to determine payoffs. This seems to suggest that decay models might be a subset of the probabilistic models. In this section we ask if information about equilibrium networks in probabilistic models gives some indication about equilibrium networks in decay models. To address this question, we compare marginal payoffs of links in both types of models.[6] In order to make easier the comparison we assume that

[6]Note that in the probabilistic model players' marginal payoffs are expected marginal payoffs. However, for both types of models, we use the term marginal payoffs to make reading easier. Moreover, we

when a player forms a link in the empty network, the marginal payoffs of this link are the same in both models, that is we set $\delta = p$.

We first show that when the initial network is minimal, that is when there is at most one path between two players in the network, the marginal payoff of a link is always at least as great in the probabilistic model as in the decay model.

Indeed, suppose that in a minimal network, denoted by $\boldsymbol{g}^1$, one player, say player i, forms a link with another player, say player j. Denote by $\boldsymbol{g}^2$ the resulting network. Either j is not observed by player i in $\boldsymbol{g}^1$ and it is obvious that the marginal payoff of the link $i\,j$ is the same in both models, or j is observed by i in $\boldsymbol{g}^1$. In the latter case, in the decay model, player i being at distance 1 from player j in $\boldsymbol{g}^2$, she obtains an amount p of resources of player j. In the probabilistic model, i accesses to the resources of j in $\boldsymbol{g}^2$ if the link $i\,j$ works, that occurs with a probability p. She also accesses to the resources of j even though the link $i\,j$ does not work. It is enough that all the links which were contained in a path from j to i in $\boldsymbol{g}^1$ work, that occurs with a positive probability. So, the amount of resources of player j obtained by player i in $\boldsymbol{g}^2$ is greater than p. With the same type of reasoning, we can show that the part of the ressources of players $k \neq j$, obtained by i in $\boldsymbol{g}^2$, is at least as great in the probabilistic model than in the decay model. The result follows. From this result, it is straightforward that *a minimally connected Nash network in the probabilistic model is also a Nash network in the decay model.*

Next what happens if the initial network is not minimal? The example which follows shows that the above result does not hold anymore.

Example 5 Let $N = \{1, 2, 3, 4\}$ be the set of players and let $\boldsymbol{g}^1$ be a network such that $E\{\boldsymbol{g}^1\} = (1\,2, 2\,3, 3\,4, 4\,1)$.

assume that players in the probabilistic model are risk neutral.

Suppose that in $\boldsymbol{g}^1$ player 1 forms a link with player 3. We can check that for some p, for instance for $p = 0.8$, the marginal payoff of this link is greater in the probabilistic model than in the decay model, whereas the converse is true for some other p, for instance for $p = 0.9$.

Recall that if the initial network is minimal, then the marginal payoff of a link is always as great in the probabilistic model as in the decay model. This difference in the result can be explained in the following way.

Suppose that the initial network, denoted by $\boldsymbol{g}^1$, is not minimal. Then, there exist in $\boldsymbol{g}^1$ at least two players, say i and j, such that there are at least two paths between these two players. Let player i form a link with player j in $\boldsymbol{g}^1$ and denote by $\boldsymbol{g}^2$ the resulting network. Although the total payoff of player i in $\boldsymbol{g}^2$ is greater in the probabilistic model than in the decay model, this does not imply that the marginal payoff of the link $i\,j$ is greater in the probabilistic model than in the decay model. Indeed, it is easy to check that, in $\boldsymbol{g}^1$, player i also gets a greater payoff in the probabilistic model than in the decay model.
Actually, when the initial network is not minimal, the difference in the marginal payoff of a link $i\,j$ depends on the architecture of the network (in particular the number of paths that exist between player i and the other players from whom i obtains resources in the initial network) and on the probability that a link works. This makes it difficult to find a general rule which orders the marginal payoff of a link in both models. Thus, when the number of players is greater than 3 and the initial network is not minimal, information about equilibrium networks in one type of models does not provide any indications about equilibrium networks in the other type of models.

6 Discussion

We now sum up the main insights obtained from the introduction of heterogeneity. Table 1 provides an overview of the results. The first column indicates the scope of strict Nash networks and the second column does the same for efficient networks. The third column indicates whether existence of Nash networks is always guaranteed.

In going from a deterministic model with homogeneous parameters to a homogeneous probabilistic link failure model, Bala and Goyal (2000 [1], [2]) find that Nash networks change from being minimally connected to super-connected. More precisely, they find that the strict Nash networks change from being empty and center-sponsored stars to being empty and connected networks. This is also true when we allow for decay with homogeneous parameters. In the Galeotti, Goyal and Kamphorst (2006, [4]) formulation, that allows for heterogeneity in values and costs in the deterministic framework, empty and minimal networks with center-sponsored stars can be supported as Nash. Moreover, the authors show that only minimal networks can be Nash. In contrast, heterogeneity in imperfect reliability models, whether it is of the heterogeneous links type à la Haller and Sarangi (2005, [6]), or heterogeneous player type as shown in this paper, always yields an "anything goes" result implying that any network can be sustained as strict Nash by an appropriate set of parameters. We find that the same analysis holds when we introduce heterogeneity in models with decay. Moreover, Table 1 shows that a similar trend holds for efficiency. The key insight here is that when heterogeneity requires agents to take into account alternative paths between agents instead of just affecting values and costs of link formation, then it is possible to obtain a richer set of networks.

Existence of equilibrium in the different models is a more tricky issue. Bala

and Goyal (2000, [1]) show the existence of equilibrium in the homogeneous deterministic framework through a constructive proof. Haller, Kamphorst and Sarangi (2005, [5]) show that in the deterministic setting there always exists a Nash network if costs of setting links are homogeneous and values of players are heterogeneous. They also prove that this result does not hold if costs of forming links are heterogeneous. Interestingly, with homogeneous link success probabilities under identical values and costs (Bala and Goyal, 2000, [2]), the existence of equilibrium remains an open question. However, with the introduction of heterogeneity of any type in imperfect reliability models, it is possible to show that a Nash equilibrium does not exist for all parameter values.
In the model of decay with homogeneous parameters we find that Nash equilibrium always exists. However, when we introduce costs heterogeneity, with either value or decay heterogeneity, then Nash networks do not always exist. Once again we see that in models involving alternative paths, non-existence is likely. Further, it is also clear that in models allowing for heterogeneity in costs of link formation we have to be cautious about existence issues.

Finally, it is important to ask whether the richness of results stems from degrees of freedom in choosing model parameters made possible by heterogeneity.[7] We note that the introduction of heterogeneity concerning only one of all parameters (values, costs, reliability, decay) dramatically increases the set of Nash networks. For instance, with heterogeneous links reliability or heterogeneous decay, each essential network can be a Nash network. Second, this result holds even if the heterogeneous parameter takes few differents values and these values are very close one from each other. Thus a key result of this paper is that the Bala and Goyal's model is not quite robust to the introduction of various types of heterogeneity.

[7]See also Haller and Sarangi (2005, [6]) for a discussion of this.

	Strict Nash networks	**Efficient Networks**	**Existence**
Models with perfect reliability and no decay			
Homogeneous Values and Costs	Empty network, Center-sponsored stars	Empty network, Center-sponsored stars Minimally connected networks	Yes
Heterogeneous Values only	Empty network and minimal networks in which every non-singleton component is a Center-sponsored star	Empty network, Minimal networks	Yes
Heterogeneous Costs only	Empty networks, Minimal networks	Empty network, Minimal networks	No
Imperfect Reliability Models			
Homogeneous Values, Costs and Reliability	Empty network, Connected networks	Empty network, Connected networks	unresolved
Homogeneous Values, Costs and Heterogeneous reliability	Essential networks	Essential networks	No
Heterogeneous Values and/or Costs and Homogeneous reliability	Essential networks	Essential networks	No
Decay Models			
Homogeneous decay and Homogeneous Values, Costs	Empty network, Connected networks	Empty network, Star networks, and Complete network	Yes
Homogeneous Decay and Heterogeneous values (or costs)	Essential networks	Essential networks	No
Heterogeneous Decay and Homogeneous values (or costs)	Essential networks	Essential networks	unresolved
Heterogeneous Decay and Costs, either homogeneous or heterogeneous Values	Essential networks	Essential networks	No

Table 1: Two-way flow models: Results.

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