

Ordinal Games*

JACQUES DURIEU^a, HANS HALLER^b, NICOLAS QUEROU^c, PHILIPPE SOLAL^a

August 2007

^a CREUSET, University of Saint-Etienne, 42023 Saint-Etienne, France.

^b Department of Economics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0316, USA.

^c Department of Economics, Queen's University, Belfast, Northern Ireland, UK.

Abstract. We study strategic games where players' preferences are weak orders which need not admit utility representations. First of all, we extend Voorneveld's concept of best-response potential from cardinal to ordinal games and derive the analogue of his characterization result: An ordinal game is a best-response potential game if and only if it does not have a best-response cycle. The characterization implies equilibrium existence results. Further, Milgrom and Shannon's concept of quasi-supermodularity is extended from cardinal games to ordinal games. We find that under certain compactness and semicontinuity assumptions, the ordinal Nash equilibria of a quasi-supermodular game form a nonempty complete lattice. Finally, we extend several set-valued solution concepts from cardinal to ordinal games in our sense.

Keywords: Ordinal Games, Potential Games, Quasi-Supermodularity, Rationalizable Sets, Sets Closed under Behavior Correspondences

JEL Classification: C72

* We thank Amoz Kats for comments and Gianni Bosi for references.

1 Introduction

The purpose of this paper is to derive general results for certain classes of ordinal games. It is motivated by the increasingly important role that ordinal properties have played in game-theoretic analysis in recent years. First of all, the introduction of supermodular games by Topkis (1979) and the subsequent analysis by Bulow, Geanakoplos and Klemperer (1985), Vives (1990), and Milgrom and Roberts (1990) gave rise to an important new strand of literature in economics and game theory. But strategic complementarity, the key ingredient of supermodularity, is essentially an ordinal concept and much of the theory of supermodular games can be formulated in ordinal terms. Second, since the seminal paper by Monderer and Shapley (1996), a sizeable strand of literature on potential games has emerged. Monderer and Shapley already distinguish between exact, weighted, and ordinal potentials for cardinal games. Kukushkin (1999) and Norde and Patrone (2001) have introduced the concept of ordinal potential for ordinal games.

Ordinality in strategic games stands for two different, not mutually exclusive concepts. On the one hand, within the confines of traditional game theory, an ordinal perspective abstracts from particular utility representations (payoff functions). It considers invariant properties with respect to utility representations. It identifies games having the same game form and identical ordinal preferences or identical best response correspondences. More generally, it investigates isomorphisms and equivalence classes of games. For the ordinal perspective of games, see the contributions of Thompson (1952), Mertens (1987, 2003), Vermeulen and Jansen (2000), and Morris and Ui (2004). On the other hand, the concept of ordinal games transcends tra-

ditional game theory and allows for players' preferences which do not admit utility representations. Many contributions to demand theory and general equilibrium theory consider incomplete or intransitive preferences. Sonnenschein (1971), Shafer (1974), Kim and Richter (1986) made pioneering contributions to demand theory with incomplete or intransitive preferences. Schmeidler (1969), Shafer and Sonnenschein (1975), Bergstrom (1976), Borglin and Keiding (1976), Shafer (1976) belong to the early contributors to general equilibrium theory with incomplete or intransitive preferences. To the extent that the work of these and subsequent authors deals with abstract economies (generalized games, pseudo-games), it applies to ordinal games as well.

Incomplete or intransitive preferences constitute an important, but not the only class of preferences without utility representations. Specifically, the present paper deals with ordinal games where players' preference relations are weak orders: Players' preferences are complete and transitive, yet need not admit utility representations. As mentioned above, both potential games and supermodular games lend themselves to ordinal analysis. Notice that in contrast to arbitrary finite games, both finite potential games and finite supermodular games always possess a Nash equilibrium in pure strategies. In both cases, a purely ordinal approach can be taken. The arguments differ from the standard equilibrium existence proofs by means of the Brouwer, Kakutani, or Fan-Glicksberg fixed point theorem, which require a topological vector space structure and continuity in that topology.

To begin with, we introduce the concept of ordinal Nash equilibrium for ordinal games. In the case of potential games, no fixed point theorem is

needed. We extend Voorneveld's (2000) concept of best-response potential from cardinal games to ordinal games in our sense and derive the analogue of his characterization result: An ordinal game is a best-response potential game if and only if it does not have a best-response cycle. As corollary, we obtain that if an ordinal game is a potential game, then it is a best-response potential game. In the case of supermodular games, one can resort to the lattice-theoretic, non-topological fixed point theorem of Zhou (1994). Next Milgrom and Shannon's (1994) concept of quasi-supermodularity is extended from cardinal games to ordinal games in our sense. We find that under certain compactness and semicontinuity assumptions, the ordinal Nash equilibria of a quasi-supermodular game form a nonempty complete lattice. As an immediate corollary, one obtains that the ordinal Nash equilibria of a finite quasi-supermodular game form a nonempty complete lattice.

Finally, we extend several set-valued solution concepts from cardinal to ordinal games in our sense. The definition of rationalizability follows Pearce (1984), with attention confined to pointwise beliefs, but not restricted to subsets of Euclidean spaces. We prove the existence of a nonempty and compact subset of rationalizable joint strategies in ordinal games where each individual strategy set is a compact Hausdorff space and all preference relations are continuous. The definition of a closed set under a behavior relation is an adaptation of Ritzberger and Weibull's (1995) concept of a closed set under a behavior correspondence, again without the restriction to subsets of Euclidean spaces. We demonstrate the existence of a minimal closed set under a behavior correspondence for the class of ordinal games where each strategy set is a compact Hausdorff space.

In sum, the contribution of this paper is two-fold: As a methodological advance, all concepts, assertions, and derivations are formulated in purely ordinal terms. Moreover, we generalize several previous results by relaxing the restrictions imposed in the literature: In addition to weaker assumptions regarding strategy spaces and preferences in some instances, finiteness of the player set is not assumed in Theorems 1, 3–5.

The next section contains the basic definitions regarding weak orders and ordinal games. We also elaborate on the fact that our assumptions on strategy sets and preferences are more general than the stated assumptions for cardinal games. Section 3 is devoted to potential games. Section 4 is about quasi-supermodular games. Section 5 is about set-valued concepts. Section 6 offers final remarks.

2 Preliminaries

We first collect some definitions and properties pertaining to preference relations, in particular weak orders. We then define ordinal games and related concepts, in particular ordinal Nash equilibria. For two nonempty sets S and Y , $\psi : S \twoheadrightarrow Y$ denotes a relation from S to Y , that is a mapping $\psi : S \rightarrow 2^Y$ that assigns to each $s \in S$ a subset $\psi(s)$ of Y . ψ is called a correspondence if $\psi(s) \neq \emptyset$ for all $s \in S$.

2.1 Weak Orders

Let X be a nonempty set. A binary relation \succeq on X is called a **weak order**, if it is transitive and strongly complete. The latter means that $x \succeq y$ or $y \succeq x$ for all $x, y \in X$. For a weak order \succeq on X , its asymmetric part \succ ,

defined by

$$x \succ z : \iff [x \succeq z \& \neg(z \succeq x)]$$

for all $x, z \in X$, is irreflexive and transitive, and its symmetric part \sim , defined by

$$x \sim z : \iff [x \succeq z \& z \succeq x]$$

for all $x, z \in X$, is an equivalence relation, that is reflexive, symmetric and transitive. A weak order \succeq is called a **total order**, if it is antisymmetric, that is $x \sim z \implies x = z$ for all $x, z \in X$. \succeq has or admits a utility representation if there exists a function $u : X \rightarrow \mathbb{R}$ such that $x \succeq y \iff u(x) \geq u(y)$ for $x, y \in X$. Then u is called a utility or payoff function representing \succeq or a **utility representation** of or for \succeq .

In case X is endowed with a topology τ , we say that \succeq is **upper semicontinuous** if the sets $\{x \in X | x \prec z\}, z \in X$, are open and \succeq is **lower semicontinuous** if the sets $\{x \in X | x \succ z\}, z \in X$, are open. \succeq is **continuous** if it is both upper and lower semicontinuous. The **order topology** on X induced by \succeq has the sets $\{x \in X | x \succ z\}, z \in X$, and $\{x \in X | x \prec z\}, z \in X$, as a subbase of open sets. By definition, \succeq is continuous in its order topology.

An element $z \in X$ is called a **maximal element** of the binary relation \succeq on X , if $\{x \in X | x \succ z\} = \emptyset$. z is a greatest element, if $z \succeq x$ for all $x \in X$. In the case of a weak order, maximal and greatest elements coincide. For convenient reference, we state the following well known fact.

Lemma 1 *Let (X, τ) be a compact topological space and \succeq be an upper semicontinuous weak order on X . Then the set of maximal elements of \succeq is nonempty and compact.*

GENERALITY OF ASSUMPTIONS. It is important to note that the lemma applies in instances where \succeq does not admit a utility representation, like in the following example.

Example 1. Let $X = [0, 1] \times \{0, 1\}$ be endowed with the following total order \succeq which is the restriction of the lexicographic order on \mathbb{R}^2 to $[0, 1] \times \{0, 1\}$:

$$\text{For } x, y \in [0, 1], \text{ with } x > y, (x, 1) \succ (x, 0) \succ (y, 1) \succ (y, 0).$$

Let τ be the order topology induced by \succeq . Then the following hold:

- (a) (X, τ) is a compact Hausdorff space.
- (b) \succeq is continuous with respect to τ .
- (c) \succeq does not have a utility representation.

PROOF. (a) For $(x, a), (y, b) \in X$ with $x > y$, $\{\chi \in X \mid \chi \succ ((x+y)/2, 0)\}$ and $\{\chi \in X \mid \chi \prec ((x+y)/2, 0)\}$ are disjoint open sets with $(x, a) \in \{\chi \in X \mid \chi \succ ((x+y)/2, 0)\}$ and $(y, b) \in \{\chi \in X \mid \chi \prec ((x+y)/2, 0)\}$. For $(x, 1), (x, 0) \in X$, $\{\chi \in X \mid \chi \succ (x, 0)\}$ and $\{\chi \in X \mid \chi \prec (x, 1)\}$ are disjoint open sets with $(x, 1) \in \{\chi \in X \mid \chi \succ (x, 0)\}$ and $(x, 0) \in \{\chi \in X \mid \chi \prec (x, 1)\}$. This shows that (X, τ) is Hausdorff.

If (X, \succeq) is order-complete, i.e. if every non-empty subset of X with an upper bound has a supremum, then every closed and bounded subset of X is compact in the order topology. See Problem 5.C in Kelley (1955). Now every subset of X is bounded. Thus, if we can show that every non-empty subset of X has a supremum, then compactness of (X, τ) is demonstrated.

For a non-empty subset A of X , let $A_1 = \{x \in [0, 1] : (x, 0) \in A \text{ or } (x, 1) \in$

$A\}$. In case $(\sup A_1, 1) \in A$, $(\sup A_1, 1)$ is the supremum of A . In case $(\sup A_1, 1) \notin A$, $(\sup A_1, 0)$ is the supremum of A . In any case, A has a supremum. This shows that (X, \succeq) is order-complete and, consequently, (X, τ) is compact.

(b) By definition, \succeq is continuous in its order topology.

(c) \succeq has a continuum of “gaps” of the form $((x, 0), (x, 1))$, $x \in [0, 1]$. Therefore, it does not have any (continuous or discontinuous) utility representation. ■ ■

Amoz Kats has suggested the following interpretation of the example. A voter has preferences over pairs (x, c) where $x \in [0, 1]$ is the platform chosen by a political candidate and $c \in \{0, 1\}$ is one of the two candidates. If the voter has the choice between two different platforms, the identity of the candidates does not matter. If both candidates offer the same platform, then the voter has a preference for candidate 1.

The relation \succeq in Example 1 is the restriction of the lexicographic order on \mathbb{R}^2 to X . The restriction of the lexicographic order to the unit square $[0, 1]^2$ creates a similar example. A further example with properties (a)-(c) can be generated by means of the well ordering principle.

Example 2. Namely, let \succ be a well order on \mathbb{R} , i.e. \succ is a total order such that every nonempty subset of \mathbb{R} has a minimum. For each $r \in \mathbb{R}$, let $I_r = \{x \in \mathbb{R} : r \succ x\}$. Let $R = \{r \in \mathbb{R} : I_r \text{ is uncountable}\}$. If $R = \emptyset$, set $\Omega = \mathbb{R}$. If $R \neq \emptyset$, set $\Omega = I_{\min R}$. Then \succ induces a well order on Ω which does not have a utility representation. Choose an element $\omega^* \notin \Omega$, set $X = \Omega \cup \{\omega^*\}$ and extend the well order to X by postulating $\omega^* \succ \omega$ for all

$\omega \in \Omega$. A well order \succeq renders (X, \succeq) order-complete so that one can follow the pattern of proof of Example 1. ■ ■

The literature has been mostly concerned with the existence of continuous or upper semi-continuous utility representations for continuous or upper semi-continuous weak orders. The key results of the seminal contributions of Eilenberg (1941), Debreu (1954), Rader (1963) can be summarized as follows: *Let X be a topological space which is second countable (has a countable base of open sets) or is separable and connected. If \succeq is a continuous weak order on X then \succeq has a continuous utility representation.*¹ Beardon *et al.* (2002) provide a classification of total orders which do not admit a utility representation. The lexicographic order is a prominent example of the so-called “planar type”. Example 2 is of the so-called “long type”. Estévez Toranzo and Hervés Beloso (1995) show that if $X \neq \emptyset$ is a non-separable metric space, then there exists a continuous weak order on X which cannot be represented by a utility function. However, this result becomes obsolete under the compactness assumption of Lemma 1: If X is a compact metric space then it is separable and second countable.

2.2 Ordinal Games

Let $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ denote an **ordinal non-cooperative game** with the following interpretation and properties:

- $N \neq \emptyset$ denotes the set of players.
- Each player $i \in N$ has a nonempty set X_i of strategies. $X = \prod_{i \in N} X_i$ represents the set of joint strategies or strategy profiles.

¹For generalizations and variations, see Rader (1963), Monteiro (1987), Candeal, Hervés, and Induráin (1998), Bosi and Mehta (2002).

- Each player $i \in N$ has a binary relation \succeq_i over the joint strategy set X , which reflects his preferences over the outcomes of the game G . Each binary relation \succeq_i is assumed to be a weak order.

We denote $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$. For a player $i \in N$ and a joint strategy $x = (x_j)_{j \in N} \in X$, we write $x_{-i} = (x_j)_{j \neq i} \in X_{-i}$ and, with slight abuse of notation, $x = (x_i, x_{-i}) \in X$. For every player $i \in N$ and every joint strategy of his opponents $x_{-i} \in X_{-i}$,

$$M_i(x_{-i}) = \{x_i \in X_i \mid \nexists z_i \in X_i : (z_i, x_{-i}) \succ_i (x_i, x_{-i})\}$$

is the set of **best responses**, that is the set of maximal strategies for i under \succeq_i against x_{-i} . Note that every maximal element is a greatest element because \succeq_i is a weak order. For every player $i \in N$ and every joint strategy $x = (x_i, x_{-i}) \in X_i \times X_{-i}$, we denote by

$$B_i(x) = \{z_i \in X_i \mid (z_i, x_{-i}) \succeq_i (x_i, x_{-i})\}$$

the **set of better responses** or **upper contour set** — that is the set of strategies that player i weakly prefers to x_i , given x or x_{-i} .

Let $B : X \rightarrow X$, $x \mapsto \prod_{i \in N} B_i(x)$ be the joint better-response relation. Let $M : X \rightarrow X$, $x \mapsto \prod_{i \in N} M_i(x_{-i})$ be the joint best-response relation which maps each joint strategy to its joint best-responses. The set of **ordinal Nash equilibria** of G is defined by

$$\mathcal{N}(G) = \{x \in X \mid x \in M(x)\}$$

When appropriate, we shall consider each strategy set X_i endowed with a topology. For the remainder of this paragraph, suppose each individual strategy space X_i is endowed with a topology τ_i and $X = \prod_i X_i$ is endowed with

the corresponding product topology. We say that \succeq_i is **upper semicontinuous** on X_i for every $x_{-i} \in X_{-i}$ if the set of better responses $B_i(x_i, x_{-i})$ is a closed subset of X_i for every $x = (x_i, x_{-i}) \in X$. Clearly, if a preference \succeq_i is continuous on X then it is upper semicontinuous on X and upper semicontinuous on X_i for every $x_{-i} \in X_{-i}$.

3 Ordinal Potential Games

Monderer and Shapley (1996) develop the concept of potential for cardinal games $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ in both cardinal and ordinal versions. Monderer and Shapley (1996), Voorneveld and Norde (1997) and Ui (2000) characterize several classes of potential games. However, their approach is biased towards the cardinal side since it requires the potential to be a real-valued function, while from a strictly ordinal viewpoint, the potential provides an order. Consequently, Kukushkin (1999) and Norde and Patrone (2001) have introduced the concept of ordinal potential for ordinal games. An ordinal game has a potential if there exists a **quasi-order** on X , that is a reflexive and transitive binary relation \succeq , containing the preferences of all players: $(x_i, x_{-i}) \succeq_i (z_i, x_{-i}) \iff (x_i, x_{-i}) \succeq (z_i, x_{-i})$ for all $i \in N$, $x_i, z_i \in X_i$, $x_{-i} \in X_{-i}$.

Voorneveld (2000) introduces and studies best-response potential games, a new class of potential games. A game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$, is a best-response potential game if there exists a real-valued function $P : X \rightarrow \mathbb{R}$ such that for every $i \in N$, $x_{-i} \in X_{-i}$, we have

$$\arg \max_{x_i \in X_i} u_i(x_i, x_{-i}) = \arg \max_{x_i \in X_i} P(x_i, x_{-i})$$

Here, we adapt his definition for ordinal games. An ordinal game $G =$

$(N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ is a **best-response potential game** if there exists a quasi-order \succeq on X , such that for every $i \in N$, $x_{-i} \in X_{-i}$, we have

$$M_i(x_{-i}) = M_{\succeq}(x_{-i})$$

where $M_{\succeq}(x_{-i})$ denotes the set of greatest elements of \succeq over X_i given $x_{-i} \in X_{-i}$. It is clear from the definition of a best-response potential that (i) each greatest element of \succeq is an ordinal Nash equilibrium of G ; (ii) the set of ordinal Nash equilibria of G coincides with the set of ordinal Nash equilibria of the ordinal game $(N, (X_i)_{i \in N}, (\succeq)_{i \in N})$ provided \succeq is a weak order on X .

3.1 Characterization of best-response potentials

Voorneveld (2000) provides a characterization of best-response potential games. We implement in the ordinal setting the ideas introduced in Voorneveld (2000) and Norde and Patrone (2001). First we need the following definition.

Let $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ be an ordinal game. A **path** in the set of joint strategies X is a sequence (x^1, x^2, \dots) of elements $x^k \in X$ such that for all $k = 1, 2, \dots$, the joint strategies x^k and x^{k+1} differ in exactly one, say the $i(k)^{th}$, component. A path is **best-response compatible** if the deviating player moves to a best response, that is

$$\forall k = 1, 2, \dots : x_{i(k)}^{k+1} \in M_{i(k)}(x_{-i(k)}^k)$$

By definition the trivial path (x^1) consisting of a single joint strategy $x^1 \in X$ is best-response compatible. A finite path (x^1, x^2, \dots, x^m) is called a **best-response cycle** if it is best-response compatible, $x^1 = x^m$, and $x^{k+1} = (x_{i(k)}^{k+1}, x_{-i(k)}^k) \succ_{i(k)} x^k$ for some $k \in \{1, \dots, m-1\}$.

Theorem 1 *An ordinal game $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ is a best-response potential game if and only if there is no best-response cycle.*

PROOF. (\Rightarrow) Assume that G is an ordinal game with best-response potential $\underline{\triangleright}$. Suppose that (x^1, x^2, \dots, x^m) is a best-response cycle. Define by \triangleright the asymmetric part of $\underline{\triangleright}$ on X , that is for all $x, z \in X$,

$$x \triangleright z : \iff [x \underline{\triangleright} z \ \& \ \neg(z \underline{\triangleright} x)]$$

By definition, $x_{i(k)}^{k+1} \in M_{i(k)}(x_{-i(k)}^k)$ for all $k \in \{1, \dots, m-1\}$ and $x^{k+1} = (x_{i(k)}^{k+1}, x_{-i(k)}^k) \succ_{i(k)} x^k$ for at least one such k . So $x^{k+1} \underline{\triangleright} x^k$ for all $k \in \{1, \dots, m-1\}$ and $x^m \underline{\triangleright} x^1$ by transitivity. But since there is at least one k such that $x^{k+1} \succ_{i(k)} x^k$ we necessarily have $x^{k+1} \triangleright x^k$ for this k by definition of a best-response potential. It follows that $x^m \underline{\triangleright} x^{k+1} \triangleright x^k \underline{\triangleright} x^1$ which in turn implies that $x^m \triangleright x^1$, contradicting the fact that $x^1 = x^m$.

(\Leftarrow) Suppose that G has no best-response cycle. Define the binary relation $(X, \underline{\triangleright})$ as follows:

$$(\forall x, z \in X) : z \underline{\triangleright} x : \iff [\exists \text{ a best-response compatible path from } x \text{ to } z]$$

First note that the binary relation $\underline{\triangleright}$ on X is reflexive and transitive; i.e., it is a quasi-order. We have to show that $M_i(x_{-i}) = M_{\underline{\triangleright}}(x_{-i})$ for every $i \in N$ and $x_{-i} \in X_{-i}$. Let $i \in N, x_{-i} \in X_{-i}$.

(a) Pick $z_i \in M_i(x_{-i})$. Since \succeq_i is strongly complete, for all $x_i \in X_i$ we have that $(z_i, x_{-i}) \succeq_i (x_i, x_{-i})$, hence the path $((x_i, x_{-i}), (z_i, x_{-i}))$ is best-response compatible and $(z_i, x_{-i}) \underline{\triangleright} (x_i, x_{-i})$. Therefore, $z_i \in M_{\underline{\triangleright}}(x_{-i})$. This observation implies that $M_i(x_{-i}) \subseteq M_{\underline{\triangleright}}(x_{-i})$.

(b) Pick $z_i \in M_{\underline{\triangleright}}(x_{-i})$. Suppose $z_i \notin M_i(x_{-i})$. Then there exists $x_i \in X_i$ such that $(x_i, x_{-i}) \succ_i (z_i, x_{-i})$. By the absence of a best-response cycle,

it cannot be the case that $(z_i, x_{-i}) \succeq (x_i, x_{-i})$, contradicting $z_i \in M_{\succeq}(x_{-i})$. We conclude that $z_i \in M_i(x_{-i})$. From this observation we get $M_i(x_{-i}) \supseteq M_{\succeq}(x_{-i})$.

The assertion that (X, \succeq) is a best-response potential for G follows from (a) and (b). ■ ■

Voorneveld (2000, Theorem 3.1) proves in a cardinal setting that a best-response potential exists if and only if: (i) X contains no best-response cycles and (ii) the quotient of (X, \succeq) can be represented by a real-valued function. Theorem 1 states that from a purely ordinal perspective, condition (ii) has nothing to do with best-response potential games. A similar remark has been made by Norde and Patrone (2000) for the class of ordinal potential games.

Finally note that if (x^1, x^2, \dots, x^m) is a best-response cycle, then it is a weak improvement cycle in the sense of Norde and Patrone (2001), that is $x^{k+1} = (x_{i(k)}^{k+1}, x_{-i(k)}^k) \succeq_{i(k)} x^k$ for all $k \in \{1, \dots, m-1\}$, $x^1 = x^m$, and $x^{k+1} = (x_{i(k)}^{k+1}, x_{-i(k)}^k) \succ_{i(k)} x^k$ for some $k \in \{1, \dots, m-1\}$. A weak improvement path is thus a path such that at each iteration one player is drawn from the population to play a better response against the current joint strategy of his opponents. Theorem 2.2 in Norde and Patrone (2001) states that an ordinal game G is a potential game if and only if G contains no weak improvement cycle. We obtain an immediate corollary — which is also an obvious consequence of the definitions:

Corollary 1 *If an ordinal game G is a potential game, then it is a best-response potential game.*

3.2 Existence of Ordinal Nash Equilibria

If we consider best-response potential games in which all but one player have a finite set of strategies, and if we equip the only infinite strategy set with a topology we obtain the following result.

Theorem 2 *Let $G = (N, (X_j)_{j \in N}, (\succeq_j)_{j \in N})$ be a best-response potential game with N finite. If for some $i \in N$, X_j is a finite set for every $j \neq i$, X_i is a compact topological space and \succeq_i is upper semicontinuous on X_i for each $x_{-i} \in X_{-i}$, then $\mathcal{N}(G)$ is nonempty.*

PROOF. Suppose that $\mathcal{N}(G)$ is empty. Let i satisfy the hypothesis. Because of the compactness and upper semicontinuity conditions, $M_i(x_{-i})$ is nonempty and compact for every $x_{-i} \in X_{-i}$, by Lemma 1. For every $j \neq i$, $M_j(x_{-j})$ is nonempty for each $x_{-j} \in X_{-j}$ by the finiteness of X_j . Pick any selection $m_i(\cdot)$ from $M_i(\cdot)$ and any x_{-i} in X_{-i} . Construct a best-response compatible path (x^1, x^2, \dots) as follows: $x^1 = (m_i(x_{-i}), x_{-i})$ and for $k = 2, 3, \dots$, if $x_i^k \notin M_i(x_{-i}^k)$, then $x^{k+1} = (m_i(x_{-i}^k), x_{-i}^k)$; otherwise $x^{k+1} = (z_{j(k)}^{k+1}, x_{-j(k)}^k)$ for some player $j(k) \neq i$ such that $z_{j(k)}^{k+1} \in M_{j(k)}(x_{-j(k)}^k)$ and $(z_{j(k)}^{k+1}, x_{-j(k)}^k) \succ_{j(k)} x^k$. Note that such a player $j(k)$ exists by the presumed emptiness of $\mathcal{N}(G)$. Since X_{-i} is a finite set and player i uses only the finite set of strategies $m_i(X_{-i})$, there exist $k, l \in \mathbb{N}$ such that $x^k = x^{k+l}$. Hence $(x^k, x^{k+1}, \dots, x^{k+l})$ is a best-response cycle which by Theorem 1 contradicts the premise that G is a best-response potential game. We conclude that G possesses at least one Nash equilibrium. ■ ■

This existence result extends earlier results by Voorneveld (1997) and Norde and Tijs (1998) which were obtained in a cardinal setting for exact potential games and generalized ordinal potential games, respectively. As an

immediate consequence, we obtain:

Corollary 2 *If a finite ordinal game G has a best-response potential, then it has an ordinal Nash equilibrium.*

4 Quasi-Supermodular Ordinal Games (Games with Strategic Complementarities)

Let X be a partially ordered set, with the reflexive, antisymmetric and transitive binary relation \geq . Given elements x and z in X , denote by $x \vee z$ the least upper bound or **join** of x and z in X , provided it exists, and $x \wedge z$ the greatest lower bound or **meet** of x and z in X , provided it exists. A partially ordered set X that contains the join and the meet of each pair of its elements is called a **lattice**. A lattice in which each nonempty subset has a supremum and an infimum is **complete**. In particular, a finite lattice is complete. If Y is a subset of a lattice X and Y contains the join and the meet with respect to X of each pair of elements of Y , then Y is a **sublattice** of X . A sublattice Y of a lattice X in which each nonempty subset has a supremum and an infimum with respect to X that are contained in Y is a **subcomplete sublattice** of X . Any finite sublattice of a lattice is subcomplete.

We now define an order on the subsets of a lattice. We use the **strong set order** \geq_s introduced by Milgrom and Shannon (1994). Let X be a lattice and let Y and Z be two subsets of X . We say that $Y \geq_s Z$ if for every $y \in Y$ and every $z \in Z$, $y \vee z \in Y$ and $y \wedge z \in Z$. We say that a relation $\rho : X \rightarrow Y$ from a lattice X to a lattice Y is **increasing** in x on X if for every $x \in X$, $\rho(x)$ is a sublattice of Y and if for $x \geq z$, $\rho(x) \geq_s \rho(z)$.

If X is a lattice partially ordered by the relation \geq , then subsets of the form $[a, b] = \{x \in X : b \geq x \geq a\}$, $[a, \infty) = \{x \in X : x \geq a\}$, or $(-\infty, b] = \{x \in X : b \geq x\}$ are sublattices of X for all $a, b \in X$. These sets and X are the **closed intervals** in X . We say that a lattice X is equipped with the **interval topology** when each closed set can be represented as the intersection of sets that are finite unions of closed intervals in X , including the empty set as the empty union of sets. In other words, the closed intervals constitute a subbase of closed sets of the interval topology.

Next consider an ordinal game. Suppose that each individual strategy set X_i is a lattice partially ordered by the relation \geq_i . Then the product sets X and X_{-i} , $i \in N$, are also lattices with respect to the canonical partial orders \geq and \geq_{-i} , $i \in N$, respectively. For instance, $x \geq y \Leftrightarrow \forall i : x_i \geq_i y_i$ for $x, y \in X$. We say that the preference \succeq_i is **quasi-supermodular** on X_i for each $x_{-i} \in X_{-i}$ if for every $x_i, z_i \in X_i$, $x_i \neq z_i$, $x_{-i} \in X_{-i}$,

$$(i) \quad (x_i, x_{-i}) \succeq_i (x_i \wedge z_i, x_{-i}) \implies (x_i \vee z_i, x_{-i}) \succeq_i (z_i, x_{-i});$$

$$(ii) \quad (x_i, x_{-i}) \succ_i (x_i \wedge z_i, x_{-i}) \implies (x_i \vee z_i, x_{-i}) \succ_i (z_i, x_{-i}).$$

We say that a preference \succeq_i satisfies the **strategic complement property** in (x_i, x_{-i}) on $X_i \times X_{-i}$ if for every $x_i, z_i \in X_i$ and $x_{-i}, z_{-i} \in X_{-i}$ with $x_i \geq_i z_i$, $x_i \neq z_i$ and $x_{-i} \geq_{-i} z_{-i}$, $x_{-i} \neq z_{-i}$,

$$(iii) \quad (x_i, z_{-i}) \succeq_i (z_i, z_{-i}) \implies (x_i, x_{-i}) \succeq_i (z_i, x_{-i});$$

$$(iv) \quad (x_i, z_{-i}) \succ_i (z_i, z_{-i}) \implies (x_i, x_{-i}) \succ_i (z_i, x_{-i}).$$

An **ordinal game** $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ is **quasi-supermodular** if for each $i \in N$, X_i is a lattice, \succeq_i is quasi-supermodular on X_i for each $x_{-i} \in X_{-i}$, and \succeq_i satisfies the strategic complement property on $X_i \times X_{-i}$.

Quasi-supermodular games in a cardinal setting were introduced by Migrom and Shannon (1994). Note that a (quasi-supermodular) game $\Gamma = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ with payoff functions $u_i : X \rightarrow \mathbb{R}$, $i \in N$, uniquely determines a (quasi-supermodular) ordinal game G . But not every ordinal quasi-supermodular game has a cardinal representation. For instance, take the pair (X, \succeq) from Example 1 as the strategy space and the preference relation of the player in a one-person game. Moreover, let $X = [0, 1] \times \{0, 1\}$ be partially ordered according to the canonical partial order on \mathbb{R}^2 . Then the one-player game is quasi-supermodular and does not have a cardinal representation.

Lemma 2 *Let $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ be a quasi-supermodular game. Then for each $i \in N$, $M_i(x_{-i})$ is increasing in x_{-i} on X_{-i} .*

PROOF. Pick $x_{-i} \geq_{-i} z_{-i}$ and $x_i \in M_i(x_{-i})$, $z_i \in M_i(z_{-i})$. The elements $x_i \wedge z_i$ and $x_i \vee z_i$ are in X_i since X_i is a lattice. We have $(z_i, z_{-i}) \succeq_i (x_i \wedge z_i, z_{-i})$. Then $(z_i \vee x_i, z_{-i}) \succeq_i (x_i, z_{-i})$ by quasi-supermodularity and, consequently, $(x_i \vee z_i, x_{-i}) \succeq_i (x_i, x_{-i})$ by the strategic complement property. Since $x_i \in M_i(x_{-i})$, it cannot be the case that $(x_i \vee z_i, x_{-i}) \succ_i (x_i, x_{-i})$. Therefore, $(x_i \vee z_i, x_{-i}) \sim_i (x_i, x_{-i})$, which implies $z_i \vee x_i \in M_i(x_{-i})$. Now suppose $(z_i, z_{-i}) \succ_i (x_i \wedge z_i, z_{-i})$. Then $(z_i \vee x_i, z_{-i}) \succ_i (x_i, z_{-i})$ by quasi-supermodularity and hence $(x_i \vee z_i, x_{-i}) \succ_i (x_i, x_{-i})$ by the strategic complement property. But this contradicts $(x_i \vee z_i, x_{-i}) \sim_i (x_i, x_{-i})$ (and $x_i \in M_i(x_{-i})$). Therefore, $(z_i, z_{-i}) \sim_i (x_i \wedge z_i, z_{-i})$, which implies $x_i \wedge z_i \in M_i(z_{-i})$.

We have shown $M_i(x_{-i}) \geq_s M_i(z_{-i})$. Finally, setting $x_{-i} = z_{-i}$ yields $x_i \vee z_i \in M_i(x_{-i})$ and $x_i \wedge z_i \in M_i(x_{-i})$ for $x_i, z_i \in M_i(x_{-i})$ — which means that $M_i(x_{-i})$ is a sublattice of X_i . This completes the proof. ■■

Theorem 3 *Let the game $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ be quasi-supermodular and each individual strategy set X_i be equipped with the interval topology. If for all $i \in N$, the set X_i is compact and the preference \succeq_i is upper semi-continuous on X_i for every $x_{-i} \in X_{-i}$, then the set of ordinal Nash equilibria of G , $\mathcal{N}(G)$, is a nonempty complete lattice.*

PROOF. Birkhoff's theorem (1967, Theorem X.20) states that a lattice is compact in its interval topology if and only if it is complete.² Hence each X_i is a complete lattice and, therefore, has a lower bound ℓ_i and an upper bound o_i ; consequently, the closed intervals of the form $[a_i, b_i]$ constitute a subbase of closed sets for the interval topology. The product $X = \prod_{i \in N} X_i$ is compact in the product topology as the product of compact spaces. X has lower bound $\ell = (\ell_i)_{i \in N}$ and upper bound $o = (o_i)_{i \in N}$ and the cylinder sets of the form $[a_i, b_i] \times \prod_{j \neq i} X_j, i \in N$, with $X_j = [\ell_j, o_j]$ for $j \in N$, constitute a subbase \mathcal{B} of closed sets for the product topology. The closed intervals of the form $[a, b] = \prod_{i \in N} [a_i, b_i]$ constitute a subbase \mathcal{B}' of closed sets for the interval topology on X . On the one hand, $\mathcal{B} \subseteq \mathcal{B}'$. On the other hand, each $[a, b] \in \mathcal{B}'$ is a closed set in the product topology, for

$$[a, b] = \prod_{i \in N} [a_i, b_i] = \bigcap_{i \in N} \left([a_i, b_i] \times \prod_{j \neq i} X_j \right).$$

²We adopt Frink's (1942) and Topkis' (1998) definition of the interval topology, which in contrast to Birkhoff's does not presume existence of a priori universal lower and upper bounds. Birkhoff's proof still applies.

This shows that \mathcal{B}' is a subbase of closed sets for both the product topology and the interval topology on X . Hence the assertion of Frink (1942, Theorem 4) holds: The product and the interval topology on X coincide. Thus, X is compact in its interval topology. By Birkhoff's theorem, X is a complete lattice.

Now let $i \in N$. The best-response relation $M_i : X_{-i} \rightarrow X_i$ is nonempty and compact-valued by Lemma 1 and is increasing with respect to the strong set order by Lemma 2. Therefore, the joint best-response relation M from X to itself, with $M(x) = \prod_{i \in N} M_i(x_{-i})$, is a compact-valued correspondence as the product of compact-valued correspondences and is increasing as the product of increasing correspondences.

Next pick any $x \in X$. Since $M(x)$ is compact in X , it is also compact as a subspace of X when X is endowed with the product topology. Hence $M(x)$ is also compact as a subspace of X when X is endowed with the interval topology because the product and the interval topology on X coincide. Consider any nonempty subset A of $M(x)$. Since X is a complete lattice, $\sup_X A$, the supremum of A in X exists. We claim $\sup_X A \in M(x)$. As observed above, the closed intervals of the form $[a, b]$ form a subbase of closed sets for both topologies on X . Therefore, the sets $M(x) \cap [a, b]$, $a, b \in X$, form a subbase of closed sets for the topological subspace $M(x)$ of X . Now consider the family of sets $M(x) \cap [a, \sup_X A]$, $a \in A$. Let F be a finite nonempty subset of A . Since $M(x)$ is a sublattice of X , $\sup_X F \in M(x)$ and $a \leq \sup_X F \leq \sup_X A$ for $a \in F$. Hence $\sup_X F$ belongs to the intersection of the sets $M(x) \cap [a, \sup_X A]$, $a \in F$. Since the family of closed sets $M(x) \cap [a, \sup_X A]$, $a \in A$, has nonempty finite intersections and $M(x)$ is compact, the entire family has a nonempty intersection. Let b belong to this intersection. Then $b \in M(x)$ and $a \leq b \leq \sup_X A$ for all $a \in A$. Hence

$b \in M(x)$ and $b = \sup_X A$, which shows our claim that $\sup_X A \in M(x)$. In an analogous way, one proves $\inf_X A \in M(x)$. Since A was an arbitrary nonempty subset, $M(x)$ is a subcomplete sublattice of X .³

To summarize, the joint best-response relation M is an increasing correspondence from the complete lattice X to itself and $M(x)$ is a nonempty and subcomplete sublattice for each $x \in X$. The assertion of the theorem follows from Zhou's fixed-point theorem (1994, Theorem 1, p. 297). ■ ■

Corollary 3 *Let $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ be a finite quasi-supermodular game. Then the set of ordinal Nash equilibria of G , $\mathcal{N}(G)$, is a nonempty complete lattice.*

5 Set-Valued Concepts

Set-valued concepts have proved to have many desirable properties in large classes of cardinal games $G = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$. Here we consider two set-valued concepts: the set of rationalizable joint strategies introduced independently by Bernheim (1984) and Pearce (1984) and the concept of minimal closed set under some behavior correspondence introduced by Ritzberger and Weibull (1995). We are going to extend the field of applications of these set-valued concepts from cardinal games to ordinal games. However, in contrast to most of the literature, our definitions involve only pure strategies. The reason is that there is no straightforward and commonly agreed upon extension of ordinal preferences from pure to mixed strategies. The modified

³Topkis (1998, p. 31), referring to Topkis (1977), states that when X is a lattice with the interval topology and X' is a sublattice of X , then X' is subcomplete if and only if X' is compact in the relative topology. We have demonstrated the “if” part in case X is a complete lattice.

concepts have similar properties as the original ones and may be of interest on their own. In fact, Basu (1992) works with pointwise beliefs or conjectures like us.

A strategy for a player is rationalizable if it survives iterated removal of strategies that are never a best response. Rationalizability is a concept that generalizes that of Nash equilibrium. Minimal closed set under some behavior correspondence is a concept that generalizes that of strict Nash equilibrium. More precisely, Basu and Weibull (1991) introduce first the concept of closed set under rational behavior (curb), a set-valued extension of the strict Nash equilibrium concept for cardinal games $G = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$. A product set of pure strategies is closed under rational behavior if it is non-empty and compact and contains the image under the best-response correspondence of every mixed joint strategy with support in this set. A curb set is minimal if it does not contain any proper subset which is curb. As noted by Basu and Weibull (1991), the set of rationalizable joint strategies is the largest tight curb set. Thus, a minimal curb set and the set of rationalizable joint strategies can be viewed as the two ends of a spectrum. Ritzberger and Weibull (1995) generalize the concept of curb set to a very large class of behavior correspondences. A product set of pure strategies is closed under some behavior correspondence if it is non-empty and compact and contains the image under the particular correspondence of every mixed joint strategy with support in this set. The class of behavior correspondences considered by the authors includes the better-response correspondence and the best-response correspondence. All these set-valued concepts have proved to be very useful to characterize stable sets of dynamic strategy adjustments (Ritzberger and Weibull, 1995; Young, 1998; Matros and Josephson, 2004).

Pearce (1984) considers only finite games. Bernheim (1984) considers games where each strategy space X_i is a compact subset of some Euclidean space and every $u_i : X \rightarrow \mathbb{R}$ is continuous. Similarly, Basu and Weibull (1991) prove the existence of minimal curb sets for the mixed extension of games $G = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ where each X_i is a compact set in some Euclidean space and every payoff function $u_i : X \rightarrow \mathbb{R}$ is continuous. Ritzberger and Weibull (1995) focus on finite games. We consider a more general setting: In Theorem 4 on the existence of rationalizable joint strategies, each strategy set is endowed with a compact Hausdorff topology and each player's preference \succeq_i is continuous on X . In Theorem 5 on the existence of minimal closed sets under a behavior relation, each individual strategy space is compact and Hausdorff. Notice that our specification cannot be reduced to the class of games $G = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ considered by Bernheim (1984) and Basu and Weibull (1991). The reason is that the classical utility representation theorems of Eilenberg (1941) and Debreu (1954, 1964) apply only to separable topological spaces that is topological spaces which include a countable dense subset.⁴

In the following subsections, we prove (i) the existence of a nonempty and compact subset of rationalizable joint strategies in ordinal games where each X_i is a compact Hausdorff space and each preference \succeq_i is continuous on X ; (ii) the existence of at least one minimal closed set under a behavior relation for the class of ordinal games where each X_i is a compact Hausdorff

⁴A more recent result due to Monteiro (1987) requires that the underlying topological space is arc-connected and the preferences \succeq_i are continuous and countably bounded. Recall that the weak order \succeq_i on X is countably bounded if there is a countable subset Z of X such that for every $x \in X$, there exist $z^1, z^2 \in Z$ such that $z^1 \succeq_i x \succeq_i z^2$.

space.

5.1 Rationalizable strategies

We shall define rationalizability via the method used by Pearce (1984). In addition, attention is confined to pointwise beliefs. For each $i \in N$, construct a sequence $H_i^k, k \in \mathbb{N}$, of subsets of X_i as follows: Let $H_i^1 = X_i$ and define H_i^k inductively for $k = 2, 3, \dots$ by

$$H_i^k = \left\{ x_i \in H_i^{k-1} \mid \exists x_{-i} \in \prod_{j \in N \setminus \{i\}} H_j^{k-1} : x_i \in M_i(x_{-i}) \right\}.$$

The set of **rationalizable strategies** of player $i \in N$ is defined as

$$R_i = \bigcap_{k \in \mathbb{N}} H_i^k$$

and a **joint strategy** $x \in X$ is **rationalizable** if $x \in R := \prod_{i \in N} R_i$.

Theorem 4 *Let $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ be an ordinal game. Suppose that for each $i \in N$, the set of strategies X_i is equipped with a compact Hausdorff topology and the preference \succeq_i is continuous on X . Then the set of joint rationalizable strategies $R \subseteq X$ is nonempty and compact.*

PROOF. We shall show by induction on k that $H_i^k, k \in \mathbb{N}$, is a nested sequence of nonempty and closed subsets of the compact space X_i for all $i \in N$. Therefore, the intersection R_i is nonempty and closed. Consequently, R_i is compact as a closed subset of a compact space.

THE INDUCTION ARGUMENT: For each $i \in N$, $H_i^1 = X_i$ is closed by definition. Consider any $k \in \mathbb{N}$ and suppose that $\prod_{i \in N} H_i^k$ is the Cartesian

product of nonempty and closed sets. Then each H_i^k is a nonempty and compact subset of the compact space X_i and $\prod_{i \in N} H_i^k$ is a nonempty and compact subspace of X . We have to show that each H_i^{k+1} is also nonempty and closed. Since for $i \in N$, the preference \succeq_i is continuous on X , its restriction to $\prod_{j \in N} H_j^k$ is continuous as well. For every $x_{-i} \in H_{-i}^k = \prod_{j \neq i} H_j^k$, the restriction of \succeq_i to $H_i^k \times \{x_{-i}\}$ is also continuous. Hence by Lemma 1, the set $M_i(x_{-i}|H_i^k) := \{x_i \in H_i^k | (x_i, x_{-i}) \succeq_i (z_i, x_{-i}) \text{ for all } z_i \in H_i^k\}$ is nonempty and compact. Hence H_i^{k+1} is nonempty as the union of the sets $M_i(x_{-i}|H_i^k)$, $x_{-i} \in H_{-i}^k$.

It remains to show that H_i^{k+1} is closed. To this end, it suffices to show that for any net in H_i^{k+1} converging to a point x_i^* in X_i , it follows that x_i^* belongs to H_i^{k+1} . Suppose that (D, \gg) is a directed set, that $x_i^\delta, \delta \in D$, is a net in H_i^{k+1} and that the net $x_i^\delta, \delta \in D$, converges to $x_i^* \in X_i$. For each $\delta \in D$, let us select an $x_{-i}^\delta \in H_{-i}^k$ such that $x_i^\delta \in M_i(x_{-i}^\delta|H_i^k)$. Then $(x_i^\delta, x_{-i}^\delta), \delta \in D$, is a net in $\prod_{j \in N} H_j^k$. Since $\prod_{j \in N} H_j^k$ is compact, there exists a subnet of $(x_i^\delta, x_{-i}^\delta), \delta \in D$, convergent to some $(x'_i, x'_{-i}) \in X$. Without restriction, we may assume that $(x_i^\delta, x_{-i}^\delta), \delta \in D$, is such a subnet. Then the net $x_i^\delta, \delta \in D$, converges to both x'_i and x_i^* . Since X_i is Hausdorff, $x'_i = x_i^*$. Because of continuity of the restriction of \succeq_i to $\prod_{j \in N} H_j^k$, the set $\mathfrak{H}_i^k \equiv \{(z', z'') \in \prod_{j \in N} H_j^k \times \prod_{j \in N} H_j^k | z' \succeq_i z''\}$ is a closed subset of $\prod_{j \in N} H_j^k \times \prod_{j \in N} H_j^k$; see Bridges and Mehta (1995, Proposition 1.6.2). Now let $z_i \in H_i^k$. For each $\delta \in D$, we have

$$(x_i^\delta, x_{-i}^\delta) \succeq_i (z_i, x_{-i}^\delta),$$

that is $((x_i^\delta, x_{-i}^\delta), (z_i, x_{-i}^\delta)) \in \mathfrak{H}_i^k$. Therefore, $((x'_i, x'_{-i}), (z_i, x'_{-i})) \in \mathfrak{H}_i^k$. Consequently, $x'_{-i} \in H_{-i}^k$ and $(x'_i, x'_{-i}) \succeq_i (z_i, x'_{-i})$. Since the latter holds for arbitrary $z_i \in H_i^k$, we obtain $x'_{-i} \in H_{-i}^k$ and $x'_i \in M_i(x'_{-i}|H_i^k)$. This implies

$x_i^* = x_i' \in H_i^{k+1}$ as desired. ■ ■

5.2 Closed sets under a behavior relation

Assume that each set of strategies X_i , $i \in N$, is a compact Hausdorff space. For the sake of convenience, we shall take X as the space of beliefs of each player $i \in N$, so that the beliefs for player i include his own strategy. In particular we shall denote the best response relation by $M_i(x)$ — though it is functionally independent of the component $x_i \in X_i$. Let Φ be the class of **behavior relations**: $\phi \in \Phi$ if $\phi = \prod_{i \in N} \phi_i : X \rightarrow X$ such that $M(x) \subseteq \phi(x)$ for every $x \in X$. More precisely, for each $i \in N$, the individual behavior relation $\phi_i : X \rightarrow X_i$ maps each joint strategy $x \in X$ to the superset $\phi_i(x)$ of player i 's best responses $M_i(x)$.⁵ For any behavior relation $\phi : X \rightarrow X$ and any nonempty product set $Z \subseteq X$, $\phi(Z)$ denotes the union of all images $\phi(z)$, $z \in Z$, i.e.

$$\phi(Z) = \bigcup_{z \in Z} \phi(z).$$

Given any behavior relation $\phi \in \Phi$, a **closed set under ϕ** is a product set $Z = \prod_{i \in N} Z_i \subseteq X$ such that

- (i) for each $i \in N$, $Z_i \subseteq X_i$ is a nonempty compact set of strategies;
- (ii) for each $i \in N$, and each belief $z \in Z$ of player i , the set Z_i contains all responses of player i against his belief: $\phi_i(Z) \subseteq Z_i$.

A closed set $Z \subseteq X$ under $\phi \in \Phi$ is called **minimal** if Z does not contain a proper subset which is closed under ϕ . The present concept generalizes

⁵In the context of Basu and Weibull (1991) and Ritzberger and Weibull (1995), behavior relations are correspondences.

the notion of strict Nash equilibrium. Indeed, consider any $x \in X$. The singleton set $\{x\}$ is a strict Nash equilibrium if and only if it is closed under some $\phi \in \Phi$, i.e. $\phi(\{x\}) = \{x\}$.

Remark 1

- (a) Basu and Weibull (1991) call a compact product set $Z \subseteq X$ closed under rational behavior (curb) if Z is closed under the combined best-response relation M . A product set is a prep set in the sense of Voorneveld (2004, 2004) if it contains at least one best reply (but not necessarily all best replies) to any consistent belief that a player may have about the strategic behavior of his opponents. While the minimal prep sets and the minimal curb sets of a game can differ, they coincide in generic finite games.
- (b) Ritzberger and Weibull (1995) call a compact product set $Z \subseteq X$ closed under better responses (cubr) if Z is closed under the combined better-response relation $B \in \Phi$. Because of the finiteness of X , the compactness requirement on Z is not restrictive.

Theorem 5 *Let $G = (N, (X_i)_{i \in N}, (\succeq_i)_{i \in N})$ be an ordinal game. Suppose that for each $i \in N$, the set of strategies X_i is equipped with a compact Hausdorff topology. Then:*

- (i) *for every $\phi \in \Phi$, there exists a minimal closed set;*
- (ii) *for every $\phi \in \Phi$, minimal closed sets are pairwise disjoint;*
- (iii) *if $Z \subseteq X$ is a minimal closed set under $\phi \in \Phi$ and $\phi(Z)$ is a compact and nonempty subset of X , then $\phi(Z) = Z$.*

PROOF. Part (i). Let $\phi \in \Phi$ and let Q_ϕ be the collection of all sets which are closed under ϕ in G . Obviously, $X \in Q_\phi$. Hence Q_ϕ is nonempty. Consider the partially ordered set (Q_ϕ, \subseteq) . By Hausdorff's maximum principle, Q_ϕ contains a maximal totally ordered subset, say (Q_ϕ^*, \subseteq) . Set

$$\tilde{Z} = \bigcap_{Z \in Q_\phi^*} Z \text{ and } \tilde{Z}_i = \bigcap_{Z \in Q_\phi^*} Z_i$$

for $i \in N$. Observe that each $Z \in Q_\phi^*$ is a closed subset of the compact Hausdorff space X . By construction, the collection of closed sets Q_ϕ^* is totally ordered by the relation \subseteq . Therefore, every nonempty finite subcollection of Q_ϕ^* has a nonempty intersection. Since X is compact, the finite intersection property holds and thus \tilde{Z} is nonempty. As the intersection of closed sets, \tilde{Z} is closed. Moreover, it is a product set, $\tilde{Z} = \prod_{i \in N} \tilde{Z}_i$. By definition, $\phi(\tilde{Z}) \subseteq \phi(Z) \subseteq Z$ for each $Z \in Q_\phi^*$ and, consequently, $\phi(\tilde{Z}) \subseteq \tilde{Z}$. This shows that \tilde{Z} is a closed set under ϕ . \tilde{Z} is necessarily minimal. For otherwise, (Q_ϕ^*, \subseteq) would not be maximal.

Part (ii). Suppose that $Z, Y \subseteq X$ are two arbitrary distinct minimal closed sets under some $\phi \in \Phi$, but $Z \cap Y \neq \emptyset$. Let $C = Z \cap Y$. By definition, $\phi(C) \subseteq C$ which contradicts that both Y and Z are minimal.

Part (iii). $\phi(Z) \subseteq Z$ implies $\phi(\phi(Z)) \subseteq \phi(Z)$. Compactness and nonemptiness of $\phi(Z)$ then implies that $\phi(Z)$ is closed with respect to ϕ . Hence $\phi(Z) \subsetneq Z$ contradicts the minimal closedness of Z with respect to ϕ . ■■

Notice that if the relation ϕ is compact-valued and upper hemicontinuous, then for any compact subset Z of X , $\phi(Z)$ is compact.⁶ In particular, finiteness of N , continuity of each \succeq_i on X plus nonemptiness and compactness of each X_i imply that $M \in \Phi$ is compact-valued and upper hemicontinuous.

Also notice that in the case of a compact Hausdorff space X , a subset of X is compact if and only if it is topologically closed. Therefore, “compact” can be replaced by “topologically closed” in the definition of a closed set under ϕ and the assertions of the theorem.

Further observe that one can define closedness of a product set $Z \subseteq X$ under an arbitrary relation $\phi: X \rightarrow X$. Then the analogue of Theorem 5 still holds true.

Finally, we obtain the analogue of an observation by Basu and Weibull (1991): The rationalizable strategies form the largest “tight” curb set. If ϕ is a behavior relation and $Z \subseteq X$ is a nonempty compact product set with $\phi(Z) = Z$, then Z is called a **tight set closed under ϕ** .

Corollary 4 *Under the hypothesis of Theorem 4,*

- (i) *the set of joint rationalizable strategies $R \subseteq X$ is a tight set closed under the best response correspondence M ;*
- (ii) *if $Z \subseteq X$ is a tight set closed under M then $Z \subseteq R$.*

⁶See Hildenbrand (1974, B.III, Proposition 3). The argument holds for relations as well.

PROOF. (i) By construction, R is a product set. By Theorem 4, R is nonempty and compact. By construction, $R = \bigcap_k M^k(X)$ where $k = 0, 1, 2, \dots$ and $M^0(S) = S$ for $S \subseteq X$. Hence $M(R) = \bigcap_k M^{k+1}(X) = \bigcap_k M^k(X) = R$, since $M^0(X) = X$. This shows that R is a tight set closed under M .

(ii) Suppose $Z \subseteq X$ is a tight set closed under M . Then Z is a compact product set with $M(Z) = Z$. Therefore, $Z = \bigcap_k M^k(Z) \subseteq \bigcap_k M^k(X) = R$. ■ ■

In general, R is not a minimal set closed under M . For instance, if the game G has two strict ordinal Nash equilibria x and x' , then $\{x\}$ and $\{x'\}$ are two disjoint minimal tight sets closed under M .

6 Final Remarks

The focus of this paper lies on games with players' preferences which are weak orders. Within this broad category of games, we analyze games with ordinal best-response potentials and quasi-supermodular games. We further provide sufficient conditions for the existence of a nonempty set of rationalizable joint strategies and of a closed set under a behavior relation.

In the context of cardinal games, it is frequently assumed that the mixed extension of a game exists, that is each cardinal utility representation can be extended to an expected utility functional on the set of joint mixed strategies. Consequently, the notion of a Nash equilibrium in mixed strategies can be adopted. Furthermore, the definitions of rationalizable joint strategies and of closed sets under a behavior correspondence may include best responses

against mixed strategies. There is no straightforward and commonly agreed upon extension of ordinal preferences from pure to mixed strategies. Therefore our analysis and definitions are confined to pure strategies. In lieu of expected utility comparisons, Fishburn (1978) and Perea *et al.* (2006) apply first-order stochastic dominance (induced by the ordinal preferences) to joint mixed strategies. This defines a partial order on joint mixed strategies.

As mentioned in the introduction, there exists a sizeable literature on generalized games with incomplete or intransitive preferences. Extensions of our analysis to generalized games and/or games with incomplete or intransitive preferences are left to future research.

References

- [1] Basu K. (1992) A Characterization of the Class of Rationalizable Equilibria for Oligopoly Games. *Economics Letters*, 40, 187-191.
- [2] Basu K., Weibull J. (1991) Strategy Subsets Closed under Rational Behavior. *Economics Letters*, 36, 141-146.
- [3] Beardon A.F., Candeal J.C., Herden G., Induráin E., Mehta, G.B. (2002) The Non-Existence of a Utility Function and the Structure of Non-Representable Preference Relations,” *Journal of Mathematical Economics*, 37, 17-38.
- [4] Bergstrom T. C. (1976) How to Discard “Free Disposability” at No Cost. *Journal of Mathematical Economics*, 3, 131-134.
- [5] Bernheim B.D. (1984) Rationalizable Strategic Behavior. *Econometrica*, 52, 1007-1028.

- [6] Birkhoff G. (1967) *Lattice Theory*. American Mathematical Society Colloquium Publication No. XXV, New York.
- [7] Borglin A., Keiding H. (1976) Existence of Equilibrium Actions and of Equilibrium: A Note on the “New” Existence Theorems. *Journal of Mathematical Economics*, 3, 313-316.
- [8] Bosi G., Mehta G.B. (2002) Existence of a Semicontinuous or Continuous Utility Function: A Unified Approach and an Elementary Proof,” *Journal of Mathematical Economics*, 38, 311-328.
- [9] Bridges D.S., Mehta G.B. (1995), *Representations of Preference Orderings*. Lecture Notes in Economics and Mathematical Systems 422, Springer-Verlag, Berlin and Heidelberg.
- [10] Bulow J.I., Geanakoplos J.D., Klemperer P.D. (1985) Multimarket Oligopoly: Strategic Substitutes and Complements. *Journal of Political Economy*, 93, 488-511.
- [11] Candeal J.C., Hervés C., Induráin E. (1998) Some Results on Representation and Extension of Preferences,” *Journal of Mathematical Economics* 29, 75-81.
- [12] Debreu G. (1954) Representation of a Preference Ordering by a Numerical Function, in Thrall R.M, Coombs C.H., Davis R.L. (eds.) *Decision Processes*, Wiley, New York, NY, 159-166.
- [13] Debreu G. (1964) Continuity Properties of Paretian Utility. *International Economic Review*, 5, 285-289.
- [14] Eilenberg S. (1941) Ordered Topological Spaces. *American Journal of Mathematics*, 63, 39-45.

- [15] Estévez Toranzo M., Hervés Beloso C. (1995) On the Existence of Continuous Preference Orderings without Utility Representations. *Journal of Mathematical Economics*, 24, 305-309.
- [16] Fishburn P.C. (1978) Non-Cooperative Stochastic Dominance Games. *International Journal of Game Theory*, 7, 51-61.
- [17] Frink O. (1942) Topology in Lattices. *Transactions of the American Mathematical Society*, 51, 569-582.
- [18] Hildenbrand W. (1974) *Core and Equilibria of a Large Economy*. Princeton University Press, Princeton, NJ.
- [19] Kelley J.L. (1955) *General Topology*. Van Nostrand, New York.
- [20] Kim T., Richter M.K. (1986) Nontransitive-Nontotal Consumer Theory, *Journal of Economic Theory* 38, 324-363.
- [21] Kukushkin N.S. (1999) Potential Games: A Purely Ordinal Approach. *Economics Letters*, 64, 279-283.
- [22] Matros A., Josephson J. (2004) Stochastic Imitation in Finite Games. *Games and Economic Behavior*, 49, 244-259.
- [23] Mertens J.F. (1987) Ordinality in Non-Cooperative Games. Mimeo, CORE DP 8728, Université Catholique de Louvain, Belgium.
- [24] Mertens J.F. (2003) Ordinality in Non Cooperative Games. *International Journal of Game Theory*, 32, 387-430.
- [25] Milgrom P., Roberts J. (1990) Rationalizability, Learning, and Equilibrium in Games with Strategic Complements. *Econometrica*, 58, 1255-1277.

- [26] Milgrom P., Shannon C. (1994) Monotone Comparative Statics. *Econometrica*, 62, 157-180.
- [27] Monderer D., Shapley L. (1996) Potential Games. *Games and Economic Behavior*, 14, 124-143.
- [28] Monteiro P.K. (1987) Some Results on the Existence of Utility Functions on Path Connected Spaces. *Journal of Mathematical Economics*, 16, 147-156.
- [29] Morris S., Ui T. (2004) Best Response Equivalence. *Games and Economic Behavior*, 49, 260-287.
- [30] Norde H., Patrone F. (2001) A Potential Approach for Ordinal Games. *TOP*, 9, 69-75.
- [31] Norde H., Tijs S. (1998) Determinateness of Strategic Games with a Potential. *Mathematical Methods of Operations Research*, 48, 377-385.
- [32] Pearce D. (1984) Rationalizable Strategic Behavior and the Problem of Perfection. *Econometrica*, 52, 1029-1050.
- [33] Perea A., Peters H., Schulteis T., Vermeulen D. (2006) Stochastic Dominance Equilibria in Two-Person Noncooperative Games. *International Journal of Game Theory*, 34, 457-473.
- [34] Rader T. (1963) The Existence of a Utility Function to Represent Preferences. *Review of Economic Studies*, 30, 229-232.
- [35] Ritzberger K., Weibull J. (1995) Evolutionary Selection in Normal Form Games. *Econometrica*, 63, 1371-1399.

- [36] Schmeidler D. (1969) Competitive Equilibria in Markets with a Continuum of Traders and Incomplete Preferences. *Econometrica*, 37, 578-585.
- [37] Shafer W.J. (1974) The Nontransitive Consumer. *Econometrica*, 42, 913-919.
- [38] Shafer W.J. (1976) Equilibrium in Economies without Ordered Preferences or Free Disposal. *Journal of Mathematical Economics*, 3, 135-137.
- [39] Shafer W.J., Sonnenschein H. (1975) Some Theorems on the Existence of Competitive Equilibrium. *Journal of Economic Theory*, 11, 83-93.
- [40] Sonnenschein H. (1971) Demand Theory without Transitive Preferences, with Applications to the Theory of Competitive Equilibrium, in Chipman J.S., Hurwicz L., Richter M.K., Sonnenschein H. (eds) *Preferences, Utility, and Demand*. A Minnesota Symposium. New York, Harcourt Brace Jovanovich, 215-223.
- [41] Thompson F.B. (1952) Equivalence of Games in Extensive Form. Research Memorandum RM-759, U.S. Air Force Project Rand, Rand, Santa Monica, CA. Reprinted in Kuhn H.W. (ed.) (1977) *Classics in Game Theory*. Princeton University Press, Princeton, NJ.
- [42] Topkis D. (1977) Topology and Subcomplete Sublattices. Mimeo.
- [43] Topkis D. (1979) Equilibrium Points in Nonzero-Sum n -Person Submodular Games. *SIAM Journal on Control and Optimization*, 17, 773-787.
- [44] Topkis D. (1998) *Supermodularity and Complementarity*. Princeton University Press, New Jersey.

- [45] Ui T. (2000) A Shapley Value Representation of Potential Games. *Games and Economic Behavior*, 31, 121-135.
- [46] Vermeulen A.J., Jansen M.J.M. (2000) Ordinality of Solutions of Non-cooperative Games. *Journal of Mathematical Economics*, 33, 13-34.
- [47] Vives X. (1990) Nash Equilibrium with Strategic Complementarities. *Journal of Mathematical Economics*, 19, 305-321.
- [48] Voorneveld M. (1997) Equilibria and Approximate Equilibria in Infinite Potential Games. *Economics Letters*, 56, 163-169.
- [49] Voorneveld M. (2000) Best-Response Potential Games. *Economics Letters*, 66, 289-295.
- [50] Voorneveld M. (2004) Preparation. *Games and Economic Behavior*, 48, 403-414.
- [51] Voorneveld M. (2005) Persistent Retracts and Preparation. *Games and Economic Behavior*, 51, 228-232.
- [52] Voorneveld M., Norde H. (1997) A Characterization of Ordinal Potential Games. *Games and Economic Behavior*, 13, 111-124.
- [53] Young P. (1998) *Individual Strategy and Social Structure*. Princeton NJ, Princeton University Press.
- [54] Zhou L. (1994) The Set of Nash Equilibria of a Supermodular Game is a Complete Lattice. *Games and Economic Behavior*, 7, 295-300.