Nonparametric Estimation of Large Auctions with Risk Averse Bidders

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Abstract

This paper explores the robustness of Guerre, Perrigne and Vuong's (2000) two-step nonparametric estimation procedure in first-price, sealed-bid auctions with n ($n \gg 1$) risk averse bidders. Based on an asymptotic approximation with precision of order $O(n^{-2})$ of the intractable equilibrium bidding function, we establish the uniform consistency with rates of convergence of Guerre, Perrigne and Vuong's (2000) two-step nonparametric estimator in the presence of risk aversion. Monte Carlo experiments show that the two-step nonparametric estimator performs reasonably well with a moderate number of bidders such as six.

Key Words: Large Auctions, First-price Auctions, Independent Private Value, Nonparametric Estimation, Risk Aversion

JEL Classification: C14, D44

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1 Introduction

In this paper, we explore the robustness of Guerre, Perrigne and Vuong's (2000) two-step nonparametric estimation procedure in first-price, sealed-bid auctions with a large number of risk averse bidders.

The seminal work by Guerre, Perrigne and Vuong (2000) has shown that the underlying distribution of bidders' values is nonparametrically identified from the observations of submitted bids in first-price, independent private value (FP-IPV) auctions with risk neutral bidders. Based on the equilibrium bidding behavior, they propose a two-step kernel-based estimator for the latent density of bidders' private values wherein the unobserved private values are estimated in the first step. The proposed two-step estimator is optimal in terms of the uniform convergence rate. As the private values are estimated from submitted bids, the best uniform convergence rate of this "indirect estimation" problem (Groeneboom, 1996) is slower than the best uniform convergence rate given by Stone (1982) when the private values are observable. However, when bidders are potentially risk averse, Campo et al. (2006) have shown that the distribution of bidders' private values and bidders' utility functions in FP-IPV auctions cannot be nonparametrically identified from observed bids. To estimate the latent density of bidders' private values, it is necessary to specify the utility function parametrically. They propose a multi-step semiparametric estimation procedure wherein the utility function is recovered parametrically in the initial steps. In deriving asymptotic properties, both works assume that the number of bidders n in each auction is fixed and the number of observed auctions L approaches infinity.

On the other hand, as n goes to infinity, it has been shown that the discrepancy between risk averse bidding behavior and risk neutral bidding behavior is of order $O(n^{-2})$ (Fibich, Gavious and Sela, 2004) and the discrepancy between strategic bidding behavior and perfectly competitive behavior, wherein bidders simply bid their value, is of order $O(n^{-1})$. In other words, as the size of an auction increases, the effect of risk aversion diminishes much faster than the rate at which the strategic bidding behavior degenerates to the price-taking behavior in perfect competition. Hence when the size of auction is large, Guerre, Perrigne and Vuong's (2000) two-step nonparametric estimator based on strategic bidding behavior may possess some robust properties against potential risk aversion. In this paper, we study the asymptotic properties of Guerre, Perrigne and Vuong's (2000) two-step nonparametric estimator allowing both the number of bidders n and the number of auctions L to approach infinity. We show that when n increases not too slowly relative to L, the two-step nonparametric estimator of the latent density of private values is consistent and attains the best uniform convergence rate given by Stone (1982) as if bidders' private values are observable.

Allowing both n and L to diverge to infinity introduces some extra complications in the analysis. Since the unknown private values are recovered from the observations of submitted bids and the estimated bid density, the smoothness of bid density and the uniform convergence rate of its estimator are crucial in determining the convergence rate of Guerre, Perrigne and Vuong's (2000) two-step estimator. As the equilibrium bid density depends on n, the derivatives of bid density that are bounded with fixed n could be unbounded as $n \to \infty$, and there is no standard result on the best uniform convergence rate for the nonparametric estimation of a density that is shifting with sample size as the bid density is here. Furthermore, when there exists observed heterogeneity across auctions, we need to estimate the density of private values conditional on the "fixed effects" characterizing heterogeneity across auctions. However, the best uniform convergence rate of the estimator for a conditional (or joint) density with observations in such a panel structure, where private values are of order $O(nL)$ and "fixed effects" variables are of order $O(L)$, has seldom been addressed in the literature. We show that the kernel estimator for the conditional density of private values given the "fixed effects" can attain the best uniform convergence rate at which the marginal density of "fixed effects" can be estimated.[1]

We conduct a Monte Carlo experiment to study the finite sample performance of Guerre, Perrigne and Vuong's (2000) two-step nonparametric estimator and get some interesting results. The two-step nonparametric estimator performs reasonably well in the presence of significant risk aversion when the number of bidders is six. In other words, an auction with six bidders can be considered as a large auction. In addition, the two-stage nonparametric estimation procedure sometimes outperforms the multi-step semiparametric estimation procedure when the utility function is misspecified.

This rest of the paper is organized as follows. Section 2 presents the first-price, sealed-bid auction model with risk averse bidders and derives the asymptotic approximation of the equilibrium bidding function. Section 3 establishes the uniform consistency with the convergence rate of Guerre, Perrigne and Vuong's (2000) two-step nonparametric estimator in large auctions with risk averse bidders. Section 4 specifies Monte Carlo experiments and reports the results. Section 5 briefly

[1] We assume that the marginal density of "fixed effects" is as smooth as the conditional density of the private values.

concludes.

2 Large Auctions with Risk Averse Bidders

Suppose there are a large number of potential buyers competing for a single, indivisible item. The number of potential buyers n ($n \gg 1$) is common knowledge[2]. In the first-price, sealed-bid auction under the independent private value (IPV) paradigm, the buyers simultaneously submit bids, and the highest bidder wins and pays his own bid to the seller. Buyer p's value v_p ($p = 1, \cdots, n$) for the auctioned item is his private information, while it is commonly known that the values are independently distributed on $[\underline{v}, \bar{v}] \subset \mathbb{R}^+$ according to a common distribution $F(\cdot)$, which is absolutely continuous with density $f(\cdot) > 0$. Each bidder is potentially risk averse with utility given by a common von Neumann-Morgenstern utility function $U(\cdot)$, which is twice continuously differentiable with $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$. The seller is assumed to be risk neutral. Moreover, we assume each bidder's initial wealth $w > 0$ is the same and commonly known.

Suppose the equilibrium bid for the pth bidder with private value v_p in an auction with n bidders is $b_p = s_n(v_p)$. Following Maskin and Riley (2000; 2003), and Athey (2001), the unique symmetric Bayesian Nash equilibrium of the corresponding game is characterized by the following differential equation in $s_n(\cdot)$

$$s_n'(v_p) = (n-1)\frac{f(v_p)}{F(v_p)}\lambda(v_p - s_n(v_p)), \tag{1}$$

where $\lambda(\cdot) = (U(w+\cdot) - U(w))/U'(w+\cdot)$. The boundary condition is given by $s_n(\underline{v}) = \underline{v}$.

In general, the equilibrium strategy is intractable without specification of a functional form for $U(\cdot)$. However, analytical approximations to the equilibrium strategy $s_n(\cdot)$ can be derived. To proceed, we need some regularity assumptions on $U(\cdot)$ and $F(\cdot)$ following Campo et al. (2006) as summarized in the following definitions. Throughout we denote the support of $*$ by $S(*)$, and the rth derivative of $*$ by $*^{(r)}$ ($r \geq 0$) with $*^{(0)} = *$.

Definition 1 *For $R \geq 1$, let $\mathcal{U}_R$ be the set of van Neumann-Morgenstern utility functions $U(\cdot)$ with initial wealth $w > 0$ such that:*

(i) $U : [0, \infty) \to [0, \infty)$*;*

[2] We assume in this paper that the reservation price is nonbinding, hence the number of potential bidders is equal to the number of actual bidders.

(ii) $U(\cdot)$ *is continuous on* $S(U)$*, and admits up to* $R+2$ *continuous bounded derivatives on* $(0,\infty)$ *with* $U'(\cdot) > 0$ *and* $U''(\cdot) \leq 0$ *on* $(0,\infty)$*.*

Definition 2 *For* $R \geq 1$*, let* $\mathcal{F}_R$ *be the set of distributions* $F(\cdot)$ *such that:*

(i) $S(F) = \{v : v \in [\underline{v}, \bar{v}]\}$*, with* $0 \leq \underline{v} < \bar{v} < \infty$*;*

(ii) $f(v) \geq c_f > 0$ *for* $v \in S(F)$*;*

(iii) $F(\cdot)$ *admits up to* $R+1$ *continuous bounded derivatives on* $S(F)$*.*

Except for the additional assumption that $w > 0$, $\mathcal{U}_R$ and $\mathcal{F}_R$ are defined similar to Campo et al. (2006) and thus have similar implications. Definition 1 requires that $\lambda(x)$ admits $R+1$ continuous bounded derivatives on $[0,\infty)$, and Definition 2 specifies the smoothness of $F(\cdot)$ and requires the corresponding density $f(v)$ to be bounded away from zero on $S(F)$. These regularity assumptions are quite weak. The additional assumption on initial wealth is to guarantee proper behavior of the utility function at the initial wealth level. To relax this assumption so that $w \geq 0$, Definition 1(ii) needs to be replaced by the stronger assumption that "$U(\cdot)$ is continuous and admits up to $R+2$ continuous bounded derivatives on $S(U)$ with $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$ on $S(U)$". The assumption on initial wealth is necessary for analytical approximation of the equilibrium bidding behavior in large auctions. Furthermore, we assume that the private values and the number of bidders are independent so that $f(v|n) = f(v)$. As noted by Guerre, Perrigne and Vuong (2000), this assumption is justified by the economic model. Otherwise, endogenous entry to the auction should be considered, which is outside the scope of this paper.

It is well known that, as the number of bidders n approaches infinity, the equilibrium bid approaches the bidder's private value under quite general conditions. Applying repeated integration by parts and the Laplace approximation (Copson, 1965) to the integral form of the differential equation (1),

$$\lambda(v_p - s_n(v_p)) = \frac{1}{F^{n-1}(v_p)} \int_{\underline{v}}^{v_p} F^{n-1}(u)\, d\left(s_n(u) + \lambda(u - s_n(u))\right),$$

we can derive the leading order deviation of the equilibrium bid from the private value. This is formally stated in the following proposition.[3] Another contribution of Proposition 1 is to characterize

[3] Fibich, Gavious and Sela (2004) have shown (2) based on the unproved claim that $s_n'(v) = 1 + O(n^{-1})$, which, in general, is not directly implied by the (uniform) convergence of $s_n(v)$. Here we take a different approach to derive the leading order deviation of $s_n(v)$ from v. The approach presented here is more rigorous as $s_n'(v) = 1 + O(n^{-1})$ is proved instead of assumed and more general as it allows us to express $s_n(v)$ as its asymptotic expansion with precision of $O(n^{-(R+1)})$ instead of just the leading order deviation.

the implied smoothness of the equilibrium bidding function as $n \to \infty$, which is used to derive the uniform convergence rate of the two-step nonparametric estimator in the next section. Let $\varsigma_n(v) = v - s_n(v)$ be the consumer surplus conditional on winning.

Proposition 1 *In a first-price IPV auction with n ($n \gg 1$) bidders, if $F(\cdot) \in \mathcal{F}_R$ and $U(\cdot) \in \mathcal{U}_R$ for $R \geq 1$, the equilibrium bid in the symmetric Bayesian Nash equilibrium is given by*

$$s_n(v) = v - \frac{1}{n}\frac{F(v)}{f(v)} + O(n^{-2}).^4 \tag{2}$$

Furthermore, we have $\varsigma_n^{(r)}(v) = O(n^{-1})$ for $1 \leq r \leq R$.

Let $G_n(\cdot)$ denote the distribution of equilibrium bids. We have $G_n(b) = F(v)$ with support $S(G_n) = \{b : b \in [\underline{v}, s_n(\bar{v})]\}$ and density $g_n(b) = f(v)/s_n'(v) = f(v) + O(n^{-1})$ by Proposition 1, where $v = s_n^{-1}(b)$. It follows from (2) that

$$v = s_n^{-1}(b) = b + \frac{1}{n}\frac{G_n(b)}{g_n(b)} + O(n^{-2}), \tag{3}$$

which represents the unobserved private value as a function of the observed bid with an error of order $O(n^{-2})$. This allows us to employ Guerre, Perrigne and Vuong's (2000) two-step nonparametric estimation procedure to recover the underlying distribution of risk averse bidders' private values with satisfactory precision when n is large.

3 Nonparametric Estimation and Robustness

3.1 Estimation Procedure and Asymptotic Properties

To clarify conceptual issues, we first consider L homogeneous auctions with n bidders in each auction. In order to implement Guerre, Perrigne and Vuong's (2000) two-step nonparametric estimation procedure, we first need to estimate the distribution of equilibrium bids $G_n(\cdot)$, which depends on the number of bidders. Hence it is important to study the implied smoothness of $G_n(\cdot)$ as $n \to \infty$. The following proposition summarizes the properties of $G_n(\cdot)$ relevant to the asymptotic properties of the nonparametric estimator.

[4] Throughout $f_n(x) = g_n(x) + O(n^p)$ or $f_n(x) = g_n(x) + o(n^p)$ means $\sup_x |f_n(x) - g_n(x)| = O(n^p)$ or $\sup_x |f_n(x) - g_n(x)| = o(n^p)$ respectively, for a pair of functions $f_n(\cdot)$ and $g_n(\cdot)$ and a constant p.

Proposition 2 *If $F(\cdot) \in \mathcal{F}_R$ and $U(\cdot) \in \mathcal{U}_R$ for $R \geq 1$, the distribution $G_n(\cdot)$ satisfies:*

(i) its support is $S(G_n) = \{b : b \in [\underline{v}, s_n(\bar{v})]\}$, with $\inf_{n \in \{2,3,\cdots\}} (s_n(\bar{v}) - \underline{v}) > 0$. Moreover, $S(G_n) \subset S(G_{n+1})$ for all $n \in \{2,3,\cdots\}$, and $\lim_{n\to\infty} S(G_n) = S(F)$;

(ii) for $b \in S(G_n)$, $g_n(b) \geq c_g > 0$ as $n \to \infty$;

(iii) if C is a closed subset of the interior of $S(G_\infty)$, then $g_n(\cdot)$ is bounded and admits up to R continuous bounded derivatives on C as $n \to \infty$.

Contrary to its counterpart with fixed n derived in Campo et al. (2006) where $g_n(\cdot)$ is smoother than $f(\cdot)$ with $R+1$ continuous bounded derivatives, Proposition 2(iii) shows that as $n \to \infty$, the uniform boundedness of the $(R+1)$th derivative of $g_n(\cdot)$ cannot be implied from the existing assumptions on the structure $[U, F]$.

Following Guerre, Perrigne and Vuong (2000), with the observations $\{B_{pl}; p = 1, \cdots, n, l = 1, \cdots, L\}$, the bid distribution $G_n(\cdot)$ and density $g_n(\cdot)$ can be nonparametrically estimated respectively by the empirical distribution and the kernel density estimator of the form

$$\tilde{G}_n(b) = \frac{1}{nL} \sum_{l=1}^{L} \sum_{p=1}^{n} \mathbf{1}(B_{pl} \leq b), \tag{4}$$

$$\tilde{g}_n(b) = \frac{1}{nLh_R} \sum_{l=1}^{L} \sum_{p=1}^{n} K_R\left(\frac{B_{pl} - b}{h_R}\right), \tag{5}$$

where h_R is a bandwidth such that $h_R = \lambda(\log(nL)/nL)^{1/(2R+1)}$ with λ being a strictly positive constant, and $K_R(\cdot)$ is a symmetric kernel of order R with a compact support and twice continuous bounded derivatives satisfying $\int K_R(b)\,db = 1$ and $\int K_R^2(b)\,db < \infty$. Note that classical asymptotic results regarding the empirical distribution and kernel estimator based on the i.i.d. assumption of observations do not apply to the current model as $n \to \infty$, because the equilibrium bid and hence its distribution depend on the number of bidders n. The uniform consistency of $\tilde{G}_n$ and $\tilde{g}_n$ with the convergence rate based on a triangular array of random variables that are independent but not identically distributed as we have here is derived in the appendix.

Because the kernel estimator is asymptotically biased at the boundaries of the support, Guerre, Perrigne and Vuong (2000) suggest trimming the observations B_{pl} that are too close to the boundaries of $S(G_n)$. However, in our case, as n increases, $S(G_n)$ is expanding such that $\lim_{n\to\infty} S(G_n) = S(F)$. Hence the kernel estimator is asymptotically biased at the boundaries of the support of

$F(\cdot)$. Denote the length of the support of $K_R(\cdot)$ by ρ. For $b = \bar{v} - \lambda\rho h_R/2$ with $\lambda \in [0,1)$, it follows that $E[\tilde{g}_n(\bar{v} - \lambda\rho h_R/2)] = \int_{(\underline{b}_n - \bar{v})/h_R + \lambda\rho/2}^{(\bar{b}_n - \bar{v})/h_R + \lambda\rho/2} K_R(u)\, g_n(\bar{v} - \lambda\rho h_R/2 + h_R u) du \to g_n(\bar{v} - \lambda\rho h_R/2) \int_{-\infty}^{\lambda\rho/2} K_R(u)\, du$ as n and L approach infinity. As $\int_{-\infty}^{\lambda\rho/2} K_R(u)\, du \neq 1$, the density estimator is asymptotically biased for $b \in (\bar{v} - \rho h_R/2, \bar{v}]$ and similarly for $b \in [\underline{v}, \underline{v} + \rho h_R/2)$. Let $B_{\min}$ and $B_{\max}$ be the minimum and maximum of the nL observed bids. The trimmed pseudo-private value is defined as

$$\hat{V}_{pl} = \begin{cases} B_{pl} + \tilde{G}_n(B_{pl}) / (n-1)\, \tilde{g}_n(B_{pl}), \\ \quad \text{if } B_{pl} \in [B_{\min} + \rho h_R/2, B_{\max} - \rho h_R/2], \\ \infty \quad \text{otherwise}, \end{cases} \tag{6}$$

for $p = 1, \cdots, n$ and $l = 1, \cdots, L$. The following proposition gives the rate at which the trimmed pseudo-private value converges to the true value on a closed inner subset of its support. The result will be used to derive the uniform convergence rate of the two-step estimator. Let $r = (nL/\log(nL))^{R/(2R+1)}$.

Proposition 3 *Suppose $F(\cdot) \in \mathcal{F}_R$ and $U(\cdot) \in \mathcal{U}_R$ for $R \geq 1$. Then, for any closed inner subset $C(V)$ of $S(F)$, we have almost surely*

$$\sup_{pl} \mathbf{1}_{C(V)}(V_{pl}) \left| \hat{V}_{pl} - V_{pl} \right| = O\left(\max(n/r, 1) n^{-2}\right).$$

Basically, the error of pseudo-private value $\hat{V}_{pl}$ comes from two sources: estimation error from $\tilde{G}_n(\cdot)/\tilde{g}_n(\cdot)$ and approximation error from ignoring the utility structure. So the uniform convergence rate of the pseudo-private value is determined by the slower convergence rate of these two types of errors. Suppose $R = 1$, then $n/r \approx n^2/L$ by ignoring the relatively small $\log(nL)$ term. So if n increases much slower than L such that $n^2/L \to 0$, then the approximation error dominates. The estimation error dominates otherwise.

With the trimmed pseudo-private values, the private value density $f(\cdot)$ can be estimated by the kernel density estimator

$$\hat{f}(v) = \frac{1}{nLh_R} \sum_{l=1}^{L} \sum_{p=1}^{n} K_R\left(\frac{\hat{V}_{pl} - v}{h_R}\right). \tag{7}$$

The following result establishes the uniform consistency of Guerre, Perrigne and Vuong's (2000) two-step estimator with its rate of convergence in homogenous auctions with risk averse bidders.

Proposition 4 *Suppose $F(\cdot) \in \mathcal{F}_R$ and $U(\cdot) \in \mathcal{U}_R$ for $R \geq 1$. Then, for any closed inner subset*

$C(V)$ of $S(F)$,

(i) if $L \to \infty$ and $(nh_R)^{-1} \to 0$, $(r/n)(nh_R)^{-1} \to 0$ as $n \to \infty$, we have almost surely

$$\sup_{v \in C(V)} \left| \hat{f}(v) - f(v) \right| = O(r^{-1});$$

(ii) if $L \to \infty$ and $(nh_R)^{-1} \to 0$, $(r/n)(nh_R)^{-1} \to \infty$ as $n \to \infty$, we have almost surely

$$\sup_{v \in C(V)} \left| \hat{f}(v) - f(v) \right| = O(n^2 h_R)^{-1};$$

(iii) if $L \to \infty$ and $(nh_R)^{-1} \to \infty$ as $n \to \infty$, we have almost surely

$$\sup_{v \in C(V)} \left| \hat{f}(v) - f(v) \right| = O(n^4 h_R^3)^{-1}.$$

Proposition 4(iii) shows that, when n does not diverge fast enough relative to L, Guerre, Perrigne and Vuong's (2000) two-step estimator may not be consistent in the presence of risk aversion given our choice of $K_R(\cdot)$ and h_R because of the overwhelming approximation error. A sufficient condition for the two-step estimator to be consistent is that $(nh_R)^{-1} \to 0$, which imposes a lower bound of the divergence rate of n in terms of L. By ignoring the relatively small $\log(nL)$ term, we have $(nh_R)^{-1} \approx L/n^{2R}$. Hence the constraint on the divergence rate of n is quite weak, especially for a smooth private value density (with larger R). On the other hand, when n goes to infinity fast enough relative to L, it is possible for the two-step nonparametric estimator to attain the uniform convergence rate $r = (nL/\log(nL))^{R/(2R+1)}$, which is the best uniform convergence rate when private values are observable. The intuition for the result is as follows. As $f(v) = g_n(s_n(v))\, s_n'(v)$, to estimate the private value density, $g_n(\cdot), s_n(\cdot)$ and $s_n'(\cdot)$ need to be estimated. When n is fixed, $s_n'(\cdot)$ is the hardest to estimate as it requires estimating $g_n'(\cdot)$. In fact, the best uniform convergence rate for estimating $s_n'(\cdot)$ determines the best rate for estimating $f(\cdot)$. However, when $n \to \infty$, it follows from Proposition 1 that $s_n'(v) = 1 + O(n^{-1})$. So if n diverges fast enough, $f(\cdot)$ can be estimated at the same best rate as $g_n(\cdot)$, which is r.

As in Guerre, Perrigne and Vuong (2000), asymptotic normality of the two-step estimator is not derived. This is because the first and second order terms in the expansion of $\hat{f}(v) - f(v)$ may be close (see the proof of Proposition 4), so the classical asymptotic normality result that relies only on the leading order term in the Taylor expansion may be imprecise. Guerre, Perrigne and Vuong (2000) suggest circumventing this drawback by establishing an exponential-type inequality, and that approach also applies to the current model. Interested readers may refer to that paper for more details.

3.2 Auctions with Heterogeneity

Now we can extend the above analysis to a more realistic model allowing heterogeneity. Heterogeneity across auctions is characterized by a vector of observed variables X_l and the number of bidders nI_l $(l = 1, \cdots, L)$, where the I_l's are strictly positive constants.[5] We assume n, but not I_l, approaches infinity for asymptotic properties. Let $\mathcal{I}$ be the set of possible values for I_l. Following Guerre, Perrigne and Vuong (2000), the latent joint distribution of (V_{pl}, X_l, I_l) for $p = 1, \cdots, nI_l$ and $l = 1, \cdots, L$ satisfies the following regularity assumptions:

Assumption A1

(i) *The $(d+1)$-dimensional vectors (X_l, I_l), $l = 1, \cdots, L$, are independently and identically distributed as $F_m(\cdot, \cdot)$ with density $f_m(\cdot, \cdot)$.*

(ii) *For each l, the variables V_{pl}, $p = 1, \cdots, nI_l$, are independently and identically distributed conditionally upon X_l as $F(\cdot|\cdot)$ with density $f(\cdot|\cdot)$.*

Assumption A2 *For $\mathcal{I}$ a bounded countable subset of $\mathbb{R}^+$ and $R \geq 1$,*

(i) $S(F) = \{(v, x) : x \in [\underline{x}, \bar{x}], v \in [\underline{v}(x), \bar{v}(x)]\}$, *with* $\underline{x} < \bar{x}$;

(ii) *for $(v, x) \in S(F)$, $f(v|x) \geq c_f > 0$, and, for $(x, i) \in S(F_m)$, $f_m(x, i) \geq c_f > 0$;*

(iii) *for each $i \in \mathcal{I}$, $f(\cdot|\cdot)$ and $f_m(\cdot, i)$ admit up to R continuous bounded partial derivatives on $S(F)$ and $S(F_m(\cdot, i))$.*

As argued by Guerre, Perrigne and Vuong (2000), we can assume that $\underline{x}$ and $\bar{x}$ are known as they can be readily estimated. X is assumed to be a vector of continuous variables.[6] The economic model implies that the private values and the number of bidders are independent conditional on X so that $f(v|x, ni) = f(v|x)$. With the smoothness of $F(\cdot|\cdot)$ specified in Assumption A2, the next proposition studies the implied smoothness of bid density $g_n(\cdot|\cdot, \cdot)$.

[5] Empirically, we can decompose the number of bidders of the lth auction arbitrarily into $n \in \{2, 3, \cdots\}$ and $I_l \in \mathbb{R}^+$. Say, let $n = \min_l\{nI_l\}$.

[6] If some X's are discrete, the following results hold with d replaced by the number of continuous variables in X.

Proposition 5 *Suppose $U(\cdot) \in \mathcal{U}_R$ for $R \geq 1$. Given A1 and A2, the conditional distribution $G_n(\cdot|\cdot,\cdot)$ satisfies:*

(i) its support $S(G_n)$ is such that $S(G_n(\cdot|\cdot,i)) = \{(b,x) : x \in [\underline{x}, \bar{x}], b \in [\underline{b}_n(x,i), \bar{b}_n(x,i)]\}$, with $\inf(\bar{b}_n(x,i) - \underline{b}_n(x,i)) > 0$. Moreover, $\bar{b}_n(x,i) \geq \bar{b}_m(x,i)$ for $n \geq m$, $\underline{b}_n(\cdot,i) = \underline{v}(\cdot)$, and $\lim_{n\to\infty} \bar{b}_n(\cdot,i) = \bar{v}(\cdot)$;

(ii) for $(b,x,i) \in S(G_n)$, $g_n(b|x,i) \geq c_g > 0$ as $n \to \infty$;

(iii) if C is a closed subset of the interior of $S(G_\infty)$, then, for each $i \in \mathcal{I}$, $g_n(\cdot|\cdot,i)$ is bounded and admits up to R continuous bounded derivatives on C as $n \to \infty$.

Proposition 5 extends Proposition 2 by allowing possible heterogeneity across auctions and has similar implications. Specially, item (iii) characterizes the uniform boundedness of g_n's derivatives as $n \to \infty$, which is used to derive asymptotic properties of the nonparametric estimator.

Following Guerre, Perrigne and Vuong (2000), using the observations $\{(B_{pl}, X_l, I_l); p = 1, \cdots, nI_l, l = 1, \cdots, L\}$, we can nonparametrically estimate $G_n(\cdot,\cdot,\cdot)$ and $g_n(\cdot,\cdot,\cdot)$ respectively by

$$\tilde{G}_n(b,x,i) = \frac{1}{nLh_G^d} \sum_{l=1}^{L} \frac{1}{I_l} \sum_{p=1}^{nI_l} \mathbf{1}(B_{pl} \leq b) K_G\left(\frac{X_l - x}{h_G}, \frac{I_l - i}{h_{GI}}\right), \quad (8)$$

$$\tilde{g}_n(b,x,i) = \frac{1}{nLh_g^{d+1}} \sum_{l=1}^{L} \frac{1}{I_l} \sum_{p=1}^{nI_l} K_g\left(\frac{B_{pl} - b}{h_g}, \frac{X_l - x}{h_g}, \frac{I_l - i}{h_{gI}}\right), \quad (9)$$

where h_G, h_{GI}, h_g, and h_{gI} are bandwidths and K_G and K_g are kernels with a compact support.

Similar to the case with homogeneous auctions, the asymptotic results of nonparametric estimators based on i.i.d. assumptions do not apply to $\tilde{G}_n$ and $\tilde{g}_n$ as $n \to \infty$ due to the dependence of the equilibrium bid distribution on n. We derive the uniform consistency with the convergence rate of $\tilde{G}_n$ and $\tilde{g}_n$ in the appendix. On the other hand, since the number of B_{pl} is of order $O(nL)$ while the number of observed auctions and hence (X_l, I_l) (which are analogous to fixed effects in a panel data model) are of order $O(L)$, the best uniform convergence rate for the nonparametric estimation of the joint density of (B_{pl}, X_l, I_l) as both n and L approach infinity has seldom been addressed in the literature. The following analysis sheds light on whether and to what extent $n \to \infty$ speeds up the convergence of the joint density estimator.

Since the kernel density estimator is biased at the boundaries of the support of $S(F)$ as we discussed in the case with homogenous auctions, we trim the observations that are too close to the boundary of $S(F)$. To this end, we need to estimate the unknown $S(F) = \{(v, x) : x \in [\underline{x}, \bar{x}], v \in [\underline{v}(x), \bar{v}(x)]\}$. Since $[\underline{x}, \bar{x}]$ is known, we only need to estimate the support $[\underline{v}(x), \bar{v}(x)]$. Let $h_\partial > 0$. Following Guerre, Perrigne and Vuong (2000), we consider the following partition of $\mathbb{R}^d$ with a generic hypercube of side h_∂:

$$\pi_{k_1,\cdots,k_d} = [k_1 h_\partial, (k_1+1) h_\partial) \times \cdots \times [k_d h_\partial, (k_d+1) h_\partial),$$

where $(k_1, \cdots, k_d)$ runs over $\mathbb{Z}^d$. The support $[\underline{v}(x), \bar{v}(x)]$ can be estimated as

$$\widehat{\bar{v}}(x) = \sup\{B_{pl}, p = 1, \cdots, nI_l, l = 1, \cdots, L; X_l \in \pi_{k_1,\cdots,k_d}\}, \tag{10}$$

$$\widehat{\underline{v}}(x) = \inf\{B_{pl}, p = 1, \cdots, nI_l, l = 1, \cdots, L; X_l \in \pi_{k_1,\cdots,k_d}\}, \tag{11}$$

where $\pi_{k_1,\cdots,k_d}$ is the hypercube containing x. And the estimator for $S(F)$ is $\hat{S}(F) \equiv \{(v, x) : x \in [\underline{x}, \bar{x}], v \in [\widehat{\bar{v}}(x), \widehat{\underline{v}}(x)]\}$.

Note that (3) can be rewritten as

$$V_{pl} = B_{pl} + \frac{1}{nI_l} \frac{G_n(B_{pl}, X_l, I_l)}{g_n(B_{pl}, X_l, I_l)} + O(n^{-2}),$$

where $G_n(b, x, i) = G_n(b|x, i) f_m(x, i)$. Guerre, Perrigne and Vuong's (2000) pseudo-private value is estimated by

$$\hat{V}_{pl} = B_{pl} + \frac{1}{nI_l - 1} \hat{\psi}(B_{pl}, X_l, I_l),$$

where

$$\hat{\psi}(b, x, i) \equiv \begin{cases} \tilde{G}_n(b, x, i) / (nI_l - 1) \tilde{g}_n(b, x, i), \\ \quad \text{if } (b, x) + S(2h_G) \subset \hat{S}(F) \text{ and} \\ \quad (b, x) + S(2h_g) \subset \hat{S}(F), \\ \infty \quad \text{otherwise,} \end{cases}$$

with $S(h_G)$ and $S(h_g)$ being the supports of $\{0 \times K_G(\cdot/h_G, 0)\}$ and $K_g(\cdot/h_g, \cdot/h_g, 0)$ respectively.

In the second step of Guerre, Perrigne and Vuong's (2000) two-step estimation approach, the density $f(v|x)$ is estimated nonparametrically by $\hat{f}(v|x) = \hat{f}(v, x) / \hat{f}(x)$ using the pseudo-sample

$\{(\hat{V}_{pl}, X_l), p = 1, \cdots, nI_l, l = 1, \cdots, L\}$, where

$$\hat{f}(v,x) = \frac{1}{nLh_f^{d+1}} \sum_{l=1}^{L} \frac{1}{I_l} \sum_{p=1}^{nI_l} K_f \left(\frac{\hat{V}_{pl} - v}{h_f}, \frac{X_l - x}{h_f} \right), \tag{12}$$

$$\hat{f}(x) = \frac{1}{Lh_X^d} \sum_{l=1}^{L} K_X \left(\frac{X_l - x}{h_X} \right), \tag{13}$$

h_f and h_X are bandwidths, and K_f and K_X are kernels with compact supports. The choice of kernels and bandwidths in the definition of the two-step nonparametric estimator are summarized in the following two assumptions:

Assumption A3

(i) *The kernels $K_G(\cdot,\cdot)$, $K_g(\cdot,\cdot,\cdot)$, $K_f(\cdot,\cdot)$ and $K_X(\cdot)$ are symmetric with bounded hypercube supports and twice continuous bounded (uniformly in I) derivatives with respect to their continuous arguments.*

(ii) $\int K_G(x,0)\,dx = 1$, $\int K_g(b,x,0)\,dbdx = 1$, $\int K_f(v,x)\,dvdx = 1$, *and* $\int K_X(x)\,dx = 1$.

(iii) $K_G(x,0)$ is of order $R+1$, *and* $K_g(b,x,0)$, $K_f(v,x)$ *and* $K_X(x)$ *are of order* R.

Assumption A4

(i) *As $L \to \infty$, the "discrete" bandwidths h_{GI} and h_{gI} vanish.*

(ii) *The "continuous" bandwidths h_G, h_g, h_f, and h_X are of the form:*

$$h_G = \lambda_G(\log L/L)^{1/(2R+d+2)},\ h_g = \lambda_g(\log L/L)^{1/(2R+d)},$$
$$h_f = \lambda_f(\log L/L)^{1/(2R+d)},\ h_X = \lambda_X(\log L/L)^{1/(2R+d)},$$

where the λ's are strictly positive constants.

(iii) *The "boundary" bandwidth is of the form $h_\partial = \lambda_\partial(\log L/L)^{1/(d+1)}$ with $\lambda_\partial > 0$, if $d > 0$.*

It follows from Hardle (1991) that h_G and h_X given in A4(ii) are optimal bandwidths given Proposition 5 and A2(iii). Hence $G_n(\cdot,\cdot,\cdot)$ and $f(\cdot)$ are optimally estimated in terms of the uniform convergence rate. If n were fixed and private values were observed, the optimal bandwidth for estimating $f(\cdot,\cdot)$ would be of order $(\log L/L)^{1/(2R+d+1)}$, which is asymptotically larger than the rate for h_f given in A4(ii). Similarly, the rate for h_g given in A4(ii) is asymptotically smaller than the optimal bandwidth with fixed n. However, our choices of h_f and h_g are optimal when n approaches infinity fast enough relative to L as shown below.

The following results establish the uniform consistency of the nonparametric estimators of $S(F)$ and $f(v|x)$ in large auctions with risk averse bidders.

Proposition 6 *Let $r_\partial = (L/\log L)^{1/(d+1)}$. Given A1, A2 and A4(iii), we have almost surely*

$$\sup_{x\in[\underline{x},\bar{x}]} \left|\widehat{\bar{v}}(x) - \bar{v}(x)\right| = O(r_\partial^{-1}), \text{ and } \sup_{x\in[\underline{x},\bar{x}]} \left|\widehat{\underline{v}}(x) - \underline{v}(x)\right| = O(r_\partial^{-1}).$$

We have shown in the case with homogeneous auctions that a sufficient condition for Guerre, Perrigne and Vuong's (2000) two-step estimator to be uniformly consistent is that n goes to infinity fast enough relative to L so that $(nh_R)^{-1} \to 0$. So the next result on the uniform convergence rate focuses on the case with $(nh_f)^{-1} \to 0$. Let $r_f = (L/\log L)^{R/(2R+d)}$.

Proposition 7 *Suppose $U(\cdot) \in \mathcal{U}_R$ for $R \geq 1$. Given A1-A4, for any closed inner subset $C(V)$ of $S(F)$,*

(i) if $L \to \infty$ and $(nh_f)^{-1} \to 0$, $(r_f/n)(nh_f)^{-1} \to 0$ as $n \to \infty$, we have almost surely

$$\sup_{v\in C(V)} \left|\hat{f}(v|x) - f(v|x)\right| = O(r_f^{-1});$$

(ii) if $L \to \infty$ and $(nh_f)^{-1} \to 0$, $(r_f/n)(nh_f)^{-1} \to \infty$ as $n \to \infty$, we have almost surely

$$\sup_{v\in C(V)} \left|\hat{f}(v|x) - f(v|x)\right| = O(n^2 h_f)^{-1}.$$

So when n approaches infinity fast enough relative to L, the two-step estimator of $f(v|x)$ can attain the best rate at which $f(x)$ can be estimated. Even though $f(v|x)$ is as smooth as $f(x)$ given A2(iii), one would expect $f(v|x)$ to be estimated with a convergence rate slower than $f(x)$ because private values are unobservable and the vector (V, X) has one more dimension than X. The counterintuitive result in Proposition 7 can be understood as follows. First, since unknown private values can be approximated by observed bids with precision of order $O(n^{-1})$ by Proposition 1, the approximation error may be trivial compared to the estimation error of the kernel estimator when n

goes to infinity fast enough relative to L. Hence, the information loss from not observing V may be negligible given the conditions in Proposition 7(i). Second, because there are $(n-1)L$ more (pseudo) observations of V than X, the noise from estimating the extra dimension of random variables in $f(v,x)$ and hence $f(v|x)$ reduces dramatically as $n \to \infty$. We show in the appendix that, when n diverges fast enough so that $(nh_g)^{-1} \to 0$ and $(nh_f)^{-1} \to 0$, kernel estimators of $g_n(b,x,i)$ and $f(v,x)$ can attain the best rate at which $\hat{f}(x)$ uniformly converges to $f(x)$.

4 Monte Carlo Experiments

We conduct the Monte Carlo experiments with 1000 replications, each consisting of three sets of observations. In set 1, we consider $L = 300$ auctions, each with $n = 3$ bidders. In set 2, we consider $L = 150$ auctions, each with $n = 6$ bidders. In set 3, we consider $L = 75$ auctions, each with $n = 12$ bidders. The total number of observations of submitted bids is 900 for each set. Bidders' private values for each replication are generated from the log-normal distribution F with parameters $(0,1)$, truncated at 0.055 and 2.5. The true utility takes the functional form $U(x) = 1 - \exp(-\theta x)$, where $\theta = 0.8$. The equilibrium bids are computed numerically by

$$b = \frac{1}{\theta} \log \frac{\int^v \exp(\theta t)\, dF(t)^{n-1}}{F(v)^{n-1}}. \tag{14}$$

We consider four different estimation procedures for each replication. Method 1 serves as the basis for comparison. We specify the functional form of utility as the true $U(\cdot)$ and adopt the semiparametric approach proposed by Campo et al. (2006). To estimate θ, we pool the observations from all 3 sets. Let $G_n(b)$ denote the distribution of bids in auctions with n bidders, v_α denote the αth percentile of F, and b_α^n denote the αth percentile of G_n. For $n \neq m$, (1) gives

$$\begin{aligned} v_\alpha - b_\alpha^n &= \frac{1}{\theta}\log[\frac{\theta}{n-1}\frac{G_n(b_\alpha^n)}{g_n(b_\alpha^n)} + 1], \\ v_\alpha - b_\alpha^m &= \frac{1}{\theta}\log[\frac{\theta}{m-1}\frac{G_m(b_\alpha^m)}{g_m(b_\alpha^m)} + 1]. \end{aligned}$$

Taking difference gives

$$b_\alpha^m - b_\alpha^n = \frac{1}{\theta}\log[\frac{\theta}{n-1}\frac{G_n(b_\alpha^n)}{g_n(b_\alpha^n)} + 1] - \frac{1}{\theta}\log[\frac{\theta}{m-1}\frac{G_m(b_\alpha^m)}{g_m(b_\alpha^m)} + 1] \tag{15}$$

With a large number of percentiles α, we can estimate θ using the empirical analogue of (15) by nonlinear least squares. Given an estimate $\hat{\theta}$ of θ, we then estimate f using the two-step kernel-based estimation procedure described above for each set of observations separately.

Method 2 investigates the consequences of model misspecification by assuming the utility is CRRA with $U(x) = x^{1-\theta}$ using the semiparametric approach proposed by Campo et al. (2006). Analogous to Model 1, we identify θ through the heterogeneity of the bid distributions across auctions with different number of bidders. With CRRA utility, for $n \neq m$, (1) gives

$$\begin{aligned} v_\alpha - b_\alpha^n &= \frac{1-\theta}{n-1} \frac{G_n(b_\alpha^n)}{g_n(b_\alpha^n)}, \\ v_\alpha - b_\alpha^m &= \frac{1-\theta}{m-1} \frac{G_m(b_\alpha^m)}{g_m(b_\alpha^m)}. \end{aligned}$$

Taking the difference gives

$$b_\alpha^m - b_\alpha^n = (1-\theta)\left(\frac{1}{n-1}\frac{G_n(b_\alpha^n)}{g_n(b_\alpha^n)} - \frac{1}{m-1}\frac{G_m(b_\alpha^m)}{g_m(b_\alpha^m)}\right). \qquad (16)$$

Evaluating the empirical analogue of (1) at a finite number of percentiles, we can recover θ using least squares. Then we estimate f nonparametrically for each set of observations separately.

Method 3 recovers f using Guerre, Perrigne and Vuong's (2000) two-step nonparametric estimation procedure without imposing any restrictions on the functional form of $U(\cdot)$. Method 4 is a one-step nonparametric estimation method using the observed bids as the pseudo-private values to estimate f directly, based on the fact that $\lim_{n\to\infty} s_n(v) = v$. Method 4 can only be justified when the number of bidders in each auction is very large and strategic bidding behavior is overwhelmed by the price-taking behavior in perfection competition. We compare the estimates from Methods 3 and 4 to understand the gains from incorporating strategic bidding behavior in the structural estimation.

Following Guerre, Perrigne and Vuong (2000), in nonparametric estimations we choose the triweight kernel $(35/32)\left(1-u^2\right)^3 \mathbf{1}(|u| \leq 1)$ for $K_g(\cdot)$ and $K_f(\cdot)$ so that $\rho_g = \rho_f = 2$. We also choose $h_g = 1.06\hat{\sigma}_b (nL)^{-1/5}$ and $h_f = 1.06\hat{\sigma}_v (nL_T)^{-1/5}$, where $\hat{\sigma}_b$ and $\hat{\sigma}_v$ are the standard deviations of the observed bids and the trimmed pseudo-private values, nL_T are the number of observations left after trimming, and 1.06 follows the rule of thumb.[7]

[7] Our choices of kernel functions and bandwidths do not follow Assumptions A3 and A4 because the gains of high order kernels in terms of a lower MISE are trivial with this sample size. (Fan and Marron, 1992)

Table 1: Intregrated Absolute Bias of Estimated Densities

	5-95th percentile			25-75th percentile		
	$n=3$	$n=6$	$n=12$	$n=3$	$n=6$	$n=12$
Method 1	0.0258	0.0279	0.0320	0.0023	0.0019	0.0017
Method 2	0.0757	0.0528	0.0427	0.0232	0.0163	0.0117
Method 3	0.0700	0.0442	0.0366	0.0242	0.0076	0.0022

Method 1: mean(θ)=0.7682, std(θ)=0.1934;
Method 2: mean(θ)=0.3835, std(θ)=0.0916.

Figures 1-4 display the true density of the private values with solid line and the 5th, 50th and 95th percentiles of the 1000 estimates of $\hat{f}(v)$ with dash-dot lines evaluated at 500 equally spaced points on $[0.055, 2.5]$. When the utility functional form is correctly specified, the mean of the semiparametric estimates in Figure 1 perfectly matches the true density on the 25-75th percentile of the distribution and the empirical pointwise 90% confidence interval becomes narrower as n increases. In the case that the utility function is misspecified, the semiparametric estimates in Figure 2 are biased upwards for small private values and biased downwards for large private values when $n = 3$. The bias reduces as n increases. Guerre, Perrigne and Vuong's (2000) two-step nonparametric estimates in Figure 3 are slightly downward biased when $n = 3$. The bias reduces as n increases to 6. The one-step nonparametric estimates in Figure 4 are very imprecise as a large part of the true density lies outside the empirical 90% confidence interval when $n = 3$. The performance of the one-step nonparametric estimates improves when $n = 12$.

To compare Methods 1-3 with higher precision, we report the integrated absolute bias evaluated respectively on the 5-95th percentile and the 25-75th percentile of the value distribution in Table 1. We use the integrated absolute bias instead of the integrated mean squared error as a measure of discrepancy because the semiparametric estimates may have larger standard error than the two-step nonparametric estimates as the former involves an additional step to estimate unknown parameters in the utility function. The integrals are evaluated by simulations. The two-step nonparametric estimates have smaller integrated absolute bias relative to the semiparametric estimates with misspecified utility function when $n = 6$ and 12. The bias of the two-step nonparametric estimator reduces much faster than the semiparametric estimates with misspecified utility function on the 25-75th percentile of the value distribution as n increases.

There are two important lessons to draw from the Monte Carlo experimental results. First, Guerre, Perrigne and Vuong's (2000) two-step nonparametric estimation procedure is quite robust

Figure 1: True and Estimated Densities of Private Values (Method 1)

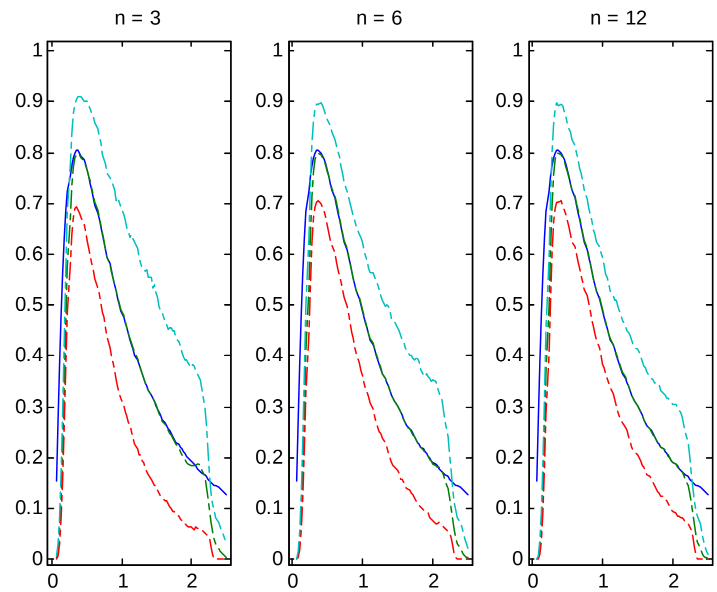


Figure 2: True and Estimated Densities of Private Values (Method 2)

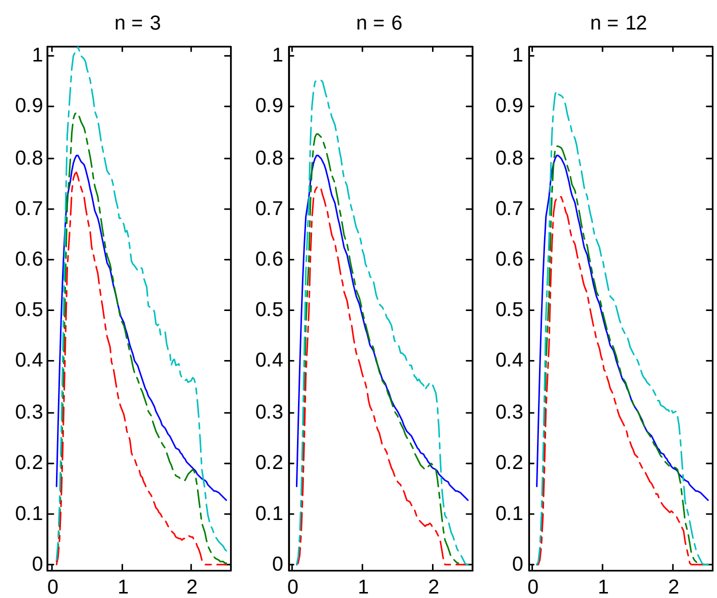


Figure 3: True and Estimated Densities of Private Values (Method 3)

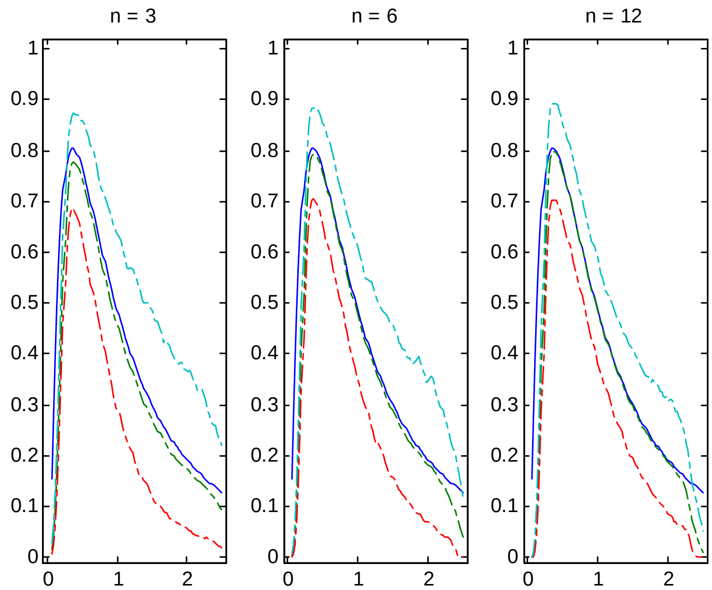
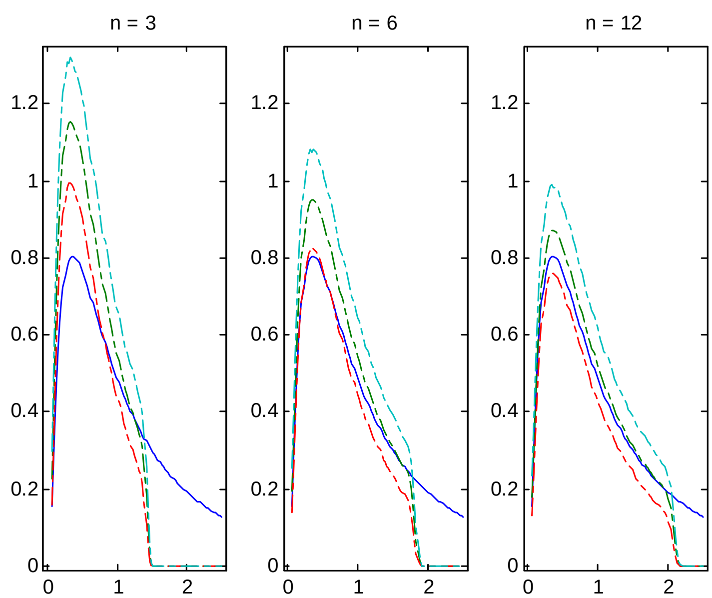


Figure 4: True and Estimated Densities of Private Values (Method 4)



with respect to risk aversion in auctions with a moderate number of bidders, and inclusion of the term $G_n(b)/(n-1)g_n(b)$ in the approximated bidding function substantially improves the performance of the estimator, as illustrated by Figures 3 and 4. Second, though the CRRA structure $U(x) = x^{1-\theta}$ is popularly assumed in the literature for many reasons, the semiparametric specification does not necessarily help to improve the fitting of $\hat{f}$. This can be understood as follows. As we discussed in Section 2, if $w = 0$, Definition 1(ii) needs to be replaced by the stronger assumption that "$U(\cdot)$ is continuous and admits up to $R+2$ continuous bounded derivatives on $S(U)$ with $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$ on $S(U)$". However, $U(x) = x^{1-\theta} \notin \mathcal{U}_R$ as $w = 0$ and $U'(0)$ is not bounded. Hence the effects of model misspecification on the equilibrium bids are $O(n^{-1})$ in this case, which dominates the errors incurred by totally ignoring risk aversion.

5 Concluding Remarks

We study the robustness of Guerre, Perrigne and Vuong's (2000) two-step nonparametric estimation procedure in large auctions with risk averse bidders. With an asymptotic approximation of the equilibrium bidding function, we show that when the number of bidders in each auction diverges not too slowly relative to the number of observed auctions, Guerre, Perrigne and Vuong's (2000) two-step kernel-based estimator is uniformly consistent on an arbitrary closed inner subset of the support of the true density and attains the best uniform convergence rate as if latent private values are observable. Monte Carlo experiments show that the two-step estimator performs reasonably well with a moderate number of bidders such as six.

One possible extension of the current work is to allow bidders to have different attitude towards risk captured by heterogeneous utility functions and initial wealths. Campo (2004) has shown that in such a model the utility functions and latent distribution of bidders' private values cannot be nonparametrically identified jointly from observed bids, and, to recover the private value distribution, it is necessary to specify the asymmetric utility structure parametrically. On the other hand, when the number of bidders is large, the effects of asymmetric risk aversion on equilibrium bids diminish. Hence asymptotic approximation of the equilibrium bidding function may provide a feasible way to implement nonparametric estimation methods in large auctions with asymmetric risk averse bidders as well, which could be of interest for future research.

A Proofs of Mathematical Properties

Proof of Proposition 1. (i) Since the equilibrium solution is symmetric in nature, we can drop the individual subscript in (1). Let $s_{RN}(\cdot)$ be the solution of the following first-order differential equation

$$s'_{RN,n}(v) = (n-1)\frac{f(v)}{F(v)}(v - s_{RN,n}(v)), \tag{17}$$

with boundary condition $s_{RN,n}(\underline{v}) = \underline{v}$. Fibich, Gavious and Sela (2004) have shown that $s_{RN,n}(v) = v + O(n^{-1})$. As $0 \leq v - s_n(v) \leq v - s_{RN,n}(v)$ for all $v \in S(F)$ (Riley and Samuelson, 1981), we can extend $\varsigma_n(v)$ to the following form

$$\varsigma_n(v) = v - s_n(v) = \frac{1}{n-1}\varsigma_{1n}(v) + o(n^{-1}), \tag{18}$$

where $\varsigma_{1n}(v) = O(1)$. As $\lambda(0) = 0$ and $\lambda'(0) = 1$, a Taylor expansion of $\lambda(\varsigma_n(v)) = \lambda(v - s_n(v))$ around 0 gives

$$\lambda(\varsigma_n(v)) = \lambda(0) + \lambda'(0)\varsigma_n(v) + \frac{1}{2}\lambda''(\tilde{x})\varsigma_n^2(v) = \varsigma_n(v) + \frac{1}{2}\lambda''(\tilde{x})\varsigma_n^2(v), \tag{19}$$

for $\tilde{x} \in [0, \varsigma_n(v)]$. Since $\lambda''(\tilde{x})$ is bounded as $n \to \infty$ by Definition 1, substitution of (18) into (19) gives

$$\lambda(\varsigma_n(v)) = \frac{1}{n-1}(\varsigma_{1n}(v) + o(1)), \tag{20}$$

which implies $s'_n(v) = O(1)$ by (1). Multiplying both sides of the differential equation (1) by $F^{n-1}(v)$ and taking integrals gives

$$\begin{aligned} s'_n(v)F^{n-1}(v) &= (n-1)f(v)F^{n-2}(v)\lambda(v - s_n(v)) \\ \int_{\underline{v}}^{v} F^{n-1}(u)\,ds_n(u) &= \int_{\underline{v}}^{v} \lambda(u - s_n(u))\,dF^{n-1}(u). \end{aligned} \tag{21}$$

Applying integration by parts to the right hand side of (21), and rearranging terms yields the integral form of the first order condition

$$\lambda(v - s_n(v)) = \frac{1}{F^{n-1}(v)}\int_{\underline{v}}^{v} F^{n-1}(u)\,d(s_n(u) + \lambda(u - s_n(u))). \tag{22}$$

Let $\phi_n(v) = s_n(v) + \lambda(v - s_n(v))$, we have

$$\begin{aligned}
\phi_n'(v) &= s_n'(v) + \lambda'(v - s_n(v))(1 - s_n'(v)) \\
&= s_n'(v) + \left(\lambda'(0) + O(v - s_n(v))\right)(1 - s_n'(v)) \\
&= 1 + \varsigma_{1n}(v) O\left(n^{-1}\right)(1 - s_n'(v)) \\
&= 1 + O\left(n^{-1}\right),
\end{aligned}$$

where the second equality holds by the mean value theorem and boundedness of $\lambda''(\cdot)$, the third equality holds because $\lambda'(0) = 1$, and the last equality holds because $\varsigma_{1n}(v) = O(1)$ and $s_n'(v) = O(1)$. We rewrite (22) in the format of Laplace integral and apply the Laplace approximation (Copson, 1965)

$$\begin{aligned}
\lambda(v - s_n(v)) &= \frac{1}{F^{n-1}(v)} \int_{\underline{v}}^{v} F^{n-1}(u)\phi_n'(u)\,du = \frac{1}{F^{n-1}(v)} \int_{\underline{v}}^{v} e^{(n-1)\ln F(u)} \phi_n'(u)\,du \\
&= \frac{\phi_n'(v)}{F^{n-1}(v)} \frac{e^{(n-1)\ln F(v)}}{(n-1)\, d\ln F(v)/dv} + o\left(n^{-1}\right) \\
&= \frac{1}{n-1}\frac{F(v)}{f(v)} + o\left(n^{-1}\right), \qquad (23)
\end{aligned}$$

where the last equality holds because $\phi_n'(v) = 1 + O\left(n^{-1}\right)$. Matching leading order terms in (20) and (23) gives $\varsigma_{1n}(v) = F(v)/f(v)$, which, together with (20) and (1), implies that $s_n'(v) = 1 + o(1)$ and $\phi_n'(v) = 1 + o\left(n^{-1}\right)$. So we can further extend $\varsigma_n(v)$ to the following form

$$\varsigma_n(v) = v - s_n(v) = \frac{1}{n}\frac{F(v)}{f(v)} + \frac{1}{n^2}\varsigma_{2n}(v) + o\left(n^{-2}\right). \qquad (24)$$

Substitution of (24) in (19) gives

$$\lambda(\varsigma_n(v)) = \frac{1}{n}\frac{F(v)}{f(v)} + \frac{1}{n^2}\varsigma_{2n}(v) + \frac{1}{2}\lambda''(\tilde{x})\left(\frac{1}{n}\frac{F(v)}{f(v)}\right)^2 + o\left(n^{-2}\right), \qquad (25)$$

Taking derivatives on both sides of (1) gives

$$s_n''(v) = (n-1)\left[\frac{d}{dv}\left(\frac{f(v)}{F(v)}\right)\lambda(v - s_n(v)) + \frac{f(v)}{F(v)}\lambda'(v - s_n(v))(1 - s_n'(v))\right].$$

Taylor approximations of $\lambda(v - s_n(v))$ and $\lambda'(v - s_n(v))$ around 0 yield $s_n''(v) = o(n)$, which implies that

$$\begin{aligned}
\phi_n''(v) &= s_n''(v) - \lambda'(v - s_n(v)) s_n''(v) + \lambda''(v - s_n(v))(1 - s_n'(v))^2 \\
&= s_n''(v) - \left(\lambda'(0) + O\left(n^{-1}\right)\right) s_n''(v) + \lambda''(v - s_n(v)) o(1) = o(1).
\end{aligned}$$

It follows from applying integration by parts and the Laplace approximation to (22) that

$$\begin{aligned}
\lambda(v - s_n(v)) &= \frac{1}{F^{n-1}(v)} \int_{\underline{v}}^{v} F^{n-1}(u) \phi_n'(u) du = \frac{1}{nF^{n-1}(v)} \int_{\underline{v}}^{v} \frac{\phi_n'(u)}{f(u)} dF^n(u) \\
&= \frac{1}{n} \frac{F(v)\phi_n'(v)}{f(v)} - \frac{1}{nF^{n-1}(v)} \int_{\underline{v}}^{v} F^n(u) d\frac{\phi_n'(u)}{f(u)} \\
&= \frac{1}{n} \frac{F(v)\phi_n'(v)}{f(v)} - \frac{1}{nF^{n-1}(v)} \int_{\underline{v}}^{v} e^{n \ln F(u)} \frac{\phi_n''(u) f(u) - \phi_n'(u) f'(u)}{f^2(u)} du \\
&= \frac{1}{n} \frac{F(v)\phi_n'(v)}{f(v)} - \frac{F^2(v)}{n^2} \frac{\phi_n''(v) f(v) - \phi_n'(v) f'(v)}{f^3(v)} + o\left(n^{-2}\right) \\
&= \frac{1}{n} \frac{F(v)}{f(v)} + \frac{F^2(v)}{n^2} \frac{f'(v)}{f^3(v)} + o\left(n^{-2}\right), \qquad (26)
\end{aligned}$$

where the last equality holds because $\phi_n'(v) = 1 + o\left(n^{-1}\right)$ and $\phi_n''(v) = o(1)$. Matching leading terms of (25) and (26) yields that

$$\varsigma_{2n}(v) = \left(\frac{f'(v)}{f(v)} - \frac{1}{2}\lambda''(\tilde{x})\right) \left(\frac{F(v)}{f(v)}\right)^2 = O(1),$$

which implies (2). Substitution of (26) into (1) gives $\varsigma_n'(v) = 1 - s_n'(v) = \frac{1}{n} \frac{d}{dv} \left(\frac{F(v)}{f(v)}\right) + o\left(n^{-1}\right)$.

(ii) First, we show by mathematical induction that

$$\begin{aligned}
\lambda(\varsigma_n(v)) &= \lambda(v - s_n(v)) = \frac{\alpha_1(v)}{n} + \cdots + \frac{\alpha_{r+1}(v)}{n^{r+1}} + o\left(n^{-(r+1)}\right), \qquad (27) \\
\varsigma_n(v) &= v - s_n(v) = \frac{\beta_1(v)}{n} + \cdots + \frac{\beta_{r+1}(v)}{n^{r+1}} + o\left(n^{-(r+1)}\right), \qquad (28)
\end{aligned}$$

where $\alpha_1(v), \cdots, \alpha_{r+1}(v)$ and $\beta_1(v), \cdots, \beta_{r+1}(v)$ are known functions invariant with n, and

$$\varsigma_n^{(r)}(v) = \frac{1}{n} \frac{d^r}{dv^r} \left(\frac{F(v)}{f(v)}\right) + o\left(n^{-1}\right), \qquad (29)$$

for $0 \leq r \leq R - 1$. We have already shown in (i) that (27)-(29) hold for $r = 0$, so we only need to show that (27)-(29) holding for $0 \leq r \leq k-1$ implies (27)-(29) hold for $r = k \leq R-1$. A Taylor expansion of $\lambda(\varsigma_n(v))$ with an integral remainder gives

$$\lambda(\varsigma_n(v)) = \lambda(0) + \lambda'(0)\varsigma_n(v) + \cdots + \frac{1}{R!}\lambda^{(R)}(0)\varsigma_n^R(v) + \int_0^{\varsigma_n(v)} \lambda^{(R+1)}(t)\frac{(\varsigma_n(v)-t)^R}{R!}dt. \quad (30)$$

Analogously, we have for $1 \leq r \leq R$

$$\lambda^{(r)}(\varsigma_n(v)) = \lambda^{(r)}(0) + \cdots + \frac{1}{(R-r)!}\lambda^{(R)}(0)\varsigma_n^{R-r}(v) + \int_0^{\varsigma_n(v)} \lambda^{(R+1)}(t)\frac{(\varsigma_n(v)-t)^{R-r}}{(R-r)!}dt. \quad (31)$$

From (1), we have

$$1 - s_n'(v) = 1 - (n-1)\frac{f(v)}{F(v)}\lambda(\varsigma_n(v)).$$

For $r \geq 2$, taking the $(r-1)$th derivatives on both sides gives

$$\begin{aligned}
\varsigma_n^{(r)}(v) &= -(n-1)\frac{d^{r-1}}{dv^{r-1}}\left(\frac{f(v)}{F(v)}\lambda(\varsigma_n(v))\right) \\
&= -(n-1)\left\{\frac{d^{r-1}}{dv^{r-1}}\left(\frac{f(v)}{F(v)}\right)\lambda(\varsigma_n(v)) + \cdots \right. \\
&\quad + \binom{r-1}{l}\frac{d^{r-1-l}}{dv^{r-1-l}}\left(\frac{f(v)}{F(v)}\right)\frac{d^l}{dv^l}\lambda(\varsigma_n(v)) + \cdots \\
&\quad \left. + \frac{f(v)}{F(v)}\frac{d^{r-1}}{dv^{r-1}}\lambda(\varsigma_n(v))\right\}, \qquad (32)
\end{aligned}$$

where by Faà di Bruno's formula,

$$\begin{aligned}
\frac{d^l}{dv^l}\lambda(\varsigma_n(v)) &= \sum\left\{\frac{l!}{m_1! + m_2! + \cdots + m_l!}\lambda^{(m_1+m_2+\cdots+m_l)}(\varsigma_n(v)) \right. \\
&\quad \left. \times \prod_{j:m_j\neq 0}\left(\frac{1}{j!}\varsigma_n^{(l)}(v)\right)^{m_j}\right\}, \qquad (33)
\end{aligned}$$

where the sum is over all l-tuples $(m_1, \cdots, m_l)$ satisfying the constraint $1m_1 + 2m_2 + \cdots + lm_l = l$. By plugging (30) and (31) into (32) and substituting $\varsigma_n(v)$ by (28), i.e.,

$$\varsigma_n(v) = \frac{\beta_1(v)}{n} + \frac{\beta_2(v)}{n^2} + \cdots + \frac{\beta_k(v)}{n^k} + o\left(n^{-k}\right),$$

where $\beta_1(v), \cdots, \beta_k(v)$ are known and invariant with n by induction assumptions, we can derive from $r=1$ to $k-1$ that

$$\varsigma_n^{(r)}(v)=\frac{\gamma_{r1}(v)}{n}+\cdots+\frac{\gamma_{r,k-r}(v)}{n^{k-r}}+o\left(n^{r-k}\right), \tag{34}$$

where $\gamma_{r1}(v),\cdots,\gamma_{r,k-r}(v)$ are known functions invariant with n and $\varsigma_n^{(r)}(v)$ is of order $O\left(n^{-1}\right)$ by the induction assumptions (29). From (33), it follows for $l\leq k-1$

$$\begin{aligned}\frac{d^l}{dv^l}\lambda(\varsigma_n(v)) &= \lambda'(\varsigma_n(v))\,\varsigma_n^{(l)}(v)+O\left(n^{-2}\right)\\ &= \frac{1}{n}\lambda'(\varsigma_n(v))\frac{d^l}{dv^l}\left(\frac{F(v)}{f(v)}\right)+o\left(n^{-1}\right)\\ &= \frac{1}{n}\frac{d^l}{dv^l}\left(\frac{F(v)}{f(v)}\right)+o\left(n^{-1}\right),\end{aligned}$$

where the second equality holds because of (29), and the last equality follows from a Taylor approximation of $\lambda'(\varsigma_n(v))$. From (30), we have $\lambda(\varsigma_n(v))=F(v)/(nf(v))+o\left(n^{-1}\right)$. Hence if $k=1$

$$\varsigma_n^{(k)}(v)=1-(n-1)\frac{f(v)}{F(v)}\lambda(\varsigma_n(v))=o(1),$$

and, if $k\geq 2$, substitution of $\lambda(\varsigma_n(v))$ and $d^l\lambda(\varsigma_n(v))/dv^l$ into (32) gives

$$\begin{aligned}\varsigma_n^{(k)}(v) &= -\frac{n-1}{n}\Big\{\frac{d^{k-1}}{dv^{k-1}}\left(\frac{f(v)}{F(v)}\right)\frac{F(v)}{f(v)}+\cdots\\ &\quad+\binom{k-1}{l}\frac{d^{k-1-l}}{dv^{k-1-l}}\left(\frac{f(v)}{F(v)}\right)\frac{d^l}{dv^l}\left(\frac{F(v)}{f(v)}\right)+\cdots\\ &\quad+\frac{f(v)}{F(v)}\frac{d^{k-1}}{dv^{k-1}}\left(\frac{F(v)}{f(v)}\right)\Big\}+o(1)\\ &= -\frac{n-1}{n}\frac{d^{k-1}}{dv^{k-1}}\left(\frac{f(v)}{F(v)}\frac{F(v)}{f(v)}\right)+o(1)=o(1),\end{aligned}$$

which implies that

$$\frac{d^k}{dv^k}\lambda(\varsigma_n(v))=\lambda'(\varsigma_n(v))\,\varsigma_n^{(k)}(v)+O\left(n^{-2}\right)=o(1).$$

It follows that

$$
\begin{aligned}
\varsigma_n^{(k+1)}(v) &= -(n-1)\frac{d^k}{dv^k}\left(\frac{f(v)}{F(v)}\lambda(\varsigma_n(v))\right) \\
&= -(n-1)\left\{\frac{d^k}{dv^k}\left(\frac{f(v)}{F(v)}\right)\lambda(\varsigma_n(v))+\cdots\right. \\
&\quad +\binom{k}{l}\frac{d^{k-l}}{dv^{k-l}}\left(\frac{f(v)}{F(v)}\right)\frac{d^l}{dv^l}\lambda(\varsigma_n(v))+\cdots \\
&\quad \left.+\frac{f(v)}{F(v)}\frac{d^k}{dv^k}\lambda(\varsigma_n(v))\right\} \\
&= o(n).
\end{aligned}
$$

As

$$
\phi_n(v) = s_n(v) + \lambda(\varsigma_n(v)) = s_n(v) + \frac{1}{n-1}\frac{F(v)}{f(v)}s_n^{(1)}(v),
$$

where the second equality follows from (1), we have

$$
\begin{aligned}
\phi_n^{(r)}(v) &= s_n^{(r)}(v) + \frac{1}{n-1}\left\{\frac{d^r}{dv^r}\left(\frac{f(v)}{F(v)}\right)s_n^{(1)}(v)+\cdots\right. \\
&\quad \left.+\binom{r}{l}\frac{d^{r-l}}{dv^{r-l}}\left(\frac{f(v)}{F(v)}\right)s_n^{(l+1)}(v)+\cdots+\frac{f(v)}{F(v)}s_n^{(r+1)}(v)\right\}.
\end{aligned}
$$

By substituting (34), it can be rewritten in the following form

$$
\phi_n^{(r)}(v) = \frac{\delta_{r1}(v)}{n} + \cdots + \frac{\delta_{r,k-r}(v)}{n^{k-r}} + o\left(n^{r-k}\right), \tag{35}
$$

where $\delta_{r1}(v), \cdots, \delta_{r,k-r}(v)$ are known functions invariant with n for $r \leq k-1$. Furthermore, we have

$$
\begin{aligned}
\phi_n^{(k)}(v) &= s_n^{(k)}(v) + \frac{d^k}{dv^k}\lambda(\varsigma_n(v)) \\
&= s_n^{(k)}(v) + \lambda'(\varsigma_n(v))\varsigma_n^{(k)}(v) + O\left(n^{-2}\right) \\
&= \mathbf{1}(k=1) + o\left(n^{-1}\right),
\end{aligned}
$$

where $\mathbf{1}(k=1)$ is an indicator of $k=1$, and

$$\begin{aligned}\phi_n^{(k+1)}(v) &= s_n^{(k+1)}(v) + \frac{d^{k+1}}{dv^{k+1}}\lambda(\varsigma_n(v)) \\ &= s_n^{(k+1)}(v) + \lambda'(\varsigma_n(v))\varsigma_n^{(k+1)}(v) + o(1) = o(1),\end{aligned}$$

since $\lambda'(\varsigma_n(v)) = \lambda'(0) + O(\varsigma_n(v)) = 1 + O(n^{-1})$, $\varsigma_n^{(k)}(v) = o(1)$, and $\varsigma_n^{(k+1)}(v) = o(n)$. Repeated integration by parts of (22) gives that

$$\begin{aligned}\lambda(\varsigma_n(v)) &= \frac{1}{n}\psi_1(v)F(v) + \cdots + \frac{(-1)^{k-1}}{n(n+1)\cdots(n+k-1)}\psi_k(v)F^k(v) \\ &\quad + \frac{(-1)^k}{n(n+1)\cdots(n+k-1)F^{n-1}(v)}\int_{\underline{v}}^{v} F^{n+k-1}(u)\psi_{k+1}(u)f(u)\,du,\end{aligned}$$

where $\psi_1(v) = \phi_n'(v)/f(v), \psi_2(v) = \psi_1'(v)/f(v), \cdots, \psi_{k+1}(v) = \psi_k'(v)/f(v)$. As $\psi_l(v)$ is a polynomial of $\phi_n'(v), \cdots, \phi_n^{(l)}(v)$, by substitution of (35), we have for $l \leq k-1$

$$\psi_l(v) = \zeta_{l0}(v) + \frac{\zeta_{l1}(v)}{n} + \cdots + \frac{\zeta_{l,k-l}(v)}{n^{k-l}} + o\left(n^{l-k}\right), \tag{36}$$

$\psi_k(v) = \zeta_{k0}(v) + o(n^{-1})$, and $\psi_{k+1}(v) = \zeta_{k+1,0}(v) + o(1)$, where $\zeta_{l0}(v), \cdots, \zeta_{l,k-l}(v)$ are known functions invariant with n. The Laplace approximation gives

$$\begin{aligned}\lambda(\varsigma_n(v)) &= \frac{1}{n}\psi_1(v)F(v) + \cdots + \frac{(-1)^{k-1}}{n(n+1)\cdots(n+k-1)}\psi_k(v)F^k(v) \\ &\quad + \frac{(-1)^k}{n(n+1)\cdots(n+k-1)F^{n-1}(v)}\int_{\underline{v}}^{v} e^{(n+k-1)\ln F(u)}\psi_{k+1}(u)f(u)\,du \\ &= \frac{1}{n}\psi_1(v)F(v) + \cdots + \frac{(-1)^{k-1}}{n(n+1)\cdots(n+k-1)}\psi_k(v)F^k(v) \\ &\quad + \frac{(-1)^k}{n(n+1)\cdots(n+k-1)^2}\psi_{k+1}(v)F^{k+1}(v) + o\left(n^{-(k+1)}\right).\end{aligned}$$

By substitution of (36), it can be rewritten in the form of (27)

$$\lambda(\varsigma_n(v)) = \frac{\alpha_1(v)}{n} + \frac{\alpha_2(v)}{n^2} + \cdots + \frac{\alpha_{k+1}(v)}{n^{k+1}} + o\left(n^{-(k+1)}\right), \tag{37}$$

with $\alpha_1, \cdots, \alpha_k(v)$ known by the induction assumptions and $\alpha_{k+1}(v)$ explicitly derived. Let

$$\varsigma_n(v) = \frac{\beta_1(v)}{n} + \frac{\beta_2(v)}{n^2} + \cdots + \frac{\beta_{k+1}(v)}{n^{k+1}} + o\left(n^{-(k+1)}\right), \tag{38}$$

where $\beta_1, \cdots, \beta_k(v)$ are known by the induction assumptions. By substituting (38) into (30) and matching leading order terms with (37), we can solve for $\beta_{k+1}(v)$. Substitute (30) and (31) into (32) and replace $\varsigma_n(v)$ by (38). Now we can derive from $r = 1$ to k that

$$\varsigma_n^{(r)}(v) = \frac{\gamma_{r1}(v)}{n} + \cdots + \frac{\gamma_{r,k+1-r}(v)}{n^{k+1-r}} + o\left(n^{r-k-1}\right),$$

where $\gamma_{r1}(v), \cdots, \gamma_{r,k+1-r}(v)$ are known functions invariant with n. Specifically, $\varsigma_n^{(k)}(v) = \gamma_{k1}(v)/n + o\left(n^{-1}\right)$. This, together with the induction assumption (29), implies that $\gamma_{k1}(v) = d^k\left(F(v)/f(v)\right)/dv^k$.

Lastly, we can show with analogous arguments that (27)-(29) holding for $r \leq R-1$ implies (29) holds for $r = R$. ■

Corollary A.1 *In a first-price IPV auction with n ($n \gg 1$) bidders, suppose $U(\cdot) \in \mathcal{U}_R$ for $R \geq 1$. Given A1 and A2, the equilibrium bid in the symmetric Bayesian Nash equilibrium is given by*

$$s_n(v, x) = v - \frac{1}{n}\frac{F(v|x)}{f(v|x)} + O(n^{-2}).$$

Furthermore, we have $\varsigma_n^{(r)}(v, x) = O(n^{-1})$ for $1 \leq r \leq R$.

Proof of Proposition 2. Let $h(v) = s_n(v) - s_m(v)$, with $n > m \geq 2$. $h(v) = 0$ implies that

$$\begin{aligned} h'(v) &= s_n'(v) - s_m'(v) \\ &= (n-1)\frac{f(v)}{F(v)}\lambda(v - s_n(v)) - (m-1)\frac{f(v)}{F(v)}\lambda(v - s_m(v)) \\ &= (n-m)\frac{f(v)}{F(v)}\lambda(v - s_n(v)) > 0, \end{aligned}$$

where the inequality holds because $n > m$ and $U(\cdot)$ is monotonically increasing. Since $h(0) = 0$, by the single crossing lemma $s_n(v) > s_m(v)$ for $\bar{v} \geq v > \underline{v}$. As $\lim_{n\to\infty} s_n(\bar{v}) = \bar{v}$ by Proposition 1, (i) follows. Next, $g_n(b) = f(v)/s_n'(v)$ with $b = s_n(v)$. Because $f(v)$ is bounded away from zero by assumption and $s_n'(v)$ is bounded with $\lim_{n\to\infty} s_n'(v) = 1$ by Proposition 1, (ii) follows. To prove (iii), we note that substitution of $g_n(b) = f(v)/s_n'(v)$ into (1) gives

$$g_n(b) = \frac{F(v)}{(n-1)(v - s_n(v))}, \tag{39}$$

with $b = s_n(v)$. Since $(n-1)(v - s_n(v)) = F(v)/f(v) + O(n^{-1})$, it follows from Proposition 1 that $\sup_{b \in C} |g_n(b)|$ is bounded as $n \to \infty$. Similarly, $g_n^{(r)}(b)$ $(r = 1, \cdots, R)$ can be derived by taking rth differentiation on both sides of (39). Using mathematical induction, the desired result follows from Proposition 1. ■

Proof of Proposition 5. Trivial extension of Proposition 2 based on Corollary A.1. ■

B Proofs of Statistical Properties

To prove Propositions B.3 and 4 we need two auxiliary lemmas on the uniform convergence of $\tilde{G}_n(\cdot)$ and $\tilde{g}_n(\cdot)$ defined by (4) and (5). Throughout $|\cdot|_{r,*}$ denotes the sup-norm of the rth derivatives of $\cdot$ on the set $*$.

Lemma B.1 *Suppose for $R \geq 1$, $F(\cdot) \in F_R$, $U(\cdot) \in U_R$, and $\tilde{G}_n(\cdot)$ is given by (4), we have almost surely*

$$\left|\tilde{G}_n(b) - G_n(b)\right|_{0,C} = O\left(1/\sqrt{nL}\right),$$

where C is an arbitrary closed inner subset of $S(G_\infty)$.

Proof. It follows from Proposition 2 that

$$\begin{aligned}
\tilde{G}_n(b) &= \frac{1}{nL}\sum_{l=1}^{L}\sum_{p=1}^{n}\mathbf{1}(B_{pl} \leq b) \\
&= \frac{1}{nL}\sum_{l=1}^{L}\sum_{p=1}^{n}\mathbf{1}(G_n(B_{pl}) \leq G_n(b)) \\
&= \frac{1}{nL}\sum_{l=1}^{L}\sum_{p=1}^{n}\mathbf{1}(u_{pl} \leq G_n(b)),
\end{aligned}$$

where $u_{pl} = G_n(B_{pl})$ is uniformly distributed on $[0,1]$ since $B_{pl} \sim G_n(\cdot)$. Let $u = G_n(b) \in [0,1]$, and $n_C = \min\{n : C \subset S_n^o\}$ where S_n^o is the interior of $S(G_n)$. n_C exists because of Proposition

2(i). Then for $n > n_C$,

$$\begin{aligned} \left|\tilde{G}_n(b) - G_n(b)\right|_{0,C} &= \left|\frac{1}{nL}\sum_{l=1}^{L}\sum_{p=1}^{n}\mathbf{1}\left(u_{pl} \le G_n(b)\right) - G_n(b)\right|_{0,C} \\ &= \left|\frac{1}{nL}\sum_{l=1}^{L}\sum_{p=1}^{n}\mathbf{1}\left(u_{pl} \le u\right) - u\right|_{0,C} \\ &= O(1/\sqrt{nL}), \end{aligned}$$

where the last step holds because the empirical distribution of uniform distribution (which does not depend on n) converges uniformly to the true distribution at the rate of $\sqrt{nL}$. ■

Lemma B.2 *Suppose for $R \ge 1$, $F(\cdot) \in F_R$, $U(\cdot) \in U_R$, and $\tilde{g}_n(\cdot)$ as given by (5), we have almost surely*

$$|\tilde{g}_n(b) - g_n(b)|_{0,C} = O(1/r),$$

where C is an arbitrary closed inner subset of $S(G_\infty)$ and $r = (nL/\log(nL))^{R/(2R+1)}$.

Proof. The proof relies on the argument of Guerre, Perrigne and Vuong's (2000) proof for the case of fixed n. However, the problem is different because, as we allow both n and L to approach infinity, the observations are from a triangular array of random variables shifting with sample size. Hence the standard consistency results based on the i.i.d. assumption of observations do not apply directly. We divide the proof into three steps. The first step studies the uniform bias of $\tilde{g}_n(\cdot)$, the second step studies its uniform variance bound, and the last step establishes the exponential-type inequality. We simplify notation by omitting the subscript R in h_R and K_R in this proof. The sup-norm is taken over the whole support of the function unless otherwise indicated.

Step 1: Uniform Bias

For any closed inner subset C of $S(G_\infty)$, let $n_C = \min\{n : C \subset S_n^o\}$ where S_n^o is the interior of $S(G_n)$. n_C exists because of Proposition 2(i). For $n > n_C$,

$$\begin{aligned} E\tilde{g}_n(b) &= E\frac{1}{nLh}\sum_{l=1}^{L}\sum_{p=1}^{n}K\left(\frac{B_{pl} - b}{h}\right) \\ &= \int K(u)\, g_n(b + hu)\, du. \end{aligned}$$

Without loss of generality, suppose $u \ge 0$. Then for $b \in C$ and L sufficiently large, $\tilde{b} \in [b, b + hu] \subset C'$, where C' is a closed inner subset of $S(G_n)$. Since $g_n(\cdot)$ admits up to R continuous bounded derivatives on any closed inner subset of $S(G_n)$, a Taylor expansion gives

$$g_n(b+hu) - g_n(b) \leq hug_n^{(1)}(b) + \cdots + \frac{(hu)^{R-1}}{(R-1)!} g_n^{(R-1)}(b) + \frac{|hu|^R}{R!} |g_n|_{R,C'} .$$

As $K(\cdot)$ is of order R, moments of order strictly smaller than R vanish. So we have

$$\begin{aligned}
|E\tilde{g}_n(b) - g_n(b)|_{0,C} &= \sup_{b\in C} \left| \int K(u)(g_n(b+hu) - g_n(b))\, du \right| \\
&\leq h^R |g_n|_{R,C'} \frac{1}{R!} \left(\int |u|^R K(u)\, du \right) \\
&= h^R |g_n|_{R,C'} M^R,
\end{aligned}$$

where $M^R = (1/R!) \int |u|^R K(u)\, du$. It follows from the definition of r and h that

$$r\, |E\tilde{g}_n(b) - g_n(b)|_{0,C} \leq \lambda^R M^R |g_n|_{R,C'} . \tag{40}$$

Step 2: Uniform Variance

For $b \in C$, we have

$$\begin{aligned}
Var(\tilde{g}_n(b)) &= Var\left(\frac{1}{nLh} \sum_{l=1}^{L} \sum_{p=1}^{n} K\left(\frac{B_{pl} - b}{h} \right) \right) \\
&= \frac{1}{nLh^2} Var\left(K\left(\frac{B-b}{h} \right) \right) \leq \frac{1}{nLh^2} E\left(K\left(\frac{B-b}{h} \right) \right)^2 \\
&= \frac{1}{nLh} \int K^2(u)\, g_n(b+hu)\, du.
\end{aligned}$$

Let $Q = \int K^2(u)\, du$, it follows that

$$|Var(\tilde{g}_n(b))|_{0,C} \leq \frac{Q\, |g_n|_0}{nLh} = \frac{Q\, |g_n|_0}{\lambda r^2 \log(nL)}. \tag{41}$$

Step 3: Exponential-type Inequality

In this step, we establish the exponential-type inequality for the probability of deviation of $\tilde{g}_n(b) - g_n(b)$ in sup-norm over C. Let C be covered by T inner intervals of the form

$$B_t \equiv B(b_t, \Delta) = \{ b \in S(G_\infty) : b \in [b_t - \Delta, b_t + \Delta] \} ,$$

where $b_t \in C$ and $\Delta > 0$. Moreover, we consider minimal coverings for C, i.e., coverings for which T is the smallest number denoted by $T(C, \Delta)$. Let

$$
\begin{aligned}
e(\iota, \tau) &= \iota + 2\tau |K|_1 + \lambda^R M^R |g_n|_{R,C'}, \\
P(\iota, \tau) &= 2T\left(C, \tau h^2 / r\right) \exp\left(-\frac{\lambda \iota^2 \log(nL)}{2Q |g_n|_0 + 4\iota |K|_0 / (3r)}\right),
\end{aligned}
$$

where ι, τ are strictly positive constants.

Step 3(a): From (40) and the triangular inequality, we obtain

$$
\begin{aligned}
& \Pr\left(r |\tilde{g}_n(b) - g_n(b)|_{0,C} > e(\iota, \tau)\right) \\
\leq\ & \Pr\left(r |\tilde{g}_n(b) - E\tilde{g}_n(b)|_{0,C} + r |E\tilde{g}_n(b) - g_n(b)|_{0,C} > e(\iota, \tau)\right) \\
\leq\ & \Pr\left(r |\tilde{g}_n(b) - E\tilde{g}_n(b)|_{0,C} > e(\iota, \tau) - \lambda^R M^R |g_n|_{R,C'}\right). \qquad (42)
\end{aligned}
$$

Let $\tilde{g}_n(b) - E\tilde{g}_n(b) = (1/nL) \sum_{i=1}^{nL} \zeta_{i,nL}(b)$, where

$$
\zeta_{i,nL}(b) = \frac{1}{h}\left[K\left(\frac{B_i - b}{h}\right) - EK\left(\frac{B - b}{h}\right)\right].
$$

As the $\zeta_{i,nL}$'s are independent zero-mean variables, it follows from (41)

$$
Var\left(r\zeta_{i,nL}\right) = nLr^2 Var(\tilde{g}_n) \leq \frac{nLQ |g_n|_0}{\lambda \log(nL)}.
$$

By the triangular inequality we have

$$
\left|r\zeta_{i,nL}\right| \leq \frac{2r |K|_0}{h} = \frac{2nL |K|_0}{\lambda r \log(nL)}.
$$

Hence the Bernstein inequality gives

$$
\begin{aligned}
& \Pr\left(r\left|\tilde{g}_n(b)-E\tilde{g}_n(b)\right|>\iota\right) \\
= \; & \Pr\left(\left|\sum\nolimits_{i=1}^{nL} r\zeta_{i,nL}(b)-\sum\nolimits_{i=1}^{nL} E\left(r\zeta_{i,nL}(b)\right)\right|>nL\iota\right) \\
\leq \; & 2\exp\left(-\frac{n^2L^2\iota^2}{2\sum_{i=1}^{nL} Var\left(r\zeta_{i,nL}\right)+4n^2L^2\iota\left|K\right|_0/\left(3\lambda r\log\left(nL\right)\right)}\right) \\
\leq \; & 2\exp\left(-\frac{\lambda\iota^2\log\left(nL\right)}{2Q\left|g_n\right|_0+4\iota\left|K\right|_0/\left(3r\right)}\right) \\
= \; & \frac{P\left(\iota,\tau\right)}{T\left(C,\tau h^2/r\right)},
\end{aligned}
$$

for any $b\in C$, ι, n, and L.

Step 3(b): Consider a minimal covering of C for some $\Delta>0$. For any $b\in B_t$, we have by the triangular inequality

$$
r\left|\tilde{g}_n(b)-E\tilde{g}_n(b)\right|\leq\sup_{1\leq t\leq T}\left|\frac{r}{nL}\sum\nolimits_{i=1}^{nL}\zeta_{i,nL}(b_t)\right|+\sup_{1\leq t\leq T}\sup_{b\in B_t}\left|\frac{r}{nL}\sum\nolimits_{i=1}^{nL}\left(\zeta_{i,nL}(b_t)-\zeta_{i,nL}(b)\right)\right|,
$$

which implies that

$$
\begin{aligned}
& \Pr\left(r\sup_{b\in C}\left|\tilde{g}_n(b)-E\tilde{g}_n(b)\right|>e\left(\iota,\tau\right)-\lambda^RM^R\left|g_n\right|_{R,C'}\right) \\
\leq \; & \Pr\left(\sup_{1\leq t\leq T}\sup_{b\in B_t}\left|\frac{r}{nL}\sum\nolimits_{i=1}^{nL}\left(\zeta_{i,nL}(b_t)-\zeta_{i,nL}(b)\right)\right|>e\left(\iota,\tau\right)-\iota-\lambda^RM^R\left|g_n\right|_{R,C'}\right) \\
& +\Pr\left(r\sup_{1\leq t\leq T}\left|\tilde{g}_n(b_t)-E\tilde{g}_n(b_t)\right|>\iota\right).
\end{aligned}
\tag{43}
$$

Since

$$
\left|\frac{1}{h}K\left(\frac{B-b_t}{h}\right)-\frac{1}{h}K\left(\frac{B-b}{h}\right)\right|\leq\frac{\Delta\left|K\right|_1}{h^2},
$$

by the mean value theorem, we have by the triangular inequality

$$
\left|\zeta_{i,nL}(b_t)-\zeta_{i,nL}(b)\right|\leq\frac{\Delta\left|K\right|_1}{h^2}+E\frac{\Delta\left|K\right|_1}{h^2}=\frac{2\Delta\left|K\right|_1}{h^2}.
$$

Step 3(c): Let $\Delta=\tau h^2/r$, it follows

$$
\sup_{1\leq t\leq T}\sup_{b\in B_t}\left|\frac{r}{nL}\sum\nolimits_{i=1}^{nL}\left(\zeta_{i,nL}(b_t)-\zeta_{i,nL}(b)\right)\right|\leq\frac{2r\Delta\left|K\right|_1}{h^2}=2\tau\left|K\right|_1.
$$

Hence

$$\begin{aligned} & \Pr\left(\sup_{1\le t\le T}\sup_{b\in B_t}\left|\frac{r}{nL}\sum\nolimits_{i=1}^{nL}\left(\zeta_{i,nL}\left(b_t\right)-\zeta_{i,nL}\left(b\right)\right)\right|>e\left(\iota,\tau\right)-\iota-\lambda^R M^R\left|g_n\right|_{R,C'}\right) \\ = & \Pr\left(\sup_{1\le t\le T}\sup_{b\in B_t}\left|\frac{r}{nL}\sum\nolimits_{i=1}^{nL}\left(\zeta_{i,nL}\left(b_t\right)-\zeta_{i,nL}\left(b\right)\right)\right|>2\tau\left|K\right|_1\right)=0. \end{aligned} \tag{44}$$

Then it follows from (42), (43), (44), and the Bernstein inequality that

$$\begin{aligned} & \Pr\left(r\left|\tilde{g}_n\left(b\right)-g_n\left(b\right)\right|_{0,C}>e\left(\iota,\tau\right)\right) \\ \le & \Pr\left(r\left|\tilde{g}_n\left(b\right)-E\tilde{g}_n\left(b\right)\right|_{0,C}>e\left(\iota,\tau\right)-\lambda^R M^R\left|g_n\right|_{R,C'}\right) \\ \le & \Pr\left(r\sup\nolimits_{1\le t\le T}\left|\tilde{g}_n\left(b_t\right)-E\tilde{g}_n\left(b_t\right)\right|>\iota\right) \\ \le & \sum\nolimits_{t=1}^{T}\Pr\left(r\left|\tilde{g}_n\left(b_t\right)-E\tilde{g}_n\left(b_t\right)\right|>\iota\right) \\ \le & P\left(\iota,\tau\right). \end{aligned}$$

The covering number $T\left(C,\Delta\right)$ is of order Δ^{-1}, as the covered set C is an interval. Hence $T\left(C,\tau h^2/r\right)=O((nL/\log\left(nL\right))^{(R+2)/(2R+1)})$. By taking ι sufficiently large, $P\left(\iota,\tau\right)$ converges as $nL\to\infty$. The desired result follows from the Borel-Cantelli lemma and the fact that $e\left(\iota,\tau\right)=O\left(1\right)$.

■

Proof of Proposition 3. The proof presented here follows Guerre, Perrigne and Vuong's (2000) proof for the risk neutrality case. Let

$$\bar{V}_{pl}=B_{pl}+\frac{1}{n-1}\psi_n\left(B_{pl}\right),$$

with $\psi_n\left(\cdot\right)=G_n\left(\cdot\right)/g_n\left(\cdot\right)$. Let $\tilde{\psi}_n\left(\cdot\right)=\tilde{G}_n\left(\cdot\right)/\tilde{g}_n\left(\cdot\right)$, with $\tilde{G}_n\left(\cdot\right)$ and $\tilde{g}_n\left(\cdot\right)$ given by (4) and (5) respectively. Since $C\left(V\right)$ is a closed inner subset of $S\left(F\right)$ and $s_n\left(\cdot\right)$ is a strictly increasing continuous function, $C(B) = C(s_n(V))$ is a closed inner subset of $S(G_n)$. From (3), we have

$$
\begin{aligned}
& \mathbf{1}_{C(V)}(V_{pl})\left|\hat{V}_{pl} - V_{pl}\right| \\
\leq\; & \mathbf{1}_{C(V)}(V_{pl})\left(\left|\hat{V}_{pl} - \bar{V}_{pl}\right| + \left|\bar{V}_{pl} - V_{pl}\right|\right) \\
=\; & \frac{\mathbf{1}_{C(B)}(B_{pl})}{n-1}\left|\tilde{\psi}_n(B_{pl}) - \psi_n(B_{pl})\right| + O(n^{-2}) \\
=\; & \frac{\mathbf{1}_{C(B)}(B_{pl})\,\mathbf{1}(\hat{V}_{pl} \neq \infty)}{n-1}\left|\tilde{\psi}_n(B_{pl}) - \psi_n(B_{pl})\right| \\
& + \frac{\mathbf{1}_{C(B)}(B_{pl})\left(1 - \mathbf{1}(\hat{V}_{pl} \neq \infty)\right)}{n-1}\left|\tilde{\psi}_n(B_{pl}) - \psi_n(B_{pl})\right| + O(n^{-2}).
\end{aligned}
$$

It is easy to see that $1_{C(B)}(B_{pl})(1 - 1(\hat{V}_{pl} \neq \infty)) = 0$ almost surely for any p and l as $n, L \to \infty$. Since $G_n(\cdot) \leq 1$ and $g_n(\cdot)$ has a positive lower bound c_g by Proposition 2(ii), we have

$$
\begin{aligned}
& \mathbf{1}_{C(B)}(B_{pl})\,\mathbf{1}(\hat{V}_{pl} \neq \infty)\left|\tilde{\psi}_n(B_{pl}) - \psi_n(B_{pl})\right| \\
=\; & \frac{\mathbf{1}_{C(B)}(B_{pl})\,\mathbf{1}(\hat{V}_{pl} \neq \infty)}{g_n\left|\tilde{g}_n\right|}\left|(\tilde{G}_n - G_n)g_n + (g_n - \tilde{g}_n)G_n\right| \\
\leq\; & \frac{\mathbf{1}_{C(B)}(B_{pl})\,\mathbf{1}(\hat{V}_{pl} \neq \infty)}{c_g\hat{c}_g}\left|(\tilde{G}_n - G_n)\left|g_n\right|_0 + (g_n - \tilde{g}_n)\right|.
\end{aligned}
$$

where $\hat{c}_g = \min\{|\tilde{g}_n(B_{pl})|\} \to c_g > 0$. It follows from Lemma B.1 and B.2 that

$$
\sup \mathbf{1}_{C(B)}(B_{pl})\,\mathbf{1}(\hat{V}_{pl} \neq \infty)\left|\tilde{\psi}(B_{pl}) - \psi(B_{pl})\right| = O(1/r).
$$

Thus if $L \to \infty$ and $r/n \to 0$ as $n \to \infty$, we have almost surely for any closed inner subset $C(V)$ of $S(F)$,

$$
\sup_{pl} \mathbf{1}_{C(V)}(V_{pl})\left|\hat{V}_{pl} - V_{pl}\right| = O(1/nr), \tag{45}
$$

and, if $L \to \infty$ and $r/n \to \infty$ as $n \to \infty$, we have almost surely for any closed inner subset $C(V)$ of $S(F)$,

$$
\sup_{pl} \mathbf{1}_{C(V)}(V_{pl})\left|\hat{V}_{pl} - V_{pl}\right| = O\left(1/n^2\right). \tag{46}
$$

■

Proof of Proposition 4. Following Guerre, Perrigne and Vuong (2000), let

$$\tilde{f}(v) = \frac{1}{nL} \sum_{l=1}^{L} \sum_{p=1}^{n} K_R \left(\frac{V_{pl} - v}{h_R} \right) \tag{47}$$

be the "infeasible" nonparametric estimator of f using the true private values V_{pl}. Let $C'(V)$ be an inner closed subset of $S(F)$ containing all hypercubes of size δ (small enough) centered at v in $C(V)$. Define $C''(V)$ analogously with respect to $C'(V)$. Hence $C(V) \subset C'(V) \subset C''(V) \subset S(F)$. For $v \in C(V)$ and n, L large enough, $\hat{f}(v)$ uses at most observations $\hat{V}_{pl}$ in $C'(V)$ and hence at most V_{pl} in $C''(V)$ by the uniform convergence of pseudo-private values $\hat{V}_{pl}$ to V_{pl} in Proposition 3. Similarly, $\tilde{f}(v)$ uses at most V_{pl} in $C''(V)$ for any v in $C(V)$. Hence we have almost surely for n, L large enough,

$$\begin{aligned}
& \left| \hat{f}(v) - \tilde{f}(v) \right| \\
= & \left| \frac{1}{nLh_R} \sum_{l=1}^{L} \sum_{p=1}^{n} \mathbf{1}_{C''(V)}(V_{pl}) \left[K_R \left(\frac{v - \hat{V}_{pl}}{h_R} \right) - K_R \left(\frac{v - V_{pl}}{h_R} \right) \right] \right| \\
\leq & \left| \frac{1}{nLh_R} \sum_{l=1}^{L} \sum_{p=1}^{n} \mathbf{1}_{C''(V)}(V_{pl}) \frac{(\hat{V}_{pl} - V_{pl})}{h_R} \frac{\partial K_R}{\partial v} \left(\frac{v - V_{pl}}{h_R} \right) \right| \\
& + \frac{1}{2nLh_R} \sum_{l=1}^{L} \sum_{p=1}^{n} \mathbf{1}_{C''(V)}(V_{pl}) \frac{(\hat{V}_{pl} - V_{pl})^2}{h_R^2} \left| \frac{\partial^2 K_R}{\partial v^2}(v) \right|_0 \\
\leq & \frac{\sup_{p,l} \mathbf{1}_{C''(V)}(V_{pl}) \left| \hat{V}_{pl} - V_{pl} \right|}{h_R} \frac{1}{nLh_R} \sum_{l=1}^{L} \sum_{p=1}^{n} \left| \frac{\partial K_R}{\partial v} \left(\frac{v - V_{pl}}{h_R} \right) \right| \\
& + \frac{\sup_{p,l} \mathbf{1}_{C''(V)}(V_{pl}) \left| \hat{V}_{pl} - V_{pl} \right|^2}{2h_R^3} \left| \frac{\partial^2 K_R}{\partial v^2}(v) \right|_0 .
\end{aligned}$$

Let

$$\tilde{K}(x) = \left| \frac{\partial K_R}{\partial v}(x) \right| / \int \left| \frac{\partial K_R}{\partial v}(u) \right| du.$$

Thus we have almost surely, as $\tilde{K}(x)$ is a well defined kernel,

$$\left| \frac{1}{nLh_R} \sum_{l=1}^{L} \sum_{p=1}^{n} \tilde{K} \left(\frac{v - V_{pl}}{h_R} \right) - f(v) \right|_0 \to 0,$$

which implies $\frac{1}{nLh_R}\sum_{l=1}^{L}\sum_{p=1}^{n}\left|\partial K_R\left(\left(v-V_{pl}\right)/h_R\right)/\partial v\right|$ converges uniformly on $C\left(V\right)$ to

$$f\left(v\right)\int\left|\frac{\partial K_R}{\partial v}\left(u\right)\right|du.$$

Hence $\frac{1}{nLh_R}\sum_{l=1}^{L}\sum_{p=1}^{n}\left|\partial K_R\left(\left(v-V_{pl}\right)/h_R\right)/\partial v\right|$ is bounded almost surely. $\left|\partial^2 K_R\left(v\right)/\partial v^2\right|_0$ is bounded by the definition of $K_R\left(\cdot\right)$. We consider the following two cases:

(i) $L\to\infty$, and $r/n\to 0$ as $n\to\infty$

From (45), we have almost surely

$$\left|\hat{f}\left(v\right)-\tilde{f}\left(v\right)\right|_{0,C(V)}=O\left(\frac{1}{n}\left(\frac{\log nL}{nL}\right)^{(R-1)/(2R+1)}\right)+O\left(\frac{1}{n^2}\left(\frac{\log nL}{nL}\right)^{(2R-3)/(2R+1)}\right).$$

If $R=1$, then $r/n\to 0$ implies that

$$\left|\hat{f}\left(v\right)-\tilde{f}\left(v\right)\right|_{0,C(V)}=O\left(\frac{1}{n}\left(\frac{\log nL}{nL}\right)^{(R-1)/(2R+1)}\right).$$

If $R\geq 2$, then $\left(2R-3\right)/\left(2R+1\right)\geq\left(R-1\right)/\left(2R+1\right)$, which also implies that

$$\left|\hat{f}\left(v\right)-\tilde{f}\left(v\right)\right|_{0,C(V)}=O\left(\frac{1}{n}\left(\frac{\log nL}{nL}\right)^{(R-1)/(2R+1)}\right)=O\left(\frac{1}{nrh_R}\right).$$

Since $r/n\to 0$ implies $1/\left(nh_R\right)\to 0$, we have almost surely

$$\begin{aligned}\left|\hat{f}\left(v\right)-f\left(v\right)\right|_{0,C(V)} &\leq \left(\left|\hat{f}\left(v\right)-\tilde{f}\left(v\right)\right|_{0,C(V)}+\left|\tilde{f}\left(v\right)-f\left(v\right)\right|_{0,C(V)}\right)\\ &= O\left(\frac{1}{nrh_R}\right)+O\left(\frac{1}{r}\right)=O\left(r^{-1}\right),\end{aligned}$$

where $|\tilde{f}\left(v\right)-f\left(v\right)|_{0,C(V)}=O\left(r^{-1}\right)$ follows from analogous arguments used in the proof for Lemma B.2.

(ii) $L\to\infty$, and $r/n\to\infty$ as $n\to\infty$

From (46), we have almost surely

$$|\hat{f}\left(v\right)-\tilde{f}\left(v\right)|_{0,C(V)}=O\left(n^2h_R\right)^{-1}+O(n^4h_R^3)^{-1}.$$

If $(nh_R)^{-1} \to 0$, then $|\hat{f}(v) - \tilde{f}(v)|_{0,C(V)} = O\left(n^2 h_R\right)^{-1}$. Hence, if $(r/n)(nh_R)^{-1} \to 0$, we have almost surely that $|\hat{f}(v) - f(v)|_{0,C(V)} = O\left(n^2 h_R\right)^{-1} + O\left(r^{-1}\right) = O\left(r^{-1}\right)$; and if $(r/n)(nh_R)^{-1} \to \infty$, we have almost surely that $|\hat{f}(v) - f(v)|_{0,C(V)} = O\left(n^2 h_R\right)^{-1} + O\left(r^{-1}\right) = O\left(n^2 h_R\right)^{-1}$. On the other hand, if $(nh_R)^{-1} \to \infty$, then $|\hat{f}(v) - \tilde{f}(v)|_{0,C(V)} = O(n^4 h_R^3)^{-1}$. We have almost surely that $|\hat{f}(v) - f(v)|_{0,C(V)} = O(n^4 h_R^3)^{-1} + O\left(r^{-1}\right) = O(n^4 h_R^3)^{-1}$. ■

Proof of Proposition 6. Trivial extension of Proposition 2 in Guerre, Perrigne and Vuong (2000). ■

To prove Proposition 7 we need an auxiliary lemma on the uniform convergence of $\tilde{G}_n(b,x,i)$ $\tilde{g}_n(b,x,i)$ and $\tilde{f}(v,x)$ defined in (8) (9) and (55).

Lemma B.3 *Suppose A1-A4 hold, and $L \to \infty$, $(nh_g)^{-1} \to 0$ as $n \to \infty$. We have almost surely*

$$\begin{aligned}
|\tilde{G}_n(b,x,i) - G_n(b,x,i)|_{0,C} &= O(1/r_G), \\
|\tilde{g}_n(b,x,i) - g_n(b,x,i)|_{0,C} &= O(1/r_g) \\
|\tilde{f}(v,x) - f(v,x)|_{0,C} &= O(1/r_f),
\end{aligned}$$

where C is an arbitrary closed inner subset of $S(G_\infty)$, $r_G = (L/\log L)^{(R+1)/(2R+d+2)}$, and $r_g = r_f = (L/\log L)^{R/(2R+d)}$.

Proof. The proof relies on the argument of Guerre, Perrigne and Vuong's (2000) proof for the case of fixed n. However the problem is different as we are interested in the asymptotic properties of the estimators allowing both n and L to approach infinity. The arguments are more involved here, because $G_n(\cdot,\cdot,\cdot)$ shifts with sample size, and (B_{pl}, X_l, I_l) and (V_{pl}, X_l) observed in the same auction are correlated. We divide the proof into three steps. The first step studies the uniform bias, the second step studies the uniform variance bound, and the last step establishes exponential-type inequality. As the proofs are similar, we only detail the proof for $\tilde{g}_n(\cdot,\cdot,\cdot)$, as it is the most different from Guerre, Perrigne and Vuong's (2000) proof. The sup-norm is taken over the whole support of the function unless otherwise indicated.

Step 1: Uniform Bias

For any closed inner subset C of $S(G_\infty)$, let $n_C = \min\{n : C \subset S_n^o\}$ where S_n^o is the interior of $S(G_n)$. n_C exists because of Proposition 5(i). For $n > n_C$,

$$\begin{aligned} E\tilde{g}_n(b,x,i) &= E\left[\frac{1}{h_g^{d+1}} K_g\left(\frac{B_p - b}{h_g}, \frac{X - x}{h_g}, 0\right) \mathbf{1}(I = i)\right] \\ &= \int\int K_g(u,y,0)\, g_n(b + h_g u, x + h_g y, i)\, dudy. \end{aligned}$$

Define $\gamma(t) = g_n(b + th_g u, x + th_g y, i) - g_n(b,x,i)$ for $t \in [0,1]$. For L large enough, $(b + th_g u, x + th_g y) \in (b,x) + S(h_g) \subset C_i'$ for $(b,x,i) \in C$ and $t \in [0,1]$, where C_i' is a closed inner subset of $S(G_n(\cdot,\cdot,i))$. Since $g_n(\cdot,\cdot,\cdot)$ admits up to R continuous bounded derivatives with

$$|\gamma|_{R,[0,1]} \leq h_g^R \|(u,y)\|^R |g_n|_{R,C'}.$$

Thus a Taylor expansion gives

$$\gamma(1) - \gamma(0) \leq \gamma^{(1)}(0) + \cdots + \frac{1}{(R-1)!}\gamma^{(R)}(0) + \frac{1}{R!}|\gamma|_{R,[0,1]},$$

where $\gamma^{(r)}(0)$ is a polynomial of order r in (u,y). As $K_g(\cdot,\cdot)$ is of order R, moments of order strictly smaller than R vanish. It follows that

$$\begin{aligned} & |E\tilde{g}_n(b,x,i) - g_n(b,x,i)|_{0,C} \\ = & \left|\int K_g(u,y,0)(\gamma(1) - \gamma(0))\, dudy\right| \leq h_g^R \frac{1}{R!} |g_n|_{R,C'} \int \|(u,y)\|^R |K_g(u,y,0)|\, dudy \\ = & h_g^R M_g^R |g_n|_{R,C'} = \lambda_g^R M_g^R |g_n|_{R,C'} / r_g, \end{aligned} \tag{48}$$

where $M_g^R = (1/R!) \int \|(u,y)\|^R |K_g(u,y,0)|\, dudy$.

Step 2: Uniform Variance

For $(b,x,i) \in C$, we have

$$
\begin{aligned}
& Var\left(\tilde{g}_n\left(b,x,i\right)\right) \\
= \quad & Var\left(\frac{1}{Lh_g^{d+1}}\sum_{l=1}^{L}\frac{\mathbf{1}\left(I_l=i\right)}{ni}\sum_{p=1}^{ni}K_g\left(\frac{B_{pl}-b}{h_g},\frac{X_l-x}{h_g},0\right)\right) \\
= \quad & \frac{1}{Lh_g^{2(d+1)}}Var\left(\frac{\mathbf{1}\left(I=i\right)}{ni}\sum_{p=1}^{ni}K_g\left(\frac{B_{pl}-b}{h_g},\frac{X_l-x}{h_g},0\right)\right) \\
\leq \quad & \frac{1}{Lh_g^{2(d+1)}}E\left(\frac{\mathbf{1}\left(I=i\right)}{ni}\sum_{p=1}^{ni}K_g\left(\frac{B_{pl}-b}{h_g},\frac{X_l-x}{h_g},0\right)\right)^2 \\
= \quad & \frac{1}{Lh_g^{2(d+1)}}E\left(\frac{\mathbf{1}\left(I=i\right)}{(ni)^2}\sum_{p=1}^{ni}K_g^2\left(\frac{B_{pl}-b}{h_g},\frac{X_l-x}{h_g},0\right)\right) \\
& +\frac{1}{Lh_g^{2(d+1)}}E\left(\frac{\mathbf{1}\left(I=i\right)}{(ni)^2}\sum_{p=1}^{ni}\sum_{q=1,q\neq p}^{ni}K_g\left(\frac{B_{pl}-b}{h_g},\frac{X_l-x}{h_g},0\right)K_g\left(\frac{B_{ql}-b}{h_g},\frac{X_l-x}{h_g},0\right)\right) \\
= \quad & \frac{1}{(ni)Lh_g^{d+1}}\int K_g^2\left(u,y,0\right)g_n\left(b+h_gu,x+h_gy,i\right)dudy \\
& +\frac{ni-1}{(ni)Lh_g^{d}}\int K_g\left(u_1,y,0\right)K_g\left(u_2,y,0\right) \\
& \times g_{n,B|(X,I)}\left(b+h_gu_1|x+h_gy,i\right)g_{n,B|(X,I)}\left(b+h_gu_2|x+h_gy,i\right)g_{n,(X,I)}\left(x+h_gy,i\right)du_1du_2dy.
\end{aligned}
$$

Let $Q_{g1}=\int K_g^2\left(u,y,0\right)dudy$ and $Q_{g2}=\int K_g\left(u_1,y,0\right)K_g\left(u_2,y,0\right)du_1du_2dy$.

$$
\left|Var\left(\tilde{g}_n\left(b,x,i\right)\right)\right|_{0,C}\leq\frac{Q_{g1}\left|g_n\right|_0}{(ni)Lh_g^{d+1}}+\frac{(ni-1)Q_{g2}\left|g_{n,B|(X,I)}^2g_{n,(X,I)}\right|_0}{(ni)Lh_g^d}. \tag{49}
$$

Step 3: Exponential-type Inequality

In this step, we establish the exponential-type inequalities for the probabilities of deviations of $\tilde{g}_n\left(b,x,i\right)-g_n\left(b,x,i\right)$ in sup-norm over C_i, where $C_i=\{(b,x):(b,x,i)\in C\}$. Let C_i be covered by T inner "balls" of the form

$$
B_{it}\equiv B_i\left(\left(b_t,x_t\right);\Delta\right)=\left\{(b,x)\in S\left(G_\infty\right):b\in\left[b_t-\Delta,b_t+\Delta\right],x\in\left[x_t-\Delta,x_t+\Delta\right]\right\},
$$

where $(b_t, x_t) \in C_i$, and $\Delta > 0$ for $t = 1, \cdots, T$. Moreover, we consider minimal coverings for C_i, i.e., coverings for which T is the smallest number denoted by $T(C_i, \Delta)$. Let

$$\begin{aligned} e_g(\iota,\tau) &= \iota + 2(d+1)\tau |K_g|_1 + \lambda_g^R M_g^R |g_n|_{R,C'}, \\ P_g(\iota,\tau) &= 2T\left(C_i, \tau h_g^{d+2}/r_g\right) \\ &\quad \times \exp\left(-\frac{\lambda_g^d \iota^2 \log L}{2Q_{g1}|g_n|_0/(nih_g) + 2(1+1/ni)Q_{g2}\left|g^2_{n,B|(X,I)} g_{n,(X,I)}\right|_0 + 4\iota |K_g|_0/(3r_g h_g)}\right), \end{aligned}$$

where ι and τ are strictly positive constants.

Step 3(a): From (48) and the triangular inequality, we obtain

$$\begin{aligned} \Pr\left(r_g |\tilde{g}_n - g_n|_{0,C} > e_g(\iota,\tau)\right) &\leq \Pr\left(r_g |\tilde{g}_n - E\tilde{g}_n|_{0,C} + r_g |E\tilde{g}_n - g_n|_{0,C} > e_g(\iota,\tau)\right) \\ &\leq \Pr\left(r_g |\tilde{g}_n - E\tilde{g}_n|_{0,C} > e_g(\iota,\tau) - \lambda_g^R M_g^R |g_n|_{R,C'}\right). \quad (50) \end{aligned}$$

Let $\tilde{g}_n(b,x,i) - E\tilde{g}_n(b,x,i) = (1/L)\sum_{m=1}^{L} \zeta_{mL}(b,x,i)$, where

$$\begin{aligned} \zeta_{mL}(b,x,i) &= \frac{1}{nih_g^{d+1}} \sum_{p=1}^{ni} \left\{ K_g\left(\frac{B_{pm}-b}{h_g}, \frac{X_m - x}{h_g}, 0\right) \mathbf{1}(I_m = i) \right. \\ &\quad \left. -E\left(K_g\left(\frac{B_p - b}{h_g}, \frac{X-x}{h_g}, 0\right)\mathbf{1}(I=i)\right)\right\}. \end{aligned}$$

As the ζ_{mL}'s are independent zero-mean variables for $m = 1, \cdots, L$, it follows from (49)

$$\begin{aligned} Var(r_g \zeta_{mL}(b,x,i)) &= Lr_g^2 Var(\tilde{g}_n) \leq \frac{Lr_g^2 Q_{g1}|g_n|_0}{(nih_g)Lh_g^d} + \frac{(ni-1)Lr_g^2 Q_{g2}\left|g^2_{n,B|(X,I)} g_{n,(X,I)}\right|_0}{(ni)Lh_g^d} \\ &= \frac{LQ_{g1}|g_n|_0}{(nih_g)\lambda_g^d \log L} + \frac{(ni-1)LQ_{g2}\left|g^2_{n,B|(X,I)} g_{n,(X,I)}\right|_0}{(ni)\lambda_g^d \log L}. \end{aligned}$$

By the triangular inequality we have

$$|r_g \zeta_{mL}(b,x,i)| \leq \frac{2r_g}{h_g^{d+1}} |K_g|_0 = \frac{2L|K_g|_0}{\lambda_g^d r_g h_g \log L}.$$

Hence the Bernstein inequality gives

$$
\begin{aligned}
& \Pr\left(r_g\left|\tilde{g}_n(b,x,i) - E\tilde{g}_n(b,x,i)\right| > \iota\right) \\
= \;& \Pr\left(\left|\sum_{m=1}^{L} r_g\zeta_{mL}(b,x,i) - \sum_{m=1}^{L} E\left(r_g\zeta_{mL}(b,x,i)\right)\right| > L\iota\right) \\
\leq \;& 2\exp\left(-\frac{L^2\iota^2}{2\sum_{m=1}^{L} Var\left(r_g\zeta_{mL}\right) + 4L^2\iota\left|K_g\right|_0 / \left(3\lambda_g^d r_g h_g \log L\right)}\right) \\
\leq \;& 2\exp\left(-\frac{\lambda_g^d\iota^2\log L}{2Q_{g1}\left|g_n\right|_0/(nih_g) + 2(1+1/ni)Q_{g2}\left|g_{n,B|(X,I)}^2 g_{n,(X,I)}\right|_0 + 4\iota\left|K_g\right|_0/(3r_g h_g)}\right) \\
= \;& \frac{P_g(\iota,\tau)}{T\left(C_i, \tau h_g^{d+2}/r_g\right)},
\end{aligned}
$$

for any $(b,x,i) \in C$, ι, n, and L.

Step 3(b): Consider a minimal covering of C for some $\Delta > 0$. For any $b \in B_t$, we have

$$
\begin{aligned}
r_g\left|\tilde{g}_n(b,x,i) - E\tilde{g}_n(b,x,i)\right|_{0,C_i} \leq \;& \sup_{1\leq t\leq T}\left|\frac{r_g}{L}\sum\nolimits_{m=1}^{L}\zeta_{mL}(b_t,x_t,i)\right| \\
& + \sup_{1\leq t\leq T}\sup_{(b,x)\in B_{it}}\left|\frac{r_g}{L}\sum\nolimits_{m=1}^{L}\left(\zeta_{mL}(b_t,x_t,i) - \zeta_{mL}(b,x,i)\right)\right|.
\end{aligned}
$$

This gives

$$
\begin{aligned}
& \Pr\left(r_g\left|\tilde{g}_n(b,x,i) - E\tilde{g}_n(b,x,i)\right|_{0,C_i} > e(\iota,\tau) - \lambda_g^R M_g^R\left|g_n\right|_{R,C'}\right) \\
\leq \;& \Pr\left(\sup_{1\leq t\leq T}\sup_{(b,x)\in B_{it}}\left|\frac{r_g}{L}\sum\nolimits_{m=1}^{L}\left(\zeta_{mL}(b_t,x_t,i) - \zeta_{mL}(b,x,i)\right)\right| > e(\iota,\tau) - \iota - \lambda_g^R M_g^R\left|g_n\right|_{R,C'}\right) \\
& + \Pr\left(r_g\sup_{1\leq t\leq T}\left|\tilde{g}_n(b_t,x_t,i) - E\tilde{g}_n(b_t,x_t,i)\right| > \iota\right). \qquad (51)
\end{aligned}
$$

For any $(b,x) \in B_{it}$, it follows from the mean value theorem

$$
\left|\frac{1}{h_g^{d+1}}K_g\left(\frac{B-b_t}{h_g}, \frac{X-x_t}{h_g}, 0\right) - \frac{1}{h_g^{d+1}}K_g\left(\frac{B-b}{h_g}, \frac{X-x}{h_g}, 0\right)\right| \leq \frac{(d+1)\Delta}{h_g^{d+2}}\left|K_g\right|_1.
$$

The triangular inequality gives

$$
\left|\left(\zeta_{mL}(b_t,x_t,i) - \zeta_{mL}(b,x,i)\right)\right| \leq \frac{(d+1)\Delta}{h_g^{d+2}}\left|K_g\right|_1 + E\frac{(d+1)\Delta}{h_g^{d+2}}\left|K_g\right|_1 = \frac{2(d+1)\Delta}{h_g^{d+2}}\left|K_g\right|_1.
$$

Step 3(c): Let $\Delta = \tau h_g^{d+2}/r_g$, it follows that

$$\sup_{1\leq t\leq T}\sup_{b\in B_t}\left|\frac{r_g}{L}\sum\nolimits_{m=1}^{L}\left(\zeta_{mL}\left(b_t,x_t,i\right)-\zeta_{mL}\left(b,x,i\right)\right)\right| \leq \frac{2r_g(d+1)\Delta}{h_g^{d+2}}\left|K_g\right|_1 = 2(d+1)\tau\left|K_g\right|_1 .$$

Hence

$$\Pr\left(\sup_{1\leq t\leq T}\sup_{b\in B_t}\left|\frac{r_g}{L}\sum\nolimits_{m=1}^{L}\left(\zeta_{mL}\left(b_t,x_t,i\right)-\zeta_{mL}\left(b,x,i\right)\right)\right| > e\left(\iota,\tau\right)-\iota-\lambda_g^R M_g^R\left|g_n\right|_{R,C'}\right) = 0. \tag{52}$$

Then it follows from (50), (51), (52), and the Bernstein inequality that

$$\begin{aligned}\Pr\left(r_g\left|\tilde{g}_n-g_n\right|_{0,C} > e_g\left(\iota,\tau\right)\right) &\leq \Pr\left(r_g\left|\tilde{g}_n\left(b,x,i\right)-E\tilde{g}_n\left(b,x,i\right)\right|_{0,C_i} > e\left(\iota,\tau\right)-\lambda_g^R M_g^R\left|g_n\right|_{R,C'}\right)\\ &\leq \Pr\left(r_g\sup_{1\leq t\leq T}\left|\tilde{g}_n\left(b_t,x_t,i\right)-E\tilde{g}_n\left(b_t,x_t,i\right)\right| > \iota\right)\\ &\leq \sum\nolimits_{t=1}^{T}\Pr\left(r_g\left|\tilde{g}_n\left(b_t,x_t,i\right)-E\tilde{g}_n\left(b_t,x_t,i\right)\right| > \iota\right) \leq P\left(\iota,\tau\right).\end{aligned}$$

As the dimension of the covered set C is $d+1$, the covering number $T\left(C,\Delta\right)$ is of order $\Delta^{-(d+1)}$. Hence $T\left(C,\tau h_g^{d+2}/r_g\right) = O(L/\log L)^{(d+1)(R+d+2)/(2R+d)}$. By taking ι sufficiently large, $P\left(\iota,\tau\right)$ converges as $L\rightarrow\infty$. The desired result follows from the Borel-Cantelli lemma and the fact that $e\left(\iota,\tau\right) = O\left(1\right)$. ■

Proof of Proposition 7. First, the uniform consistency of pseudo-private values follows from similar arguments as used in the proof of Proposition 3. If $L\rightarrow\infty$, $r_g/n\rightarrow 0$ as $n\rightarrow\infty$, we have almost surely for any closed inner subset $C\left(V\right)$ of $S\left(F\right)$,

$$\sup\nolimits_{pl}\mathbf{1}_{C(V)}(V_{pl},X_l)\left|\hat{V}_{pl}-V_{pl}\right| = O\left(1/nr_g\right), \tag{53}$$

and, if $L\rightarrow\infty$, $r_g/n\rightarrow\infty$ as $n\rightarrow\infty$, we have almost surely for any closed inner subset $C\left(V\right)$ of $S\left(F\right)$,

$$\sup\nolimits_{pl}\mathbf{1}_{C(V)}(V_{pl},X_l)\left|\hat{V}_{pl}-V_{pl}\right| = O\left(1/n^2\right). \tag{54}$$

To establish the uniform consistency of the two-step estimator, let

$$\tilde{f}\left(v,x\right) = \frac{1}{nLh_f^{d+1}}\sum_{l=1}^{L}\frac{1}{I_l}\sum_{p=1}^{nI_l}K_f\left(\frac{V_{pl}-v}{h_f},\frac{X_l-x}{h_f}\right) \tag{55}$$

be the "infeasible" nonparametric estimator of f using the true private values V_{pl}. Let $C'(V)$ be an inner closed subset of $S(F)$ containing all hypercubes of size δ (small enough) centered at (v,x) in $C(V)$. Define $C''(V)$ analogously with respect to $C'(V)$. Hence $C(V) \subset C'(V) \subset C''(V) \subset S(F)$. For $(v,x) \in C(V)$ and n, L large enough, $\hat{f}(v,x)$ uses at most observations $(\hat{V}_{pl}, X_l)$ in $C'(V)$ and hence at most (V_{pl}, X_l) is in $C''(V)$ by the uniform convergence of pseudo-private values $\hat{V}_{pl}$ to V_{pl}. Similarly, $\tilde{f}(v,x)$ uses at most (V_{pl}, X_l) in $C''(V)$ for any (v,x) in $C(V)$. Hence we have almost surely for n, L large enough,

$$\begin{aligned}
& \left|\hat{f}(v,x) - \tilde{f}(v,x)\right| \\
= & \left|\frac{1}{nLh_f^{d+1}} \sum_{l=1}^{L} \frac{1}{I_l} \sum_{p=1}^{nI_l} \mathbf{1}_{C''(V)}(V_{pl}, X_l) \left[K_f\left(\frac{\hat{V}_{pl}-v}{h_f}, \frac{X_l-x}{h_f}\right) - K_f\left(\frac{V_{pl}-v}{h_f}, \frac{X_l-x}{h_f}\right)\right]\right| \\
\leq & \left|\frac{1}{nLh_f^{d+1}} \sum_{l=1}^{L} \frac{1}{I_l} \sum_{p=1}^{nI_l} \mathbf{1}_{C''(V)}(V_{pl}, X_l) \frac{(\hat{V}_{pl}-V_{pl})}{h_f} \frac{\partial K_f}{\partial v}\left(\frac{V_{pl}-v}{h_f}, \frac{X_l-x}{h_f}\right)\right| \\
& + \frac{1}{2nLh_f^{d+1}} \sum_{l=1}^{L} \frac{1}{I_l} \sum_{p=1}^{nI_l} \mathbf{1}_{C''(V)}(V_{pl}, X_l) \frac{(\hat{V}_{pl}-V_{pl})^2}{h_f^2} \left|\frac{\partial^2 K_f}{\partial v^2}\left(v, \frac{X_l-x}{h_f}\right)\right|_0 \\
\leq & \frac{\sup_{p,l} \mathbf{1}_{C''(V)}(V_{pl}, X_l)\left|\hat{V}_{pl}-V_{pl}\right|}{h_f} \frac{1}{nLh_f^{d+1}} \sum_{l=1}^{L} \frac{1}{I_l} \sum_{p=1}^{nI_l} \left|\frac{\partial K_f}{\partial v}\left(\frac{V_{pl}-v}{h_f}, \frac{X_l-x}{h_f}\right)\right| \\
& + \frac{\sup_{p,l} \mathbf{1}_{C''(V)}(V_{pl}, X_l)\left|\hat{V}_{pl}-V_{pl}\right|^2}{2h_f^3} \frac{1}{Lh_f^d} \sum_{l=1}^{L} \left|\frac{\partial^2 K_f}{\partial v^2}\left(v, \frac{X_l-x}{h_f}\right)\right|_0 .
\end{aligned}$$

The two sums may be viewed as kernel estimators and hence uniformly bounded on $C(V)$. We consider the following two cases:

(i) $L \to \infty$, and $r_f/n \to 0$ as $n \to \infty$

From (53), we have almost surely

$$\left|\hat{f}(v,x) - \tilde{f}(v,x)\right|_{0,C(V)} = O\left(\frac{1}{n}\left(\frac{\log L}{L}\right)^{(R-1)/(2R+d)}\right) + O\left(\frac{1}{n^2}\left(\frac{\log L}{L}\right)^{(2R-3)/(2R+d)}\right).$$

If $R = 1$, then $r_f/n \to 0$ implies that

$$\left|\hat{f}(v,x) - \tilde{f}(v,x)\right|_{0,C(V)} = O\left(\frac{1}{n}\left(\frac{\log nL}{nL}\right)^{(R-1)/(2R+1)}\right).$$

If $R \geq 2$, then $(2R-3)/(2R+d) \geq (R-1)/(2R+d)$, which also implies that

$$\left|\hat{f}(v,x) - \tilde{f}(v,x)\right|_{0,C(V)} = O\left(\frac{1}{n}\left(\frac{\log nL}{nL}\right)^{(R-1)/(2R+d)}\right) = O\left(\frac{1}{nr_f h_f}\right).$$

Since $r_f/n \to 0$ implies $1/(nh_f) \to 0$, we have almost surely

$$\begin{aligned}\left|\hat{f}(v,x) - f(v,x)\right|_{0,C(V)} &\leq \left(\left|\hat{f}(v,x) - \tilde{f}(v,x)\right|_{0,C(V)} + \left|\tilde{f}(v,x) - f(v,x)\right|_{0,C(V)}\right) \\ &= O\left(\frac{1}{nr_f h_f}\right) + O\left(\frac{1}{r_f}\right) = O(r_f^{-1}),\end{aligned}$$

where $|\tilde{f}(v,x) - f(v,x)|_{0,C(V)} = O(r_f^{-1})$ follows from Lemma B.3.

(ii) $L \to \infty$, and $r_f/n \to \infty$ as $n \to \infty$

From (54), we have almost surely

$$|\hat{f}(v,x) - \tilde{f}(v,x)|_{0,C(V)} = O\left(n^2 h_f\right)^{-1} + O(n^4 h_f^3)^{-1} = O\left(n^2 h_f\right)^{-1},$$

as $(nh_f)^{-1} \to 0$. Hence, if $(r_f/n)(nh_f)^{-1} \to 0$, we have almost surely that $|\hat{f}(v,x) - f(v,x)|_{0,C(V)} = O\left(n^2 h_f\right)^{-1} + O(r_f^{-1}) = O(r_f^{-1})$; and if $(r_f/n)(nh_f)^{-1} \to \infty$, we have almost surely that $|\hat{f}(v,x) - f(v,x)|_{0,C(V)} = O\left(n^2 h_f\right)^{-1} + O(r_f^{-1}) = O\left(n^2 h_f\right)^{-1}$. ■

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