Collusive Equilibrium in Cournot Oligopolies with Unknown Costs

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Abstract: This paper studies collusive equilibrium in infinite horizon repeated games with discounting in which the firms play a quantity setting Cournot game each period when the firms do not know the costs of the other firms. We find that there are pooled Perfect Bayesian equilibrium that are collusive in which a firm plays exactly the same strategy irrespective of its realized cost. There are also separating equilibrium with signaling as well as with communication, in which the firms produce the optimal incentive compatible vector. In the separating equilibrium with signaling, the firms play the strictly separating Bayesian Nash equilibrium in period 1 and then from period 2 onwards produce the optimal incentive compatible collusive quantity vector. In the separating equilibrium with communication the firms produce the optimal incentive compatible quantity vector from period 1 onwards after an initial communication phase. We show that among these equilibria, the separating equilibrium with communication gives the highest expected discounted sum of joint profits.

Keywords: Oligopoly, Collusion, Repeated Games, Private Information, Folk Theorem, Pooling Equilibrium, Separating Equilibrium, Perfect Bayesian Equilibrium.

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1 Introduction

The fact that firms may not know each others costs is not uncommon. For instance a firm may receive a privately observed technology shock that permanently changes its cost structure for the foreseeable future. The other firms may have some idea of some aspects of the technology shock and about the costs of its rivals, but may not have all the relevant information. It is true that publicly held firms publish reports about their revenue and profits and thus one might argue that the costs of the firms are never really private information but have to be revealed on a periodic basis. However, this is certainly not true of partnerships and firms that are privately held, as these do not report costs or even profits, in their annual statements[1]. Even with publicly held firms it can be argued that not everything about the costs of the firms are made explicit. Indeed, for such firms one may even pose the question a little differently. Since the firms report their costs, is there an incentive for the firms to misreport their true costs. Would this then work against some form of tacit collusion? or would the nature of tacit collusion itself be modified in order to discourage the misrepresentation of the costs of the firms. Thus it seems it would be of interest to investigate whether collusive behavior is possible in a Cournot Oligopoly of quantity setting firms who have private information about their costs.

In what follows we discuss the nature of collusive equilibrium when the firms receive a privately observed technology shock that changes the cost of a firm for a sufficiently long duration of time. We do this in the context in which in each period the firms play a Cournot quantity setting game. Since the cost of the firms are now private information, one can view a situation like this as one in which the firms play a Cournot quantity-setting game repeatedly over many periods without knowing the costs of the other firms. Since the firms do not know each others costs, firms cannot play the Cournot-Nash quantities of the stage game, and therefore, the method of using trigger-price strategies with its reversion to Cournot-Nash equilibrium strategies cannot be used to deter deviations from the collusive output. This can be done only if there is some method of eliciting information about the true costs of the firms. However, playing the Cournot-Nash equilibrium quantities on the basis of what is reported will not work as firms with

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[1] For example law firms have traditionally never declared their profits or their cost. Barred from being public firms, they have no incentive to disclose their financial results. See for example page 54 of the *Economist*, August 9, 2003.

the high costs will typically want to report low costs. Thus our setting is different from the one, for instance, in Green and Porter [1984], in which the firms do not observe each others output, but because they know each others payoff function, the firms can decide on the Cournot-Nash equilibrium output, and therefore, can respond to public signals by reverting to playing the Cournot-Nash output if the public signal indicates that with high probability there has been a deviation. Similarly, in Abreu, Pearce and Stachetti [1986] in which the firms again respond to public signals, the firms know each others payoff and thus can compute the continuation payoffs as a function of the public signals.

The works that are closest to the present work are the papers by Athey and Bagwell [2001] and [2004] as well as the paper by Athey, Bagwell and Sanchirico [2004][2]. All of these papers deal with the issue of private information and does so within an explicitly dynamic framework with multiple periods. In these cases the firms play an infinite-horizon version of the Bertrand price-setting game, in which the prices are perfectly observed, but the cost of the firms are subject to i.i.d. shocks every period. They show that the firms can collude at the efficient price by agreeing on appropriate splits of the market share. The high cost firm is willing to give up market share because it expects higher expected profit in the future. The result they obtain, however, depends critically on the fact that the firm that has high cost in the present will draw a "shock" in the future that will make it a low cost firm. In the case of the model presented here, because the high cost firm will remain a high cost firm, there seems to be little incentive for the high cost firm to reveal that information. If information is revealed at all the incentive to reveal information will have to be different. The setting of Athey and Bagwell [2004] is somewhat closer to what we have here. In Athey and Bagwell [2004] there is persistence in the costs and in the limit there can be perfect persistence. This part of their analysis thus covers the case in which the firms receive a privately observed technology shock that persists for a long duration of time as in this paper. Their result is that if the distribution of costs is log concave and the firms are sufficiently patient then the optimal collusive scheme entails price rigidity; firms set the same price and share the market equally, regardless of their respective costs. Productive efficiency can be achieved under some circumstances, but such equilibria are not optimal. It should, however, be noted that the firms play a Bertrand price-setting game in each period. Further, and this could

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[2] A paper by LaCasse [1999] addresses a similar question as in this paper with two cost types but does not explicitly discuss collusive behavior.

be the major element that drives their results, the demand side is given by a unit mass of identical consumers with a fixed reservation price r, such that $r > \bar{\theta}$, where $\bar{\theta}$ is the highest possible cost. This is different from having a downward sloping demand schedule as the firms know that the optimally collusive price is r irrespective of the privately observed costs of the firms. In case the demand is given by the usual downward sloping demand curve, the optimally collusive price depends on the realized costs of the firms, that is, if $\hat{p}(\theta)$ is the optimal collusive price when the realized costs of the firms are θ, then $\hat{p}(\bar{\theta}) > \hat{p}(\underline{\theta})$ if $\bar{\theta} > \underline{\theta}$.

The situation that we analyze here is one in which the firms not only do not know each others costs, but also face a downward sloping demand curve so the firms also do not know the optimal collusive quantity and price. An issue in this case is to find what kind of collusive agreements can be implemented, that is, are there quantity vectors such that if a firm tries to deviate from producing the output assigned to it, then it can be credibly deterred from doing so. In this case, one possible way to deter deviations is to use punishment phases in which a deviating firm is punished for all possible cost configurations[3]. Therefore, it seems that some form of minimaxing should work. But one needs to find minimaxing strategies that would work for all possible realizations of the costs of the firms. For the oligopolistic games this difficulty is overcome by using fairly draconian punishment schemes. Thus, while the method of reverting to Cournot-Nash strategies seems not to work, the method of minimaxing deviators as in Fudenberg and Maskin [1986] presents an alternative method of punishing deviators that seems to work quite well.

Some recent work have analyzed the role that private information plays in an explicitly dynamic framework. Thus Cole and Kocherlakota [2001] analyzes a class of games with hidden actions and hidden states. Kennan [2001] examines repeated bargaining in which the buyer's valuation is determined by a two-state Markov chain and this valuation is private information to the buyer. It should be noted that in both Cole and Kocherlakota [2001] as well as Kennan [2001], the private information is generated every period by a random shock to the state that is privately observed by some of the players but not by all. Another work that is also very closely related to the literature

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[3] In Athey and Bagwell [2004], given that r is the optimally collusive price for all realized costs of the firms, an equilibrium can be played in which the firms either set the price p equal to r or in case of a deviation set the price equal to $\bar{\theta}$. In the case of the quantity setting firms in a market with the usual downward sloping demand curve, the optimal collusive quantity vector or price is not known.

on collusion in infinite horizon Oligopoly games with private information is the one by Hanazono and Yang [2006]. This work analyzes collusive behavior when the firms receive private signals about independently and identically distributed demand shocks. It thus analyzes situations in which the firms have private information about the demand side of the market.

In this paper we show that the infinite horizon game in which the firms have private information about their costs and play the oligopoly stage game repeatedly over an infinite horizon, have a fairly large equilibrium set, some of which are also collusive. In these equilibria, the quantity choices could either be independent of the information about the costs, or be completely determined by them. In section 4 we analyze pooling equilibrium in which the quantity vector is independent of the realized costs as are the minimaxing strategies used to punish deviators. We call these equilibria *pooling equilibrium* as the firms do not have to ever reveal their costs and produce the same output irrespective of their cost. In section 5, we analyze a completely different kind of equilibrium in which the strategies of the firms depend on the realized cost. We call these collusive equilibria *strictly separating*, as different realizations of cost lead to different strategies. In this type of equilibrium the quantity vector depends critically on the realized cost, as does the subsequent play of the game. We show that one can find strictly separating equilibrium in which the firms produce a quantity vector that maximizes the joint profit of the firms subject to some incentive constraints. These optimal incentive compatible collusive outputs are produced after the firms signal their cost in period 1 by playing a strictly separating Bayesian Nash equilibrium. We also show that if the firms communicate prior to producing their output, then there is a separating equilibrium in which the firms play their optimal incentive compatible quantity vector from period 1 onwards. In section 6 we compare the expected discounted sum of the joint profits of the firms from the optimal pooling equilibrium to those from the optimal strictly separating equilibrium with signaling and the optimal separating equilibrium with communication. We find that the expected joint profits from the separating equilibrium with communication is at least as large as those from the either the optimal separating equilibrium with signaling or the optimal pooling equilibrium. One also notes that whether the firms are in a pooling equilibrium or in a separating equilibrium, the equilibrium prices and outputs are stable over time, indicating the classic sort of price rigidity that is common under collusion.

2 The Oligopoly with unknown costs

There are n firms. The marginal cost of firm i is some constant c_i. This is known only to firm i and is thus private information to the firm. The other firms know that c_i takes finitely many values $\{c_i(1), c_i(2), \cdots, c_i(k_i)\}$. We will denote the set of possible costs of a firm by C. There is a common probability distribution over the set of possible marginal costs of the firms given by μ. Thus, μ is a probability distribution over $\mathcal{C} = C \times C \times \cdots \times C$[4]. We will call an element $\mathbf{c}$ of $\mathcal{C}$ a *cost profile*. Given the private information of firm i, that its marginal cost is c_i, the conditional distribution about the cost function of the other firms is given by $\mu(\mathbf{c}|c_i)$. Thus the belief of firm i about the distribution of the costs of the other firms, given that its own cost is c_i is $\mu(\mathbf{c}|c_i)$.

The firms all produce the same identical product and the inverse demand function $p(q)$ satisfies $p'(q) < 0$. Each firm observes the output vector $q = (q_1, \cdots, q_n)$ every period and the resulting market price $p(q_1 + \cdots + q_n)$.

Firm i observes its own profit, which is a function of its cost c_i and is given by

$$\pi_i(\sum_i q_i, c_i) = p(q_1 + \cdots + q_n)q_i - c_i q_i.$$

Firm i does not know the profit of the other firms and only knows that the cost of the other firms are distributed according to the conditional distribution $\mu(\mathbf{c}|c_i)$.

We make the following assumptions about the demand.

Assumption 1 *There is a $\bar{q}$ such that $p(\bar{q}) = 0$ and $p(0) < \infty$.*

and

Assumption 2 *There is a $\mathring{q}$ such that*

$$p(\mathring{q}).\ \mathring{q}_i - \bar{c}\ \mathring{q}_i > 0$$

for all $i = 1, \cdots, n$.

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[4] Even though we have denoted the set of possible costs of the firms as being the same, the set of actual costs of the firms can be different. If a particular firm's cost never take certain values in C, then that is reflected in the fact that the probability μ of those cost profiles is zero. Thus, if we have two firms, and one firm's cost can take the values $c_H > c_L > 0$ and the other firm's cost is given by c_H, then the probability distribution μ will satisfy the condition that $\mu(c_H, c_L) = \mu(c_L, c_L) = 0$.

where $\bar{c}$ is the highest possible marginal cost of a firm. This last assumption guarantees that there is sufficient demand in the market for a firm to operate at profit even if all the firms find that they have the highest possible marginal cost. With these assumptions the following holds.

Proposition 1 *Every firm's profit can be pushed down to* 0 *by the other firms independently of the firm's type.*

Proof: Consider q_{-i} such that $\sum_{j \neq i} q_j = \bar{q}$. Then $p(q) \leq 0$ so that $\pi_i(q, c_i) = 0 - c_i q_i \leq 0$ for any $c_i \in C$. ■

3 The infinite horizon game

The infinite horizon game is generated by repeating the incomplete information stage game of the oligopoly with unknown costs over an infinite horizon. Before the repeated game is played, the firms get to know their own marginal costs. The firms thus have private information about their costs which for firm i is randomly drawn from C according to the joint distribution μ on $\mathcal{C}$. After the firms get to know their marginal costs, the firms play the quantity setting Cournot game repeated over an infinite horizon. Thus, the infinite horizon sequential game is a game with imperfect information. The strategy of a firm i in this sequential game is a sequence $\{\sigma_{it}\}_{t=1}^{\infty}$ such that

$$\sigma_{it} : H_{t-1} \times C \to [0, \bar{q}]$$

where H_{t-1} is the set of histories of the game until period $t-1$ and an $h_{t-1} \in H_{t-1}$ is given by $h_{t-1} = \{q^1, q^2, \cdots, q^{t-1}\}$, where q^t is the quantity vector in time period t. That is, h_{t-1} is a history that consists of quantity choices of the firms until period $t-1$. The action chosen by a firm i in period t thus depends on the past history of quantities chosen by the firms and the cost c_i of the firm. A strategy combination will be denoted by $\sigma = \{\sigma_t\}_{t=1}^{\infty}$, where $\sigma_t = \{\sigma_{1t}, \cdots, \sigma_{nt}\}$.

The payoff of a firm i when a strategy combination σ is used is given by

$$\pi_i^{\infty}(\sigma, c_i) = \sum_{t=1}^{\infty} \delta^{t-1} \pi_i(\sigma_{1t}(h_{t-1}, c_1), \cdots, \sigma_{nt}(h_{t-1}, c_n), c_i)$$

Thus the payoff of firm i in the sequential game is the discounted sum of the single period profits. Since the profits depend on the true marginal costs and the quantity choices that are observed by the firms, the firms see their own true payoffs, but may not know the true payoffs of the other firms. Given a history of actions h_t, the discounted sum of payoffs of player i from time period $t+1$ onwards, when the strategy combination σ is used, will be denoted by

$$\pi_i^{\infty}(\sigma|_{h_t,c_i}) = \sum_{\ell=t}^{\infty} \delta^{\ell-1} \pi_{i\ell}(\sigma_{1\ell}(h_{\ell-1}, c_1), \cdots, \sigma_{n\ell}(h_{\ell-1}, c_n), c_i).$$

The payoff given in the preceding equation is the actual payoff that would be realized given the actual cost profile of the firms. The expected payoff, after a history h_t, given that the cost of firm i is c_i, is

$$\sum_{c \in C} \mu(\mathbf{c}|h_t, c_i) \pi_i^{\infty}(\sigma|_{h_t,c_i})$$

where $\mu(\mathbf{c}|h_t, c_i)$ is the conditional distribution over the costs of the players $j \neq i$ given the history h_t and given the private information c_i of firm i. It denotes the belief of firm i about the costs of the other firms, after firm i has observed the history h_t and c_i.

The equilibrium concept that we use here is that of a Perfect Bayesian equilibrium since we discuss equilibrium in a game with incomplete and hence imperfect information. A **Perfect Bayesian equilibrium** is a strategy combination that continues to be an optimal strategy for every player given any history and the updated beliefs of the players given that history, when the beliefs are updated using Bayes' rule. Thus, if σ is the strategy combination, then if $\mu(h_t|\sigma, \mathbf{c_{-i}}, c_i)$ is the probability of the history h_t given σ and the cost profile $(\mathbf{c_{-i}}, c_i)$ of the firms, then the probability of the cost profile being $\mathbf{c} = (\mathbf{c_{-i}}, c_i)$ is

$$\mu(\mathbf{c}|h_t, \mathbf{c_i}, \sigma) = \frac{\mu(h_t|\sigma, \mathbf{c_{-i}}, c_i)\mu(\mathbf{c}|h_{t-1}, \sigma, c_i)}{\sum_{\mathbf{c'} \in \mathcal{C}} \mu(h_t|\sigma, \mathbf{c'_{-i}}, c_i)\mu(\mathbf{c'}|h_{t-1}, \sigma, c_i)}. \tag{1}$$

Thus, in every period the belief of every firm about the cost profile of the other firms is updated, after observing the most recent history and given the private information of the firm, using Bayes' rule or equivalently, by the conditional probability of $\mathbf{c} \in \mathcal{C}$ given (h_t, σ, c_i).

Definition 1 *Given the strategy combination* $\sigma^\star = (\sigma_1^\star, \cdots, \sigma_n^\star)$, *the assessment* $(\sigma^\star, \mu^\star)$ *is a* **Perfect Bayesian** *equilibrium of the infinite horizon game if*
(i) $\mu^\star(.)$ *is a system of beliefs that is determined by* $\sigma^\star$ *according to the rule given in (1),*

and

(ii) for every player i, for every time period t and for every history h_t of actions up to time period t,

$$\sum_{\mathbf{c}\in\mathcal{C}} \mu^{\star}(\mathbf{c}|h_t, c_i, \sigma^{\star})\pi_i^{\infty}(\sigma^{\star}|_{h_t,c_i}) \geq \sum_{\mathbf{c}\in\mathcal{C}} \mu^{\star}(\mathbf{c}|h_t, c_i, (\sigma_i, \sigma^{\star}_{-i}))\pi_i^{\infty}((\sigma_i, \sigma^{\star}_{-i})|_{h_t,c_i})$$

for every $(\sigma_i|_{h_t,k_i})$.

Note that the strategies are conditioned on the private information of the firm, and the history of the actions. For a detailed discussion of Perfect Bayesian equilibrium and Sequential Equilibrium one may refer to Fudenberg and Tirole [1991] and for a discussion of Sequential equilibrium see Kreps and Wilson [1982]. A Perfect Bayesian equilibrium will be called a **pooling equilibrium** if the strategies of the firms with different costs are the same. That is no matter what the cost of the firm, the firm plays the same strategy. A Perfect Bayesian equilibrium will be called a **separating equilibrium** if the strategy of a firm depends on its cost. It will be called a **strictly separating equilibrium** if the equilibrium strategy of a firm varies strictly with its cost.

4 Pooling Equilibrium, the Folk Theorem and Collusion

In this section we show that every quantity vector that gives positive profits is a quantity vector associated with a pooling perfect Bayesian equilibrium for the right discount factor. This set of quantity vectors is the set of quantity vectors in which firms make positive profits even if their cost is the highest possible cost $\bar{c}$. We will show that given the cost profile of the firms, which will remain undisclosed in these pooling equilibria, any quantity vector in this set is the quantity vector of a perfect Bayesian equilibrium and a firm plays the same strategy irrespective of its realized cost. One can view this result as a version of the folk theorem for these repeated games with incomplete information.

The following definition states what we mean by a pooling perfect Bayesian equilibrium.

Definition 2 *A perfect Bayesian equilibrium $(\sigma^{\star}, \mu^{\star})$ is a pooling perfect Bayesian equilibrium if for every $c_i, c_i' \in C$, we have $\sigma_i^{\star}(c_i) = \sigma_i^{\star}(c_i')$ for every firm i.*

In other words in a pooling equilibrium, the strategy of a firm is exactly the same, irrespective of the actual realized cost.

Let

$$Q = \{\hat{q} = (\hat{q}_1, \cdots, \hat{q}_n) : \pi_i(\hat{q}, c_i) > 0 \, \text{for all } c_i \text{ and for all } i\}.$$

Thus Q is the set of quantity vectors that would allow every firm to make positive profits even if the realized marginal costs of the firms are the highest possible for each type. Since we will show that the quantity vectors in Q can be implemented as a pooling equilibrium, it is important to analyze the nature of the quantity vectors in Q. In fact, a question of interest is whether Q is always nonempty and whether there are quantity vectors in Q that are on the profit possibility frontier. Recall that the profit possibility frontier is the set of all quantity vectors, such that it is not possible to increase the profit of any one firm, without lowering the profit of some other firm. It is reasonable to assume that if the firms collude, then they would probably collude on a quantity vector on the profit possibility frontier.

Proposition 2 *$Q \neq \emptyset$. Further, any quantity vector $q = (q_1, \cdots, q_n)$ that satisfies the condition $p(q) > \bar{c}$ is in Q.*

Proof: By assumption 2, there is a quantity vector $\mathring{q}$ such that

$$p(\mathring{q}).\, \mathring{q}_i - \bar{c}\, \mathring{q}_i > 0.$$

Since

$$p(\mathring{q}).\, \mathring{q}_i - c_i.\, \mathring{q}_i > p(\mathring{q}).\, \mathring{q}_i - \bar{c}\, \mathring{q}_i > 0$$

it follows that $\pi_i(\mathring{q}, c_i) > 0$ for all c_i and for all $i = 1, \cdots, n$. Thus $\mathring{q} \in Q$. Hence, $Q \neq \emptyset$.

The observation that any quantity vector q that satisfies the condition $p(q) > \bar{c}$ is in Q, follows from noting that for all possible cost profiles, such a q will satisfy the condition that

$$p(q)q_i - c_i q_i \geq p(q)q_i - \bar{c}q_i > 0$$

thus showing that q is in Q. ∎

Thus at a quantity vector in the set Q, all the firms will make positive profits, even if the marginal costs of all the firms are at the highest possible level $\bar{c}$, and so all the firms will make positive profits at all possible marginal costs.

Lemma 1 *For every* $\hat{q}$ *in* Q *there is a vector* $\epsilon^i = (\epsilon_1^i, \cdots, \epsilon_n^i) >> 0$ *and a quantity vector* $\hat{q}^{i,\epsilon}$ *such that*

$$\pi_i^{i,\epsilon}(c_i) = \pi_i(\hat{q}^{i,\epsilon}, c_i) \leq \pi_i(\hat{q}, c_i) - \epsilon_i^i, \text{ and } \pi_j^{i,\epsilon}(c_j) = \pi_i(\hat{q}^{i,\epsilon}, c_j) \geq \pi_j(\hat{q}, c_j) + \epsilon_j^i$$

for all c_i *and* c_j.

Proof: Given a $\hat{q}$ in Q, define the quantity vector $\hat{q}^{i,\epsilon}$ as

$$\hat{q}_j^{i,\epsilon} = \hat{q}_j + \frac{\epsilon}{n-1}, \text{ and } \hat{q}_i^{i,\epsilon} = \hat{q}_j - \epsilon.$$

Then, clearly

$$\sum_{i=1}^n \hat{q}_i^{i,\epsilon} = \sum_{i=1}^n \hat{q}_i$$

and

$$\begin{aligned} \pi_i^{i,\epsilon}(c_i) &= [p(\hat{q}) - c_i]\hat{q}_i^{i,\epsilon} \\ &= [p(\hat{q}) - c_i](\hat{q}_i - \epsilon) = \pi_i(\hat{q}, c_i) - \epsilon[p(\hat{q}) - c_i]. \end{aligned} \tag{2}$$

Also

$$\begin{aligned} \pi_j^{i,\epsilon}(c_j) &= [p(\hat{q}) - c_j]\hat{q}_i^{i,\epsilon} \\ &= [p(\hat{q}) - c_j](\hat{q}_j + \frac{\epsilon}{n-1}) = \pi_j(\hat{q}, c_j) + \frac{\epsilon}{n-1}[p(\hat{q}) - c_j]. \end{aligned} \tag{3}$$

Define $\epsilon_i^i = \min_{c_i \in C}\{\epsilon[p(\hat{q}) - c_i]\}$ and $\epsilon_j^i = \min_{c_j \in C}\{\frac{\epsilon}{n-1}[p(\hat{q}) - c_j]\}$. The result then follows from equations (2) and (3). ■

The next result shows that every quantity vector in Q is the quantity vector of a pooling equilibrium. This result can be viewed as a version of the folk theorem.

Theorem 1 (Pooling Equilibrium and the Folk Theorem) *For every output vector* $\hat{q}$ *in* Q*, there is a* $\delta(\hat{q}) < 1$ *such that for all* $\delta \geq \delta(\hat{q})$*,* $\hat{q}$ *is the output vector of a pooling Perfect Bayesian equilibrium.*

Proof: The claim is that the strategy combination $(\sigma_1^\star, \cdots, \sigma_n^\star)$ described below is a pooling equilibrium.
(i)$\sigma_{it}^\star(h_{t-1}) = \hat{q}_i$ if the past history has been the output vector $\{\hat{q}_i\}_{i=1}^n$.
(ii) If a firm i produces $q_i \neq \hat{q}_i$ in any period t and all the other firms had produced $\hat{q}_j$ in all previous periods, then all firms $j \neq i$ produce the output $\frac{1}{n-1}\bar{q}$ for a length of time

T_i, from time $t+1$ onwards. This is a phase I punishment strategy.
(iii) If there are no deviations during a phase I punishment by any of the firms $j \neq i$, then after the length of time T_i, the firms produce the output vector $\{\hat{q}_j^{i,\epsilon}\}_{j=1}^n$ as defined in lemma 1.
(iv) If a firm $j \neq i$ deviates during a phase I punishment, then firms $\ell \neq j$ produce the output $\frac{1}{n-1}\bar{q}$ for a length of time T_j. Such a punishment is a phase II punishment.
(v) After a phase II punishment the firms produce the quantity vector $\{\hat{q}_i^{j,\epsilon}\}_{i=1}^n$.
(vi) Finally, if a firm ℓ deviates from the quantity vector $\{\hat{q}_i^{j,\epsilon}\}_{i=1}^n$, then the other firms play the phase I punishment for a length of time T_ℓ, and then produce the quantity vector $\{\hat{q}_i^{\ell,\epsilon}\}_{i=1}^n$.

We now proceed to show that the strategy profile $\sigma^\star$ is an equilibrium irrespective of the realized cost of the firm.

Let M_i be the maximum "gain" a firm can make by deviating in any period irrespective of its cost. If firm i deviates in any period then its maximum payoff in the subsequent periods, if it is has cost c_i, is

$$M_i + \delta^{T_i} \sum_{\nu=1}^{\infty} \delta^{\nu-1} \pi_i^{i,\epsilon}(\hat{q}, c_i)$$

as for a length of time T_i firm $i's$ profit is zero or less every period (see proposition 1). If firm i does not deviate, its payoff in the subsequent periods is

$$\sum_{\nu=1}^{\infty} \delta^{\nu-1} \pi_i(\hat{q}, c_i).$$

Therefore, from lemma 1 and the construction of the strategy profile, firm i does not gain from a deviation if

$$\sum_{\nu=1}^{\infty} \delta^{\nu-1} \pi_i(\hat{q}, c_i) \geq M_i + \delta^{T_i} \sum_{\nu=1}^{\infty} \delta^{\nu-1} [\pi_i(\hat{q}, c_i) - \epsilon_i^i]. \tag{4}$$

That is,

$$\frac{1-\delta^{T_i}}{1-\delta} \pi_i(\hat{q}, c_i) \geq M_i - \frac{\delta^{T_i}}{1-\delta} \epsilon_i^i. \tag{5}$$

Now note that in (5) the expression $\frac{1-\delta^{T_i}}{1-\delta} \to T_i$ as $\delta \to 1$, therefore, there is a $\delta_1 : 0 < \delta_1 < 1$ and T_i sufficiently large for which equation (4) is satisfied for all $c_i \in C$. Choose T_i so that

$$\frac{1-\delta^{T_i}}{1-\delta} \min_{c_i} \pi_i(\hat{q}, c_i) \geq M_i - \frac{\delta^{T_i}}{1-\delta} \epsilon_i^i. \tag{6}$$

Thus, phase I punishments can deter a firm from deviating irrespective of its cost.

Now consider a deviation made by a firm j during a phase I punishment. Let M_j be the maximum payoff firm j can get in a single period, and L_j the maximum loss every period that firm j sustains during a phase I punishment. Then firm $j's$ payoff after deviating is less than or equal to

$$M_j + \delta^{T_j} \sum_{\nu=1}^{\infty} [\pi_j(\hat{q}, c_j) - \epsilon_j^j],$$

and if a firm j does not deviate, the payoff in the subsequent periods is:

$$\delta^{T_i - t} \sum_{\ell=1}^{\infty} [\pi_j(\hat{q}, c_j) + \epsilon_j^i] - \sum_{\nu=1}^{T_i - t} L_j.$$

Therefore, firm j does not gain by deviating during a phase I punishment if

$$\delta^{T_i - t} \sum_{\nu=1}^{\infty} \delta^{\nu-1} [\pi_j(\hat{q}, c_j) + \epsilon_j^i] - \sum_{\nu=1}^{T_i - t} L_j \geq M_j + \delta^{T_j} \sum_{\nu=1}^{\infty} \delta^{\nu-1} [\pi_j(\hat{q}, c_j) - \epsilon_j^j]. \tag{7}$$

This reduces to

$$\begin{aligned} \delta^{T_i - t} \frac{1 - \delta^{T_j - T_i + t}}{1 - \delta} \pi_j(\hat{q}, c_j) &\geq M_j - \frac{\delta^{T_i - t}}{1 - \delta} \epsilon_j^i - \epsilon_j^j \frac{\delta^{T_j}}{1 - \delta} + L_j \frac{1 - \delta^{T_i - t}}{1 - \delta} \\ &\geq M_j - \frac{\delta^{T_i - t}}{1 - \delta} \epsilon_j^i - \epsilon_j^j \frac{\delta^{T_j}}{1 - \delta} + L_j \frac{1 - \delta^{T_i}}{1 - \delta}. \end{aligned} \tag{8}$$

In equation (8), for a given T_i and a given T_j, as $\delta \to 1$, the expression

$$\delta^{T_i - t} \frac{1 - \delta^{T_j - T_i + t}}{1 - \delta}$$

goes to $T_j + t - T_i$ and the expressions $\frac{\delta^{T_j}}{1-\delta}$ and $\frac{\delta^{T_i - t}}{1-\delta}$ both go to ∞. Further, the expression $\frac{1-\delta^{T_i}}{1-\delta}$ goes to T_i. Hence, there is a $\delta_{j2} : 0 < \delta_{j2} < 1$ such that equation (8) holds for all $\delta > \delta_{j2}$ and for all $c_j \in C$. Again choose T_j such that an expression like (8) holds for all c_j. That is, choose T_j such that

$$\delta^{T_i - t} \frac{1 - \delta^{T_j - T_i + t}}{1 - \delta} \min_{c_j} \pi_j(\hat{q}, c_j) \geq M_j - \frac{\delta^{T_i - t}}{1 - \delta} \epsilon_j^i - \epsilon_j^j \frac{\delta^{T_j}}{1 - \delta} + L_j \frac{1 - \delta^{T_i}}{1 - \delta}. \tag{9}$$

Thus, for all such discount factors $\delta > \delta_{j2}$ firm j does not gain from deviating during a phase II punishment irrespective of its cost.

Finally, suppose that firm $\ell \neq j$ deviates during a phase II punishment, then the other firms play a phase II punishment for firm ℓ and then produce the output vector

$\{\hat{q}_i^{\ell,\epsilon}\}_{i=1}^n$. As in the case of the deviation during a phase II punishment by firm j, firm ℓ cannot gain from deviating if the discount factor $\delta_{\ell 2}$ is sufficiently high.

Choose $\bar{\delta} = \max\{\delta_1, \{\delta_{j2}\}_{j\neq i}\}$, then if $\delta > \bar{\delta}$, firm i does not gain from deviating. Hence for $\delta > \bar{\delta}$, no firm can gain by deviating. This establishes the claim.

We now show that the strategy combination $\sigma^\star$ is a Perfect Bayesian equilibrium. We first note that since $(\sigma^\star|h_t, c_i) = (\sigma^\star|h_t, c_i')$ for every c_i, c_i', therefore from (1) we have $\mu(\mathbf{c}|h_t, c_i, \sigma^\star) = \mu(\mathbf{c}|h_{t-1}, c_i, \sigma^\star)$ for all $t \geq 1$. Hence, $\mu(\mathbf{c}|h_t, c_i, \sigma^\star) = \mu(\mathbf{c}|c_i)$ so that

$$\begin{aligned}\sum_{\mathbf{c}\in\mathcal{C}} \mu(\mathbf{c}|h_t, c_i, \sigma^\star)\pi_i^\infty(\sigma^\star|h_t, c_i) &= \pi_i^\infty(\sigma^\star|h_t, c_i)\sum_{\mathbf{c}\in\mathcal{C}}\mu(\mathbf{c}|c_i) \\ &= \pi_i^\infty(\sigma^\star|h_t, c_i). \qquad (10)\end{aligned}$$

Since we have already shown that for any h_t, c_i and for all strategy σ_i of firm i, $\pi_i^\infty(\sigma^\star|h_t, c_i) \geq \pi_i^\infty((\sigma_i, \sigma_{-i}^\star|h_t, c_i)$, it now follows from (10) that

$$\begin{aligned}\sum_{\mathbf{c}\in\mathcal{C}} \mu(\mathbf{c}|h_t, c_i, \sigma^\star)\pi_i^\infty(\sigma^\star|h_t, c_i) &= \pi_i^\infty(\sigma^\star|h_t, c_i) \\ &\geq \sum_{\mathbf{c}\in\mathcal{C}} \mu(\mathbf{c}|h_t, c_i, (\sigma_i, \sigma_{-i}^\star))\pi_i^\infty((\sigma_i, \sigma_{-i}^\star)|h_t, c_i). \qquad (11)\end{aligned}$$

But (11) then shows that $\sigma^\star$ is a Perfect Bayesian equilibrium. It is by construction a pooling equilibrium. This thus concludes the proof. ■

In these pooling equilibrium, the firms interact without revealing their costs to their rivals, and there is simply no change in beliefs from the ones held initially. The result holds because of the way the players can be minimaxed without the other players' knowing their costs by simply working with the quantity vector that the firms have agreed to produce. The strength of the result in theorem 1 comes from the observation that one can get the result to work for any distribution of the marginal costs, as long as the support of the distribution has a reasonable upper bound, or alternatively, when the demand in the market remains sufficiently high.

While theorem 1 gives us some idea of the set of quantity vectors that would be produced in a pooling equilibrium it does not tell us anything about whether any of these quantity vectors are collusive. The next result shows that under some mild conditions all possible optimal collusive quantity vectors are in Q. Let $\hat{q}(\underline{\mathbf{c}})$ be the quantity vector that maximizes the joint profits of the firms when the marginal costs of all the firms are $\underline{c}$, the lowest possible, and $p(\hat{q}(\underline{\mathbf{c}}))$ be the optimal collusive price in this case. The result that follows shows that if $p(\hat{q}(\underline{\mathbf{c}})) > \bar{c}$ then all the collusive quantity vectors are in Q.

Proposition 3 *Let* $\hat{q}(\underline{\mathbf{c}})$ *denote a quantity vector that maximizes the joint profit of the firms when the marginal costs of the firms are all given by* $\underline{\mathbf{c}}$*. Then if*

$$p(\hat{q}(\underline{\mathbf{c}})) - \bar{c} > 0 \tag{12}$$

the quantity vectors q *that satisfy*

$$\pi_i(q|c_i) \geq \pi_i(\hat{q}(\underline{\mathbf{c}})|c_i)$$

for all $i = 1, \cdots, n$ *are in* Q*.*

Proof: We observe that if the condition in (12) holds, then for the cost profile $\mathbf{c} = (c_1, \cdots, c_n)$ and the quantity vector q,

$$\pi_i(q|c_i) = p(q)q_i - c_i q_i \geq p(\hat{q}(\underline{\mathbf{c}}))\hat{q}_i(\underline{\mathbf{c}}) - c_i\hat{q}_i(\underline{\mathbf{c}}) > p(\hat{q}(\underline{\mathbf{c}}))\hat{q}_i(\underline{\mathbf{c}}) - \bar{c}\hat{q}_i(\underline{\mathbf{c}}) > 0$$

for all $i = 1, \cdots, n$. But this shows that the output vector q is in Q. ■

Theorem 1 and proposition 3 show that the set of quantity vectors that would be produced in a Perfect Bayesian equilibrium is large and consists of almost any quantity vector at which the firms can make positive profits. It also seems that collusion is a distinct possibility. However, even though the set Q may contain collusive quantity vectors, an important issue here is whether the firms can agree on the optimal collusive quantities given their true marginal costs. Agreeing on which quantity vector is the optimal collusive quantity vector can be problematic in these cases. As the following example shows, the difficulty may lie in determining the true optimal collusive output vector of the firms when the marginal costs are not known. The results in the next section thus focus on finding equilibrium in which the firms will have an incentive to set output levels according to their true costs.

Example 1 *An example*[5] *illustrating the incentive-compatibility issues involved in inferring the true marginal costs.*

Consider an oligopoly game in which

$$p(Q) = 10 - Q$$

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[5] Notice that the example satisfies the condition of proposition 3.

and the marginal cost of firm 1 is either $c_1 = 1$ or $c_1 = 2$ with

$$Prob(c_1 = 1) = Prob(c_1 = 2) = 0.5$$

and similarly, the marginal cost of firm 2 is either $c_2 = 1$ or $c_2 = 2$ with

$$Prob(c_2 = 1) = Prob(c_2 = 2) = 0.5.$$

It can be checked that the joint profit maximizing quantity vector when the marginal cost of the firms are $c_1 = 1$ and $c_2 = 1$ is $\hat{q}_1 = \hat{q}_2 = 2.25$[6]. The profit of each firm in this case is 10.125. If, however, the marginal cost of firm 2 is $c_2 = 2$ instead, and firm 2 reveals that its marginal cost $c_2 = 2$, then firm 1 would want to choose a quantity vector that allows it to make profits higher than 10.125. In fact, in such a case firm 1 may want to produce the entire profit maximizing output of 4.5 units. Firm 2 would in this case be left with either a very small output or zero. If on the other hand firm 2 claimed that its marginal cost was indeed only 1, rather than 2, its profit would be $(10 - 4.5) \times 2.25 - 2 \times 2.25 = 7.875$ considerably more than what it could hope to get if it revealed its true marginal cost. ■

This example shows why firms may find it difficult to know which quantity vector in Q is the true optimal collusive quantity vector. If a firm is a high-cost firm, it would want to claim that it is a low-cost firm. The best the firms can hope to do in this case is choose some quantity vector in Q and collude in producing that quantity vector without ever truly knowing whether that was the most collusive. This incentive problem raises the question of whether there is a way of inferring the true marginal costs of the firms and whether this can be done as part of an equilibrium in which the firms collude optimally given some incentive constraints. The next set of results deal with these issues.

5 Separating Equilibrium

In the previous section we characterized the set of pooling equilibrium in the repeated Cournot Oligopoly. We also showed that even though collusion is possible, it can be

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[6] Notice that in the case the marginal costs of the firms are $c_1 = c_2 = 1$, the price under optimal collusion is 5.5. If the marginal costs of the firms are $c_1 = c_2 = 2$, then the optimal collusive quantities are $\hat{q}_1 = \hat{q}_2 = 2$ and the optimal collusive price is 6. This shows that these oligopoly games are different from the class of oligopoly games analyzed in Athey and Bagwell [2004]. In their setting, in addition to the oligopoly game being a Bertrand price setting game, the optimal collusive price is independent of the marginal costs of the firms.

difficult to find the optimal collusive quantity vector. We now proceed to investigate whether there are separating equilibrium in which the firms produce the optimal collusive quantity vectors. We show here that under some relatively mild conditions there are strictly separating equilibrium in which the firms are able to collude optimally, under some incentive constraints, in a way that is consistent with their true marginal cost. A concept that plays an important role in the construction of the optimal collusive strictly separating Perfect Bayesian equilibrium is the Bayesian Nash equilibrium of the single-period game.

Definition 3 *An n-vector of quantity choices* $\{q_1^\star(.), \cdots, q_n^\star(.)\}$*, where a quantity choice is a function* $q_i^\star : C^i \to [0, \bar{q}]$*, is a* **Bayesian Nash Equilibrium** *of the game if for each firm* i *and cost* c_i *of firm* i*, we have*

$$q_i^\star(c_i) \in \text{argmax} \sum_{\mathbf{c} \in \mathcal{C}} \mu(\mathbf{c}|c_i)[\{p(\sum_{j \neq i} q_j^\star(c_j) + q_i) - c_i q_i\}]. \tag{13}$$

We will say a Bayesian Nash equilibrium is a **strictly separating** Bayesian Nash equilibrium if $q_i^\star(c_i) \neq q_i^\star(c_i')$, whenever $c_i \neq c_i'$. The next definition defines a strictly separating perfect Bayesian equilibrium of the infinite-horizon game.

Definition 4 *A Perfect Bayesian equilibrium* $(\sigma^\star, \mu^\star)$ *will be said to be a* **strictly separating** *Perfect Bayesian equilibrium if it is a Perfect Bayesian equilibrium and, in addition, satisfies the condition that for all* i *and any pair* (c_i, c_i') *from* C

$$\sigma_i^\star|_{h_t, c_i} \neq \sigma_i^\star|_{h_t, c_i'},$$

whenever $c_i \neq c_i'$.

Thus in a strictly separating equilibrium a firm's equilibrium strategy is conditioned on its cost, and if the costs differ, then so does the strategy. In a strictly separating Bayesian equilibrium, firms with different costs will play differently, and thus will tend to reveal information about their costs, since the other firms would be able to infer the cost of a firm from observing the output choices of a firm. We will show that under very mild conditions, the infinite horizon game has a strictly separating equilibrium, and this equilibrium is generated by first playing a strictly separating Bayesian equilibrium of the single-period game of incomplete information. The first lemma shows that the Cournot quantity setting game with private information always has a Bayesian Nash Equilibrium.

Lemma 2 *If the inverse demand function* $p : [0, \bar{q}] \to \mathbb{R}_+$ *is concave, then the single-period Cournot quantity-setting game has a Bayesian Nash Equilibrium.*

Proof: For every firm i with cost c_i consider the correspondence

$$B_i^{c_i} : \Pi_{\sum_{j \neq i} \# \mathcal{C}}[0, \bar{q}] \to [0, \bar{q}]$$

defined as

$$B_i^{c_i}(q_j(c_j)) = \{q_i^\star(c_i) | q_i^\star(c_i) \in \text{argmax} \textstyle\sum_{\mathbf{c} \in \mathcal{C}} \mu(\mathbf{c}|c_i) \pi_i(\mathbf{c}|c_i)\}$$

where $\pi_i(\mathbf{c}|c_i) = [p(\sum_{j \neq i} q_j^\star(c_j)) + q_i) - c_i]q_i$ is the profit of firm i given a cost profile $\mathbf{c} \in \mathcal{C}$. Since the inverse demand function $p(.)$ is concave it follows that $\pi_i(\mathbf{c}|c_i)$ is concave, and hence, $\sum_{\mathbf{c} \in \mathcal{C}} \mu(\mathbf{c}|c_i) \pi_i(\mathbf{c}|c_i)$ is concave in q_i. Thus it can be checked that $B_i^{c_i}$ is a nonempty-valued, compact-valued and convex-valued correspondence that is upper semicontinuous.

Therefore, since C is finite, the correspondence

$$B : \Pi_{n \# \mathcal{C}}[0, \bar{q}] \to \Pi_{n \# \mathcal{C}}[0, \bar{q}]$$

defined as

$$B((q_i(c_i)_{c_i \in C}, \{q_j(c_j)_{c_j \in C}\}_{j \neq i}) = \Pi_{\{c_j \in C | j = 1 \cdots n\}} B_j^{c_j}$$

is a nonempty-valued, compact-valued and convex-valued correspondence that is upper semicontinuous and defined on a compact subset of an Euclidean space. Thus the correspondence B has a fixed point $(q_i^\star(c_i)_{c_i \in C}, \{q_j^\star(c_j)_{c_j \in C}\}_{j \neq i})$. It is clear that this is a Bayesian Nash Equilibrium of the single-period game. ■

The next lemma shows that the Bayesian Nash equilibrium of the single-period game is a strictly separating equilibrium.

Lemma 3 *If the inverse demand function* $p(.)$ *is twice differentiable and satisfies the condition that* $p'(.) < 0$ *and* $p''(.) \leq 0$*, then a Bayesian Nash equilibrium of the quantity-setting game is a strictly separating Bayesian Nash Equilibrium.*

Proof: We first note that the condition on the inverse demand function shows that it is concave. Thus the profit-maximizing quantity choice, for every possible cost level $c_i \in C$, is given by the first order condition

$$\mu(\mathbf{c}|c_i)[p'(\sum_{j \neq i} q_j^\star(c_j) + q_i)q_i + p(\sum_{j \neq i} q_j^\star(c_j) + q_i)] = c_i.$$

Thus if $c_i' > c_i$, for every cost profile $\mathbf{c} \in \mathcal{C}$, we have

$$\begin{aligned} & p'(\sum_{j \neq i} q_j^\star(c_j) + q_i^\star(c_i))q_i^\star(c_i) + p(\sum_{j \neq i} q_j^\star(c_j) + q_i^\star(c_i)) \\ < \quad & p'(\sum_{j \neq i} q_j^\star(c_j) + q_i^\star(c_i'))q_i^\star(c_i') + p(\sum_{j \neq i} q_j^\star(c_j) + q_i^\star(c_i')). \end{aligned} \tag{14}$$

Now consider the function

$$p'(Q_{-i} + q)q + p(Q_{-i} + q) : [0, \bar{q}] \to \mathbb{R}.$$

The first derivative of this function with respect to q is given by

$$p''(Q_{-i} + q)q + p'(Q_{-i} + q) + p'(Q_{-i} + q) = p''(Q_{-i} + q)q + 2p'(Q_{-i} + q) < 0,$$

since $p'(.) < 0$ and $p'' \leq 0$. Thus the function is a strictly decreasing function of q. From this and from (14) it now follows that

$$q_i^\star(c_i') < q_i^\star(c_i). \tag{15}$$

But this shows that the Bayesian Equilibrium is a strictly separating equilibrium. ■

Given that there exists a strictly separating Bayesian Nash equilibrium one could be led to believe that it may be possible to play so that the firms play the Bayesian Nash equilibrium in period 1, and then play collusively based on the information revealed in period 1, when the firms play according to the strictly separating Bayesian Nash equilibrium. However, if these collusive output vectors do not satisfy some incentive constraints, then a high-cost firm may have the incentive to pretend to be a low-cost firm, and play as a low-cost firm, as example 1 shows. While such behavior may benefit the high cost firm, it can also lead to lower joint profits. Therefore, there is a need to impose incentive constraints when deciding the output vector of the firms.

Let

$$q : C \times \cdots \times C \to \mathbb{R}_+^n$$

denote an assignment of quantity vectors as a function of the realized cost profile $\mathbf{c} = (c_1, \cdots, c_n)$. This assignment of quantity vectors will be said to be *incentive compatible* if the assignment of quantity vectors satisfies the following constraints. For all $i = 1, \cdots, n$, and for $c_i \in C$,

$$\pi_i(q(\mathbf{c})|c_i) \geq \pi_i(q(\mathbf{c_{-i}}, c_i')|c_i) \tag{16}$$

for all $c_i' \neq c_i$. That is, an assignment of quantity vector is incentive compatible if none of the firms have an incentive to claim that its cost is different from that of the true cost c_i. An assignment of quantity vectors

$$\tilde{q} : C \times \cdots \times C \to \mathbb{R}^n_+$$

will be said to be *optimal incentive compatible assignment of quantity vectors* if it is an incentive compatible assignment of quantity vectors that solves

$$\text{maximize} \sum_{\mathbf{c} \in \mathcal{C}} \mu(\mathbf{c}) \pi(q(\mathbf{c})|\mathbf{c})$$

such that

$$\pi_i(q(\mathbf{c})|c_i) \geq \pi_i(q(\mathbf{c_{-i}}, c_i')|c_i) \tag{17}$$

for all $i = 1, \cdots, n$ and $c_i' \neq c_i$, where $\pi(q(\mathbf{c}))$ denotes the joint profits (the sum of the profits) of the firms when the cost profile is $\mathbf{c}$, and the firms produce the quantity vector $q(\mathbf{c})$. $\pi_i(.)$ as before denotes the profit of the individual firm i. Thus, an optimal incentive compatible quantity vector maximizes the expected joint profits of the firms, subject to the constraint that none of the firms would want to play like a firm with a different cost structure.

Lemma 4 *Assume that $p'(.) < 0$ and $p''(.) < 0$, then there exists an optimal incentive compatible assignment vector $\tilde{q} : C \times \cdots \times C \to \mathbb{R}^n_+$. Such an optimal incentive compatible quantity vector will satisfy the condition that for every cost profile $\mathbf{c} = (c_1, \cdots, c_n)$, $\tilde{q}(\mathbf{c})$ maximizes*[7]

$$\pi(q|\mathbf{c}) = \sum_{i=1}^{n} \pi_i(q|c_i)$$

subject to

$$\pi_i(q|c_i) \geq \pi_i(\tilde{q}(\mathbf{c_{-i}}, c_i')|c_i), \tag{18}$$

Proof: Given a cost profile $\mathbf{c}$, let $q(\mathbf{c_{min}})$ denote the quantity vector that maximizes

$$\sum_{i=1}^{n} \pi_i(q|c_i = c_{min}).$$

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[7] This shows that the optimal incentive compatible assignment quantity vector is interim efficient. That is, it is optimal given the assignment of quantities as a function of the realized costs, and the private information received by the firms. Hence, a firm has no reason to change its assigned output even after it receives the private information and gets to know the cost of the other firms.

We first show that $q(\mathbf{c_{min}})$ is incentive compatible, that is, it satisfies the constraints given by (16).

First note that if $c'_i \geq c_{min}$, then $q(\mathbf{c_{min}}) = q(\mathbf{c'_{min}})$, and

$$\pi_i(q(\mathbf{c_{min}})|c_i) = \pi_i(q(\mathbf{c'_{min}})|c_i \neq c'_i). \tag{19}$$

Now if $c'_i < c_{min}$ then $c'_{min} < c_{min}$. In this case we observe that

$$q_i(\mathbf{c'}) = \frac{1}{n}\hat{Q}(\mathbf{c'_{min}})$$

where $\hat{Q}(\mathbf{c'_{min}})$ maximizes

$$(p(Q) - c'_{min})Q,$$

the joint profit when the firms all have cost equal to c'_{min}.

Further,

$$\pi_i(q_i(\mathbf{c'})) = p(\hat{Q}(\mathbf{c'_{min}}))\frac{1}{n}\hat{Q}(\mathbf{c'_{min}}) - c_i\frac{1}{n}\hat{Q}(\mathbf{c'_{min}})$$

so that

$$\frac{\partial \pi_i(\hat{Q}(\mathbf{c'_{min}}))}{\partial Q} = p'(\hat{Q}(\mathbf{c'_{min}}))\frac{1}{n}\hat{Q}(\mathbf{c'_{min}}) + \frac{1}{n}p(\hat{Q}(\mathbf{c'_{min}})) - \frac{1}{n}c_i.$$

Since the first order condition for maximizing $(p(Q) - c'_{min})Q$ is

$$p'(\hat{Q}(\mathbf{c'_{min}}))\hat{Q}(\mathbf{c'_{min}}) + p(\hat{Q}(\mathbf{c'_{min}})) - c'_{min} = 0$$

we have

$$\frac{\partial \pi_i(\hat{Q}(\mathbf{c'_{min}})|c_i)}{\partial Q} = p'(\hat{Q}(\mathbf{c'_{min}}))\hat{Q}(\mathbf{c'_{min}}) + p(\hat{Q}(\mathbf{c'_{min}})) - c_i < 0 \text{ for } c'_{min} < c_i.$$

As $p'(.) < 0$ and $p''(.) < 0$, it follows that

$$\frac{\partial^2 \pi_i(Q)}{\partial Q^2} = p''(Q)Q + 2p'(Q) < 0.$$

Since $\hat{Q}(c^1) > \hat{Q}(c^2)$, whenever $c^1 < c^2$, therefore for $c'_{min} < c_{min}$, we have

$$\pi_i(q(\mathbf{c_{min}})|c_i) > \pi_i(q(\mathbf{c'_{min}})|c_i). \tag{20}$$

From (19) and (20) it follows that the quantity vector $q(\mathbf{c_{min}})$ satisfies the incentive constraints given by (16).

Therefore, the set of quantity vectors that satisfy the constraints in (16) is a non-empty compact subset of $\mathbb{R}_+^{nk}$, where k is the total number of possible cost profiles. Thus the problem given by (17) has a solution. This proves the existence of an optimal incentive compatible assignment of quantity vectors.

In order to prove the last claim, suppose there is a cost profile $\mathbf{c}'' \in \mathcal{C}$ and a quantity vector $\bar{q}(\mathbf{c}'')$ for which

$$\pi(\bar{q}|\mathbf{c}'') > \pi(\tilde{q}(\mathbf{c}'')|\mathbf{c}'')$$

and

$$\pi_i(\bar{q}(\mathbf{c}'')|c_i'') \geq \pi_i(\tilde{q}(\mathbf{c}''_{-\mathbf{i}}, c_i')|c_i''),$$

for all i and all $c_i' \in C$. Then this would imply that

$$\sum_{\{\mathbf{c}' \in \mathcal{C} | \mathbf{c}' \neq \mathbf{c}\}} \mu(\mathbf{c}')\pi(\tilde{q}|\mathbf{c}') + \mu(\mathbf{c}'')\pi(\bar{q}|\mathbf{c}'') > \sum_{\mathbf{c} \in \mathcal{C}} \mu(\mathbf{c})\pi(\tilde{q}|\mathbf{c}).$$

A contradiction to the fact that the assignment vector $\tilde{q}(.)$ solves (18). Thus every optimal incentive compatible assignment of quantity vectors $\tilde{q}(.)$ also solves (19). ■

The optimal incentive compatible vector $\tilde{q}(\mathbf{c})$ can be viewed as the most collusive output vector for the firms given that the incentive constraints have to hold. Example 1 shows that this cannot be avoided, because if an agreement does not satisfy the incentive constraints, a high-cost firm would always have an incentive to pretend to be a low-cost firm. The second claim in lemma 4 is also important as it shows that the optimal incentive compatible vector is interim efficient as the joint profits of the firms is a maximum, given the optimal incentive compatible assignment of quantity vectors, and the realized cost profile.

5.1 Strictly Separating Equilibrium with Signaling

We show here that the optimal incentive compatible assignment vector can be produced in a strictly separating Bayesian Nash equilibrium in which there would be a correct inference in equilibrium about the true costs of the firms. In this equilibrium the firms play the strictly separating Bayesian equilibrium in period 1 and then produce the optimal incentive compatible output vector from period 2 onwards. The first period in which the firms play the strictly separating Bayesian Nash equilibrium of the single-period game is played to signal the true costs of the firms. The result shows that the firms are able to successfully infer the true costs of the firms.

Theorem 2 *If the inverse demand function $p(.)$ satisfies the conditions that $p'(.) < 0$ and $p''(.) \leq 0$, then for every realized cost profile $\mathbf{c} = (c_1, \cdots, c_n)$, there is a $\underline{\delta} < 1$ such that for all $\delta > \underline{\delta}$ there is a strictly separating Perfect Bayesian equilibrium in which, from period 2 onwards, the firms produce the optimal incentive compatible quantity vector $\tilde{q}(c_1, \cdots, c_n)$.*

Proof: The claim is that the strategy combination $(\sigma_1^\star, \cdots, \sigma_n^\star)$ described below is the strategy combination of a strictly separating perfect Bayesian equilibrium.
(i) $\sigma_{i1}^\star(c_i) = q_i^\star(c_i)$ for all $i = 1, \cdots, n$.
(ii)$\sigma_{i2}^\star(h_1, c_i) = \tilde{q}_i(c_1, \cdots, c_n)$ where $(c_1, \cdots, c_n)$ satisfies the condition that $c_i = (q_i^\star)^{-1}(\sigma_{i1})$ for all $i = 1, \cdots, n$.
(iii) If a firm i produces $q_i \neq \sigma_{i2}$ in any period $t \geq 3$, and all the other firms had produced σ_{j2} in all previous periods, then all firms $j \neq i$ produce the output $\frac{1}{n-1}\bar{q}$ for a length of time T_i. This is a phase I punishment strategy.
(iv) If there are no deviations during a phase I punishment by any of the firms $j \neq i$, then after the length of time T_i, the firms produce the output vector $\{q_j^{i,\epsilon}\}_{j=1}^n$(see lemma 1) such that

$$\pi_j(q_j^{i,\epsilon}(c_1, \cdots, c_n)|c_j) = \pi_j(\tilde{q}(c_1, \cdots, c_n)|c_j) + \frac{\epsilon_i}{n-1}$$

and

$$\pi_i(q_j^{i,\epsilon}(c_1, \cdots, c_n)|c_i) = \pi_i(\tilde{q}(c_1, \cdots, c_n)|c_i) - \epsilon_i.$$

(v) If a firm $j \neq i$ deviates during a phase I punishment, then firms $\ell \neq j$ produce the output $\frac{1}{n-1}\bar{q}$ for a length of time T_j. This is a Phase II punishment.
(vi) After a phase II punishment the firms produce the quantity vector $\{q_i^{j,\epsilon}\}_{i=1}^n$.
(vii) Finally, if a firm ℓ deviates from the quantity vector $\{q_i^{j,\epsilon}\}_{i=1}^n$, then the other firms play the phase I punishment for a length of time T_ℓ, and then produce the quantity vector $\{q_i^{\ell,\epsilon}\}_{i=1}^n$.

We now show that $\sigma^\star$ is the strategy combination of a strictly separating equilibrium. We first show that if all the firms play their Bayesian Nash equilibrium output vector $q_i^\star(c_i)$ in the first period, then the strategy profile $\sigma^\star$ is an equilibrium. We then argue that the optimal strategy of the firms is to indeed produce the output vector $(q_i^\star(c_i))$ in the first period.

Let M_i be the maximum "gain" a player can make by deviating in any period irrespective of its cost. If firm i deviates in any period, then its payoff in the subsequent periods is given by at most

$$M_i + \delta^{T_i} \sum_{\nu=1}^{\infty} \delta^{\nu-1} [\pi_i(\tilde{q}(c_1, \cdots, c_n)|c_i) - \epsilon_i],$$

as for a length of time T_i firm $i's$ profit is zero every period (see proposition 1). If firm i does not deviate its payoff in the subsequent periods is

$$\sum_{\nu=1}^{\infty} \delta^{\nu-1} [\pi_i(\tilde{q}(c_1, \cdots, c_n)|c_j = c_j)].$$

Therefore, firm i does not gain from a deviation if

$$\sum_{\nu=1}^{\infty} \delta^{\nu-1} [\pi_i(\tilde{q}(c_1, \cdots, c_n)|c_i)] \geq M_i + \delta^{T_i} \sum_{\nu=1}^{\infty} \delta^{\nu-1} [\pi_i(\tilde{q}(c_1, \cdots, c_n)|c_i) - \epsilon_i]$$

or

$$\sum_{\nu=1}^{T_i} \delta^{\nu-1} \pi_i(\tilde{q}(c_1, \cdots, c_n)|c_i) \geq M_i - \delta^{T_i} [\sum_{\nu=1}^{\infty} \delta^{\nu-1} \epsilon_i].$$

That is,

$$\frac{1-\delta^{T_i}}{1-\delta} \pi_i(\tilde{q}(c_1, \cdots, c_n)|c_i) \geq M_i - \frac{\delta^{T_i}}{1-\delta} \epsilon_i. \tag{21}$$

Now note that in (21) the expression $\frac{1-\delta^{T_i}}{1-\delta} \to T_i$ as $\delta \to 1$, therefore, there is a $\delta_1 : 0 < \delta_1 < 1$ such that for T_i sufficiently large, equation (21) is satisfied for all cost profiles $(c_1, \cdots, c_n)$. Thus, phase I punishments can deter a firm from deviating.

Now consider a deviation made by a firm j during a phase I punishment. Let M_j be the maximum "gain" firm j can make irrespective of its cost and L_j the maximum "loss" firm j can sustain every period during a phase I punishment. Then firm $j's$ payoff after deviating is at most

$$M_j + \delta^{T_j} \sum_{\nu=1}^{\infty} [\pi_j(\tilde{q}(c_1, \cdots, c_n)|c_j)) - \epsilon_j],$$

and if a firm j does not deviate, the payoff in the subsequent periods is

$$\delta^{T_i - t} \sum_{\ell=1}^{\infty} [\pi_j(\tilde{q}(c_1, \cdots, c_n)|c_j) + \frac{\epsilon_j}{n-1}] - \sum_{\nu=1}^{T_i - t} L_j.$$

Therefore, firm j does not gain by deviating during a phase I punishment if

$$\delta^{T_i - t} \sum_{\nu=1}^{\infty} \delta^{\nu-1} [\pi_j(\tilde{q}(c_1, \cdots, c_n)|c_j) + \frac{\epsilon_j}{n-1}] - \sum_{\nu=1}^{T_i - t} L_j$$
$$\geq M_j + \delta^{T_j} \sum_{\nu=1}^{\infty} \delta^{\nu-1} [\pi_j(\tilde{q}(c_1, \cdots, c_n)|c_j) - \epsilon_j].$$

This reduces to

$$\begin{aligned}\delta^{T_i-t}\frac{1-\delta^{T_j-T_i+t}}{1-\delta}\pi_j(\tilde{q}(c_1,\cdots,c_n)|c_j) &\geq M_j-\frac{\delta^{T_i-t}}{1-\delta}\frac{\epsilon_j}{n-1}-\epsilon_j\frac{\delta^{T_j}}{1-\delta}+\sum_{\nu=1}^{T_i-t}L_j\\ &\geq M_j-\frac{\delta^{T_i-t}}{1-\delta}\frac{\epsilon_j}{n-1}-\epsilon_j\frac{\delta^{T_j}}{1-\delta}+\sum_{\nu=1}^{T_i}L_j.\end{aligned} \tag{22}$$

In equation (22), given T_i for a given T_j as $\delta \to 1$, the expression

$$\frac{1-\delta^{T_j-T_i+t}}{1-\delta}$$

goes to T_j+t-T_i and the expressions $\frac{\delta^{T_j}}{1-\delta}$ and $\frac{\delta^{T_i-t}}{1-\delta}$ go to ∞. Further, $\frac{1-\delta^{T_i}}{1-\delta} \to T_i$. Hence, there is a $\delta_{j2} : 0 < \delta_{j2} < 1$ such that equation (22) holds for all $\delta > \delta_{j2}$ and for all $(\mathbf{c}|c_j)$. For all such discount factors $\delta > \delta_{j2}$, firm j does not gain from deviating irrespective of its cost.

Finally, suppose that firm $\ell \neq j$ deviates during a phase II punishment, then the other firms play a phase II punishment for firm ℓ, and then produce the output vector $\{q_i^{\ell,\epsilon}\}_{i=1}^n$. As in the case of firm j, firm ℓ cannot gain from deviating if the discount factor is $\delta > \delta_{\ell 2}$.

Choose $\bar{\delta} = \max\{\delta_1, \{\delta_{j2}\}_{j\neq i}\}$, then firm i does not gain from deviating for $\delta > \bar{\delta}$. Thus no firm can gain by deviating from $\sigma^\star$ if $\delta > \bar{\delta}$.

We now show that the strategy combination $\sigma^\star$ is a Perfect Bayesian equilibrium strategy of the infinite horizon game given the beliefs $\mu(\mathbf{c}|h_t, c_i, \sigma^\star)$ generated by the $\sigma^\star$. In period 1, the expected payoff of firm i, given that it knows its marginal cost is c_i is given by

$$\sum_{\mathbf{c}\in\mathcal{C}}\mu(\mathbf{c}|c_i)\pi_i^{\infty}(\sigma^\star, c_i) = \sum_{\mathbf{c}\in\mathcal{C}}\mu(\mathbf{c}|c_i)[\pi_i((q_j^\star)_{j=1}^n) + \frac{\delta}{1-\delta}(\pi_i(\tilde{q}(\mathbf{c}|c_i))].$$

For $t \geq 2$ we have already shown that players cannot gain by deviating from $\sigma^\star$ in any period t, no matter what the beliefs are of the firms about the costs of the other firms. Further, from lemma 4 we know that since $\tilde{q}(\mathbf{c}|c_i)$ is incentive compatible, playing according to the true costs of the firms in period 1 would be optimal in the game from period 2 onwards. Since the firms maximize expected payoffs when they play the strictly separating Bayesian equilibrium, therefore, the strategy $\sigma^\star$ together with the beliefs $\mu(\mathbf{c}|h_t, c_i, \sigma^\star)$ (see (1)) is a Perfect Bayesian equilibrium of the infinite horizon game.

The fact that it is a strictly separating equilibrium follows from observing that $\sigma^\star$ involves playing a single period Bayesian Nash equilibrium in period 1. Since the conditions of lemma 3 are satisfied it follows from that result that the single period Bayesian Nash equilibrium is a strictly separating Bayesian Nash equilibrium. Therefore, $\sigma^\star$ is a strictly separating equilibrium. ∎

5.2 Equilibrium with Communication

In theorem 2 we showed that there is a strictly separating equilibrium in which the firms play a strictly separating Bayesian Nash equilibrium in period 1 to signal their costs. However, the Bayesian Nash equilibrium of the single-period game frequently gives lower profits to the firms, than the optimal incentive compatible outputs, especially in those cases in which collusion is most likely to be profitable for the firms. This, therefore, raises the question as to whether there is an equilibrium in which the firms can collude, without having to play the Bayesian equilibrium of the single-period game in period 1.

Consider the firms playing the infinite horizon game as described in section 3 but with an initial communication phase in which the firms have to report their cost to all the other firms. Thus, before choosing the output, firm i has to send a report $r_i : C \to C$. Given the reported costs, the firms then choose their individual strategies for the infinite horizon game. The next result shows that there is a strictly separating equilibrium in which the firms report their costs truthfully and then produce the optimal incentive compatible quantity vector given the cost reported by the firms.

Theorem 3 *If the inverse demand function $p(.)$ satisfies the conditions that $p'(.) < 0$ and $p''(.) \leq 0$, then for every realized cost profile $\mathbf{c} = (c_1, \cdots, c_n)$, there is a $\underline{\delta} < 1$ such that for all $\delta > \underline{\delta}$, there is a strictly separating Perfect Bayesian equilibrium, in which the firms produce the optimal incentive-compatible collusive quantity vector $\tilde{q}(\mathbf{c})$ from period 1 onwards, and in the initial communication period, the firms truthfully report their realized costs.*

Proof: The claim is that the strategy combination $(\sigma_1^\star, \cdots, \sigma_n^\star)$ described below is the strategy combination of a strictly separating Perfect Bayesian equilibrium.
(i) $\sigma_{i0}^\star(c_i) = r_i^\star(c_i) = c_i$ for all $i = 1, \cdots, n$ and $c_i \in C$.
(ii)$\sigma_{i1}^\star(r_1^\star(c_1), \cdots, r_n^\star(c_n)) = \tilde{q}_i(r_1^\star(c_i), \cdots, r_n^\star(c_n))$.
(iii) If a firm i produces $q_i \neq \sigma_{i2}$ in any period $t \geq 2$, and all the other firms had produced

σ_{j2} in all previous periods, then all firms $j \neq i$ produce the output $\frac{1}{n-1}\bar{q}$ for a length of time T_i. This is a phase I punishment strategy.

(iv) If there are no deviations during a phase I punishment by any of the firms $j \neq i$, then after the length of time T_i, the firms produce the output vector $\{q_j^{i,\epsilon}\}_{j=1}^n$(see lemma 1) such that

$$\pi_j(q_j^{i,\epsilon}(c_1, \cdots, c_n)|c_j) = \pi_j(\tilde{q}(c_1, \cdots, c_n)|c_j) + \frac{\epsilon_i}{n-1}$$

and

$$\pi_i(q_j^{i,\epsilon}(c_1, \cdots, c_n)|c_i) = \pi_i(\tilde{q}(c_1, \cdots, c_n)|c_i) - \epsilon_i.$$

(v) If a firm $j \neq i$ deviates during a phase I punishment, then firms $\ell \neq j$ produce the output $\frac{1}{n-1}\bar{q}$ for a length of time T_j. This is a Phase II punishment.

(vi) After a phase II punishment the firms produce the quantity vector $\{q_i^{j,\epsilon}\}_{i=1}^n$.

(vii) Finally, if a firm ℓ deviates from the quantity vector $\{q_i^{j,\epsilon}\}_{i=1}^n$ then the other firms play the phase I punishment for a length of time T_ℓ and then produce the quantity vector $\{q_i^{\ell,\epsilon}\}_{i=1}^n$.

Using arguments similar to those used to prove theorem 2 it can be shown that from period 1 onwards, the strategy profile $\sigma^\star$ is a Perfect Bayesian equilibrium. Further, since the assignment vector $\tilde{q}(.)$ is incentive compatible, the firms do not gain from misreporting the realized cost. Thus, reporting the true cost is an optimal strategy for the firms. The strategy profile $\sigma^\star$ is, therefore, a Perfect Bayesian equilibrium. ■

6 Stationary Equilibria and Optimal Collusion

We have seen that the infinite horizon game has pooling equilibria as well as strictly separating equilibria with signaling as well as with communication. Here we investigate which among these different types of equilibrium gives the largest expected joint profit to the firms. We show that the expected joint profits from the separating equilibrium with communication is at least as large as the expected discounted profits from the separating equilibrium with signaling as well as from the expected joint profits from the pooling equilibrium. Therefore, the separating equilibrium with communication in which the firms produce the optimal incentive compatible output vector from period one onwards is optimal among the stationary equilibria, that is, among those equilibria in which the firms produce the same quantity vector every period[8].

Theorem 4 [9] *Assume that the realizations of the cost of the firms are independent of each other and that the inverse demand function satisfies the condition that* $p' < 0$ *and* $p'' < 0$*. Then the expected discounted sum of joint profits in a separating equilibrium with communication, in which the firms produce the optimal incentive compatible quantity vector, gives the maximum expected discounted joint profits among all pooling and separating equilibria.*

Proof: We first show that the expected discounted sum of the joint profits when the firms produce the optimal incentive compatible quantity vector in a separating equilibrium with communication is at least as large as that from the optimal pooling equilibrium. Let $\hat{q}$ be the quantity vector that maximizes the expected joint profit from a pooling equilibrium. Since $\hat{q}$ is the quantity vector from a pooling equilibrium it follows that $\hat{q}(\mathbf{c}) = \hat{q}(\mathbf{c}')$ for all cost profiles $\mathbf{c}, \mathbf{c}' \in \mathcal{C}$. Therefore, $\hat{q}$ trivially satisfies the incentive compatibility constraints given by (16). In addition, it also satisfies the additional constraints that for all $i = 1, \cdots, n$

$$\hat{q}_i(\mathbf{c}) = \hat{q}_i(\mathbf{c}')$$

for all $\mathbf{c}' \neq \mathbf{c}$, $\mathbf{c}', \mathbf{c} \in \mathcal{C}$. Therefore,

$$\sum_{\mathbf{c}\in\mathcal{C}} \mu(\mathbf{c})\pi(\tilde{q}(\mathbf{c})|\mathbf{c}) \geq \sum_{\mathbf{c}\in\mathcal{C}} \mu(\mathbf{c})\pi(\hat{q}|\mathbf{c}). \tag{23}$$

From this it quickly follows that the present value of the expected joint profits from the separating equilibrium with communication, given by $\sum_{\mathbf{c}\in\mathcal{C}} \mu(\mathbf{c})\frac{\pi(\tilde{q}(\mathbf{c})|\mathbf{c})}{1-\delta}$ is greater than or equal to the expected joint profits from the optimal equilibrium, given by $\sum_{\mathbf{c}\in\mathcal{C}} \mu(\mathbf{c})\frac{\pi(\hat{q}|\mathbf{c})}{1-\delta}$.

Next we show that the expected joint profits from the separating equilibrium with communication, in which the firms produce the optimal incentive compatible output vector from period one onwards, is at least as large as the expected joint profits from the strictly separating equilibrium with signaling. We will prove this by showing that the

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[8] Since the firms produce the same quantity vector in every period, these are also the set of equilibria with rigid prices.

[9] The result is also true when the inverse demand curve is linear and exactly the same proof goes through. The condition that the realizations of the cost are independent is only used to show that the expected joint profit from optimal incentive compatible quantities is at least as large as the expected joint profit from the Bayesian Nash equilibrium outputs.

expected joint profit from the Bayesian Nash equilibrium cannot exceed the expected joint profit from the quantity vector that maximizes the expected joint profit.

We first note that since $p''(.) < 0$, the inverse demand function $p(.)$ is jointly concave in the quantity vector q. This is verified by noting that $\frac{\partial^2 p}{\partial q_i^2} = \frac{\partial^2 p}{\partial q_i q_j} = p''$ for all $i \neq j$ so that the Hessian is given by

$$\begin{bmatrix} p'' \cdots p'' \\ p'' \cdots p'' \\ \cdots \\ p'' \cdots p'' \end{bmatrix}.$$

Since $p'' < 0$, the first principal minor of this Hessian is negative. Since it is also true that the higher order principal minors are all zero, it follows that $p(.)$ is concave.

Using the fact that the inverse demand function $p(.)$ is concave, the expected profit of firm i from the Bayesian Nash equilibrium, when the realized cost is c_i is

$$\begin{aligned} E\pi_i^\star(c_i) &= \sum_{\mathbf{c}\in\mathcal{C}} \mu(\mathbf{c}|c_i)[p(\sum_{j\neq i} q_j^\star(c_j), q_i^\star(c_i)) - c_i]q_i^\star(c_i) \\ &\leq [p(\sum_{\mathbf{c}\in\mathcal{C}} \mu(\mathbf{c}|c_i)\sum_{j\neq i} q_j^\star(c_j), q_i^\star(c_i)) - c_i]q_i^\star(c_i) \\ &= [p(\sum_{j\neq i}\sum_{c_j\in C} \mu(c_j|c_i)q_j^\star(c_j), q_i^\star(c_i)) - c_i]q_i^\star(c_i). \end{aligned} \tag{24}$$

Therefore the expected profit of firm i from the Bayesian Nash equilibrium

$$\begin{aligned} \sum_{c_i\in C} \mu(c_i)E\pi_i^\star(c_i) &\leq \sum_{c_i\in C} \mu(c_i)[p(\sum_{j\neq i}\sum_{c_j\in C} \mu(c_j|c_i)q_j^\star(c_j), q_i^\star(c_i)) - c_i]q_i^\star(c_i) \\ &\leq [p(\sum_{j\neq i}\sum_{c_j\in C} \mu(c_j|c_i)q_j^\star(c_j), \sum_{c_i\in C}\mu(c_i)q_i^\star(c_i)) - \bar{c}_i]\sum_{c_i\in C}\mu(c_i)q_i^\star(c_i) \end{aligned} \tag{25}$$

where $\bar{c}_i = \sum_{c_i\in C}\mu(c_i)c_i$. Since the realizations of the cost of the firms are independent, $\mu(c_j|c_i) = \mu(c_j)$ for all $j \neq i$. Let

$$\hat{\hat{q}}_i = \sum_{c_i\in C}\mu(c_i)q_i^\star(c_i),$$

and $\hat{\hat{q}} = (\hat{\hat{q}}_1, \cdots, \hat{\hat{q}}_n)$, then (25) can be rewritten as

$$\begin{aligned} \sum_{c_i\in C} \mu(c_i)E\pi_i^\star(c_i) &\leq [p(\sum_{j\neq i}\sum_{c_j\in C} \mu(c_j|c_i)q_j^\star(c_j), \sum_{c_i\in C}\mu(c_i)q_i^\star(c_i)) - \bar{c}_i]\sum_{c_i\in C}\mu(c_i)q_i^\star(c_i) \\ &= [p(\hat{\hat{q}}) - \bar{c}_i]\hat{\hat{q}}_i. \end{aligned} \tag{26}$$

From (26) we have that the expected joint profit from the Bayesian Nash equilibrium

$$E\pi^\star \leq \sum_{i=1}^{n}[p(\hat{\hat{q}}) - \bar{c}_i]\hat{\hat{q}}_i \leq E\pi(\hat{q}) \tag{27}$$

where $\hat{q}$ is the quantity vector that maximizes the expected joint profit of the firms.

From (23) and (27) it now follows that the present value of the expected joint profits from the separating equilibrium with communication given by

$$\sum_{\mathbf{c}\in\mathcal{C}} \mu(\mathbf{c})\frac{\pi(\tilde{q}(\mathbf{c})|\mathbf{c})}{1-\delta}$$

is at least as large as the expected joint profits from the separating equilibrium with signaling given by

$$\sum_{\mathbf{c}\in\mathcal{C}} \mu(\mathbf{c})\pi^\star + \sum_{\mathbf{c}\in\mathcal{C}} \mu(\mathbf{c})\frac{\delta\pi(\tilde{q}|\mathbf{c})}{1-\delta}.$$

This completes the proof. ■

The following example illustrates some of these results.

Example 2 *The discounted sum of profits from the* **optimal separating equilibrium** *and the* **optimal pooling equilibria**.

Consider the oligopoly game of example 1 in which

$$p(Q) = 10 - Q$$

and the marginal cost of firm 1 is either $c_1 = 1$ or $c_1 = 2$ with

$$Prob(c_1 = 1) = Prob(c_1 = 2) = 0.5$$

and similarly, the marginal cost of firm 2 is either $c_2 = 1$ or $c_2 = 2$ with

$$Prob(c_1 = 1) = Prob(c_1 = 2) = 0.5.$$

In this case the Bayesian Nash equilibrium outputs of the firms are

$$q_1^\star(c_1 = 1) = 3.0833,\ q_1^\star(c_1 = 2) = 2.5833,\ q_2^\star(c_2 = 1) = 3.0833,\ \text{and}\ q_2^\star(c_2 = 2) = 2.5833.$$

As a result the profit of the firms when they play the Bayesian Nash equilibrium are

$$\pi_1^\star(c_1 = 1, c_2 = 1) = \pi_2^\star(c_1 = 1, c_2 = 1) = 8.7362,$$

$$\pi_1^\star(c_1 = 1, c_2 = 2) = 10.2779, \quad \pi_2^\star(c_1 = 1, c_2 = 2) = 6.0279,$$

$$\pi_1^\star(c_1 = 2, c_2 = 1) = 6.0279, \quad \pi_2^\star(c_1 = 2, c_2 = 1) = 10.2779,$$

and

$$\pi_1^\star(c_1 = 2, c_2 = 2) = \pi_2^\star(c_1 = 2, c_2 = 2) = 7.3195.$$

The expected joint profit of the firms from the Bayesian Nash equilibrium quantities is therefore

$$E\pi^\star = 0.25 \times 17.4724 + 0.5 \times 16.3058 + 0.25 \times 14.639 = 16.1808. \tag{28}$$

The optimal incentive compatible collusive output of the firms are

$$\hat{q}_1(c_1 = 1, c_2 = 1) = 2.25, \ \hat{q}_2(c_1 = 1, c_2 = 1) = 2.25,$$

$$\hat{q}_1(c_1 = 2, c_2 = 2) = 2, \ \hat{q}_2(c_1 = 2, c_2 = 2) = 2,$$

$$\hat{q}_1(c_1 = 1, c_2 = 2) = 2.1869, \ \hat{q}_2(c_2 = 2) = 2.155,$$

and

$$\hat{q}_1(c_1 = 2, c_2 = 1) = 2.155, \ \hat{q}_2(c_1 = 2, c_2 = 1) = 2.1869.$$

The optimal incentive compatible profits are

$$\hat{\pi}_1(c_1 = 1, c_2 = 1) = \hat{\pi}_2(c_1 = 1, c_2 = 1) = 10.125,$$

$$\hat{\pi}_1(c_1 = 1, c_2 = 2) = 10.1867, \quad \hat{\pi}_2(c_1 = 1, c_2 = 2) = 7.8809,$$

$$\hat{\pi}_1(c_1 = 2, c_2 = 1) = 7.8809, \quad \hat{\pi}_2(c_1 = 2, c_2 = 1) = 10.1867,$$

and

$$\hat{\pi}_1(c_1 = 2, c_2 = 2) = \hat{\pi}_2(c_1 = 2, c_2 = 2) = 8.$$

The *expected joint profit from the* **optimal incentive compatible quantity vector** is

$$E\pi(\tilde{q}) = 0.25 \times 20.25 + 0.5 \times 18.07 + 0.25 \times 16 = 18.10. \tag{29}$$

The output that maximizes the expected joint profit when the firms produce a pooled quantity vector is given by

$$p'(Q)Q + p(Q) = 0.5c_L + 0.5c_H = 1.5.$$

This gives $\hat{Q} = 4.25$ so that $\hat{q}_1 = \hat{q}_2 = 2.125$. The *expected joint profit of the firms in the case of the* **pooled quantity vector** is, therefore,

$$\begin{aligned} E\pi(\hat{q}) &= 0.25 \times (5.75 - 1) \times 4.25 + 0.5 \times [(5.75 - 1) \times 2.125 + (5.75 - 2) \times 2.125] \\ &\quad +0.25 \times (5.75 - 2) \times 4.25 = 18.0625. \qquad (30) \end{aligned}$$

From (28), (29) and (30) it follows that the expected discounted joint profits from the separating equilibrium with communication given by

$$\frac{18.10}{1-\delta}$$

is strictly greater than the expected discounted joint profits from the optimal separating equilibrium with signaling, which is

$$16.1808 + \frac{\delta}{1-\delta} 18.10,$$

as well as from the expected discounted joint profits from the optimal pooling equilibrium, given by

$$\frac{18.0625}{1-\delta}.$$

An interesting pooling equilibrium is one in which both the firms decide to play the pooling equilibrium assuming that each is low-cost firm.[10] Then if both are indeed low-cost firms, the profit of the firms are 10.125 each, whereas if a firm is a high-cost firm, its profit is 7.875. The expected joint profit of the firm is, therefore,

$$0.25 \times 20.25 + 0.5 \times (10.125 + 7.875) + 0.25 \times 15.75 = 18.0. \qquad (31)$$

The expected joint profit is clearly not as high as from the optimal pooling equilibrium and, therefore, not as large as from the optimal separating equilibrium with communication. ■

The separating equilibrium with communication is clearly optimal for the firms in an ex ante sense. What is, however, also of interest is that in the optimal separating equilibrium with communication, the firms would produce the optimal collusive output

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[10] This might in many cases be quite appealing to the firms because if a firm is indeed low-cost it would get the same profit as it would get if the other firm was low-cost and they produced the optimal collusive output. A high-cost firm may also be happy with such an arrangement as it avoids the possibility of having to share the market with a low-cost firm.

of the complete information case if the realized costs are identical. In that case they would also be ex post optimal. It is also worth mentioning that the optimal separating equilibrium with communication is not only optimal ex ante but also optimal in the interim, that is, after the firms receive their private information. It is, however, not optimal ex post for every possible realizations of the costs. The present example is one in which there is a great deal of symmetry between the firms as each firm can be a high-cost or a low-cost firm with equal probability. We know from the complete information case, collusion is most likely in these symmetric cases, and this seems to be true here too.

In case the situation is not symmetric, theorem 4 of course will still hold, but optimal collusion of the sort that is possible in the symmetric cases may not work out as smoothly. Consider the case in example 2, but with the probability distribution given by $\mu(L, L) = 0.05$, $\mu(L, H) = 0.9$, $\mu(H, H) = 0.05$, and $\mu(H, L) = 0$, so that firm 1 is very likely to be low-cost while firm 2 is almost certainly high-cost. In this case, the quantity vector that maximizes the expected joint profit is $\hat{q}_1 = 4.5$ and $\hat{q}_2 = 0$. The optimal incentive compatible quantity vectors are

$$\tilde{q}_1(L, H) = 4.5,\ \tilde{q}_2(L, H) = 0,$$

$$\tilde{q}_1(H, H) = 3.94,\ \tilde{q}_2(H, H) = 0.06,$$

and

$$\tilde{q}_1(L, L) = 4.365,\ \tilde{q}_2(L, L) = 0.135.$$

Clearly, the expected joint profit from the separating equilibrium with communication is higher than the expected joint profit from the pooling equilibrium. However, firm 2 gets very little of the joint profit in either case[11] and the firms could revert to playing a non-collusive equilibrium in which the firms play the single period Bayesian Nash equilbrium in every period (see for example Lemma 4, Chakrabarti [2005]).

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[11] This is also what happens in the complete information case in which one firm is a low-cost firm and the other is a high-cost firm. Optimal collusion would mean that the high-cost firm produces zero output in which case firm 2 would never agree to collude, and the firms could end up playing the Nash equilibrium every period.

7 Conclusion

The results reported here draw quite heavily on the fact that the stage game is an oligopoly game. Proposition 1, for instance, is quite crucial as it shows that the firms can use a common minimaxing strategy to minimax deviating firms that is independent of the cost of the firm. This is, of course, not the case for games in general. Depending on the nature of the private information, it may not be possible to find a common minimaxing strategy.

Communication of some sort seems to be useful in many cases when there is private information. This is clearly the case in Compte [1998] and Kandori and Matsushima [1998]. Even when there is complete information it is generally understood that firms have to agree on the collusive output vector and some bargaining may take place prior to deciding on the collusive output vector. The reporting of a firm's cost can then be part of the same communication process. In the case of the separating equilibrium of theorem 3, a preplay communication may actually have the advantage of cutting back on the signaling cost of producing the strict Bayesian Nash equilibrium output in period 1. If that is so then the separating equilibrium of theorem 3 becomes very attractive for colluding firms.

The other issue that one may raise here is whether the choice of a constant marginal cost is crucial for the results reported here. If one had U-shaped marginal costs, a version of proposition 1 would still be valid, as the price can again be driven down to low levels so that a deviator makes zero profits. Also, the non-deviators can be rewarded with a slightly larger market share in the future and the deviator can be punished by a slightly lower market share in the future after the minimaxing stage is over. Thus, it seems that the assumption of constant marginal cost as opposed to U-shaped marginal costs is not that critical.

An assumption that we have retained throughout the paper is that there are a finite number of possible costs of the firms. Thus one might ask as to what would happen to the results presented here if the marginal costs of the firms were drawn from a continuum of possible values. It seems that this would have no major impact on the results section 4. However, this would affect the existence result on the Bayesian Nash equilibrium and the incentive constraints would be hard to deal with as there would be infinitely many of them. It is thus unclear as to what would happen to the results of section 5. Clearly, the incentive compatibility conditions would have to be modified. In this case it may be that the result would hold only with a certain “noise,” and the exact nature of the result may have to be significantly rethought. This is thus left for future work.

The equilibria that we have discussed here are all stationary equilibria, namely equilibria in which the firms play the same strategy in each period if there is no deviation. Even the signaling equilibrium is mostly stationary, as after period 1, the firms produce the same optimal incentive compatible output in each period. There are non-stationary equilibrium in these games, as shown in Chakrabarti [2005]. Would a non-stationary equilibrium lead to higher expected joint profits for the firms? The non-stationary equilibrium in Chakrabarti [2005] require the firms to play the Bayesian Nash equilibrium of the single-period game for several periods, until there is no incentive for a firm to play differently than dictated by its true cost. But this then leads to lower joint profits in the first periods, and unless the firms are very patient, the discounted sum of profits will be smaller. The other approach would be to give a firm the incentive to reveal its cost early; for instance by giving it a large market share, and then play according to the information inferred. But the incentives have to be right, and if one gives away too much market share in the beginning, it would be hard to recoup the losses later, especially with discounting. The optimal incentive compatible quantity vectors may thus provide the right balance of incentives.

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