Frustration and Anger in Games*

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Abstract

Frustration, anger, and aggression have important consequences for economic and social behavior, concerning for example monopoly pricing, contracting, bargaining, traffic safety, violence, and politics. Drawing on insights from psychology, we develop a formal approach to exploring how frustration and anger, via blame and aggression, shape interaction and outcomes in economic settings.

KEYWORDS: frustration, anger, blame, belief-dependent preferences, psychological games

JEL codes: C72, D03

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1 Introduction

Anger can shape economic outcomes in important ways. Consider three cases:

> **Case 1:** Petroleum costs sky-rocketed in 2006. Did gas stations hold prices steady to avoid accusations of price gouging? Where prices rose, did this cause road rage?
>
> **Case 2:** When local football teams that are favored to win instead lose, the police get more reports of husbands assaulting wives (Card & Dahl 2011). Do unexpected losses spur thus vented frustration?
>
> **Case 3:** Following Sovereign Debt Crises (2009-), some EU countries embarked on austerity programs. Was it because citizens lost benefits that some cities experienced riots?

Traffic safety, pricing, domestic violence, political landscapes: the cases illustrate situations where anger has important consequences. However, to carefully assess how anger may shape social and economic interactions, one needs a theory that predicts outcomes based on the decision-making of anger-prone individuals and that also accounts for the strategic consideration of such individuals' behavior by their co-players. We develop such a theory.

Insights from psychology about both the triggers of anger and its consequences for behavior are evocative. The behavioral consequences of emotions are called "action tendencies," and the action tendency associated with anger is aggression. Angry players may be willing to forego material gains to punish others, or be predisposed to behave aggressively when this serves as a credible threat, and so on. But while insights of this nature can be gleaned from psychologists' writings, their analysis usually stops with the individual rather than going on to assess overall economic and social implications. We take the basic insights about anger that psychology has produced as input and inspiration for the theory we develop.[1]

[1] The relevant literature is huge. A source of insights and inspiration for us, is the *International Handbook of Anger* (Potegal, Spielberger & Stemmler 2010), which offers a cross-disciplinary perspective reflecting "affective neuroscience, business administration, epidemiology, health science, linguistics, political science, psychology, psychophysiology, and sociology" (p. 3). The non-occurrence of "economics" in the list may indicate that our approach is original!

Anger is typically anchored in frustration, which occurs when someone is unexpectedly denied something he or she cares about.[2] We assume (admittedly restrictively; cf. Section 7) that people are frustrated when they get less material rewards than they expected beforehand. Moreover, they then become hostile towards whomever they blame. There are several ways that blame may be assigned (cf. Alicke 2000) and we present three distinct approaches, captured by distinct utility functions. While players motivated by simple anger (SA) become generally hostile when frustrated, those motivated by anger from blaming behavior (ABB) or by anger from blaming intentions (ABI) go after others more discriminately, asking who caused, or who intended to cause, their dismay.

To provide general predictions, we develop a notion of polymorphic sequential equilibrium (PSE). Players correctly anticipate how others behave on average, yet different types of the same player may have different plans in equilibrium. This yields meaningful updating of players' views of others' intentions as various subgames are reached, which is crucial for a sensible treatment of how players consider intentionality as they blame others. We apply this solution concept to the aforementioned utility functions, explore properties, and compare predictions.

A player's frustration depends on his beliefs about others' choices. The blame a player attributes to another may depend on his beliefs about others' choices or beliefs. For these reasons, all our models find their intellectual home in the framework of psychological game theory; see Geanakoplos, Pearce & Stacchetti (1989), Battigalli & Dufwenberg (2009).

Several recent studies inspire us. Most are empirical, indicative of hostile action occurring in economic situations, based on either observational or experimental data.[3] A few studies present theory, mostly with the purpose of explaining specific data patterns (Rotemberg 2005, 2008, 2011; Akerlof 2013; Passarelli & Tabellini 2013). Our approach differs in that we do not start with data, but with notions from psychology which we incorporate into general games, and we are led to use assumptions which differ substantially (Section 7 elaborates, in regards to Rotemberg's work). Brams (2011), in his book *Game Theory and the Humanities*, building on his earlier (1994) "theory of moves," includes negative emotions like anger in the analysis of sequential

[2]Psychologists often refer to this as "goal-blockage;" cf. p.3 of the (*op.cit.*) *Handbook*.

[3]See Anderson & Simester (2010) and Rotemberg (2005, 2011) on pricing, Card & Dahl on domestic violence, and Carpenter & Matthews (2012), Gurdal, Miller & Rustichini (2014), and Gneezy & Imas (2014) for experiments.

interaction. Specifically, he considers players who alternate in changing the state of a two-by-two payoff matrix, with payoffs materializing only at the terminal state. His analysis differs from ours because it is restricted to this class of games and he does not assume that anger is belief-dependent.

We develop most of our analysis for a two-period setting described in Section 2. Section 3 defines frustration. Section 4 develops our notions of psychological utility. Section 5 introduces and explores equilibrium behavior. Section 6 generalizes the analysis to multistage games. Section 7 concludes. Proofs of results are collected in an online appendix.

2 Setup

We begin by describing the rules of interaction (the game form), and then we define beliefs.

2.1 Game form

We consider a finite two-stage game form describing the rules of interaction and the consequences of players' actions. The set of players is I. To ease notation, we assume that all players take actions simultaneously at each stage. Thus, nodes are histories of action profiles $a^t = (a_i^t)_{i \in I}$; $h = \varnothing$ is the empty history (the root of the game), $h = (a^1)$ is a history of length one, which may be terminal or not, and $h = (a^1, a^2)$ is a history of length 2, which is terminal. H is the set of non-terminal histories and Z is the set of terminal histories. The set of feasible actions of i given $h \in H$ is $A_i(h)$. This set is a singleton if i is not active given h. Thus, for $h \in H$, $I(h) = \{i \in I : |A_i(h)| > 1\}$ is the set of active players given h. In a perfect information game $I(h)$ is a singleton for each $h \in H$. We omit parentheses whenever no confusion may arise. For example, we may write $h = a^1$ instead of $h = (a^1)$, and $h = (a_i^1, a_j^2)$ if i (resp. j) is the only first (resp. second) mover. Finally, we let $A(h) = \times_{i \in I} A_i(h)$ and $A_{-i}(h) = \times_{j \neq i} A_j(h)$.

We assume that the material consequences of players' actions are determined by a profile of monetary payoff functions $(\pi_i : Z \to \mathbb{R})_{i \in I}$. This completes the description of the game form, if there are no chance moves. If the game contains chance moves, we augment the player set with a dummy player c (with $c \notin I$), who selects a feasible action at random. Thus, we consider an augmented player set $I_c = I \cup \{c\}$, and the sets of first and second movers may include c: $I(\varnothing), I(a^1) \subseteq I_c$. If the chance player is active at $h \in H$, its move is described by a probability density function $\sigma_c(\cdot|h) \in \Delta(A_c(h))$.

The following example, to which we will return in our discussion of blame, is here employed to illustrate our notation:

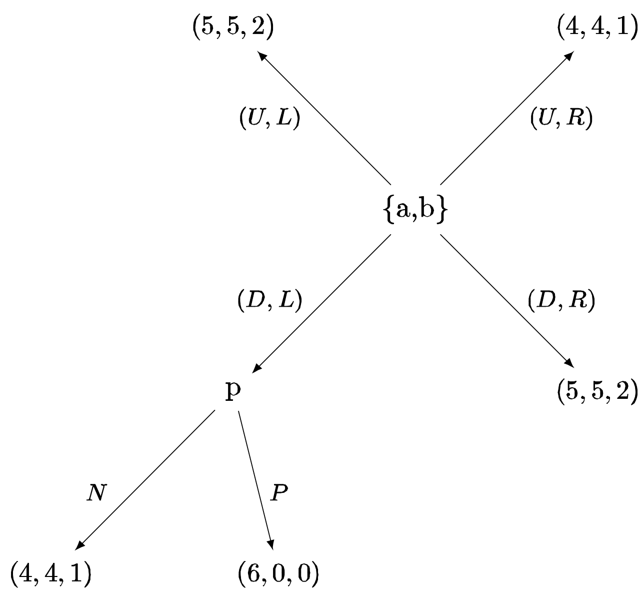

Figure A. Asymmetric punishment.

Example 1 Ann and Bob (a and b in Figure A) move simultaneously in the first stage. Penny the punisher (p in Figure A) may move in the second stage; by choosing P she then decreases π_b (while π_a increases). See Figure A. Profiles of actions and monetary payoffs are listed according to players' alphabetical order. We have:

$$H = \{\varnothing, (D, L)\}, \; Z = \{(U, L), (U, R), (D, R), ((D, L), N), ((D, L), P)\},$$
$$I(\varnothing) = \{a, b\}, \; I((D, L)) = \{p\},$$
$$A_a(\varnothing) = \{U, D\}, \; A_b(\varnothing) = \{L, R\}, \; A_p((D, L)) = \{N, P\}. \quad \blacktriangle$$

2.2 Beliefs

It is conceptually useful to distinguish three aspects of a player's beliefs: beliefs about co-players' actions, beliefs about co-players' beliefs, and the player's plan which we represent as beliefs about own actions. Beliefs are defined conditional on each history. Abstractly denote by Δ_{-i} the space of co-players' beliefs (the formal definition is given below). Player i's beliefs can be compactly described as conditional probability measures over paths and beliefs of others, *i.e.*, over $Z \times \Delta_{-i}$. Events, from i's point of view, are subsets of $Z \times \Delta_{-i}$. Events about behavior take form $Y \times \Delta_{-i}$, with $Y \subseteq Z$; events about beliefs take form $Z \times E_{\Delta_{-i}}$, with $E_{\Delta_{-i}} \subseteq \Delta_{-i}$.[4]

Personal histories To model how i determines the subjective value of feasible actions, we add to the commonly observed histories $h \in H$ also personal histories of the form (h, a_i), with $a_i \in A_i(h)$. In a game with perfect information, $(h, a_i) \in H \cup Z$. But if there are simultaneous moves at h, then (h, a_i) is not a history in the standard sense. As soon as i irreversibly chooses action a_i, he observes (h, a_i), and can determine the value of a_i using his beliefs conditional on this event (i knows in advance how he is going to update his beliefs conditional on what he observes). We denote by H_i the set of histories of i —standard and personal— and by $Z(h_i)$ the set of terminal successors of h_i.[5] The standard precedence relation $\prec$ for histories in $H \cup Z$ is extended to H_i in the obvious way: for all $h \in H$, $i \in I(h)$, and $a_i \in A_i(h)$, it holds that $h \prec (h, a_i)$ and $(h, a_i) \prec (h, (a_i, a_{-i}))$ if i is not the only active player at h. Note that $h \prec h'$ implies $Z(h') \subseteq Z(h)$, with strict inclusion if at least one player (possibly chance) is active at h.

First-order beliefs For each $h_i \in H_i$, player i holds beliefs $\alpha_i(\cdot|Z(h_i)) \in \Delta(Z(h_i))$ about the actions that will be taken in the continuation of the game. The system of beliefs $\alpha_i = (\alpha_i(\cdot|Z(h_i)))_{h_i \in H_i}$ must satisfy two properties. First, the rules of conditional probabilities hold whenever possible: if $h_i \prec h_i'$ then for every $Y \subseteq Z(h_i')$

$$\alpha_i(Z(h_i')|Z(h_i)) > 0 \Rightarrow \alpha_i(Y|Z(h_i')) = \frac{\alpha_i(Y|Z(h_i))}{\alpha_i(Z(h_i')|Z(h_i))}. \tag{1}$$

[4] Δ_{-i} turns out to be a compact metric space. Events are Borel measurable subsets of $Z \times \Delta_{-i}$. We do not specify terminal beliefs of i about others' beliefs, as they are not relevant for the models in this paper.

[5] That is, $H_i = H \cup \{(h, a_i) : h \in H, i \in I(h), a_i \in A_i(h)\}$. The definition of $Z(h_i)$ is standard for $h_i \in H$; for $h_i = (h, a_i)$ we have $Z(h, a_i) = \bigcup_{a_{-i} \in A_{-i}(h)} Z(h, (a_i, a_{-i}))$.

We use obvious abbreviations to denote conditioning events and the conditional probabilities of actions: for all $h \in H$, $a = (a_i, a_{-i}) \in A_i(h) \times A_{-i}(h)$,

$$\begin{aligned}
\alpha_i(a|h) &= \alpha_i\left(Z(h,a)|Z(h)\right), \\
\alpha_{i,i}(a_i|h) &= \sum_{a'_{-i} \in A_{-i}(h)} \alpha_i(a_i, a'_{-i}|h), \\
\alpha_{i,-i}(a_{-i}|h) &= \sum_{a'_i \in A_i(h)} \alpha_i(a'_i, a_{-i}|h).
\end{aligned}$$

Note that $\alpha_i(a_i|h) = \alpha_i\left(Z(h,a_i)|Z(h)\right)$, and that (1) implies $\alpha_i\left(a^1, a^2|\varnothing\right) = \alpha_i\left(a^2|a^1\right)\alpha_i(a^1|\varnothing)$.

With this, we can write in a simple way our second requirement, that i's beliefs about the actions simultaneously taken by the co-players are independent of i's action: for all $h \in H$, $i \in I$, $a_i \in A_i(h)$, and $a_{-i} \in A_{-i}(h)$,

$$\alpha_{i,-i}(a_{-i}|h) = \alpha_{i,-i}(a_{-i}|h, a_i). \tag{2}$$

Properties (1)-(2) imply

$$\alpha_i(a_i, a_{-i}|h) = \alpha_{i,i}(a_i|h)\alpha_{i,-i}(a_{-i}|h).$$

Thus, α_i is made of two parts, what i believes about his own behavior and what he believes about the behavior of others. The array of probability measures $\alpha_{i,i} \in \times_{h \in H}\Delta\left(A_i(h)\right)$ is —technically speaking— a behavioral strategy, and we interpret it as the **plan** of i. The reason is that the result of i's contingent planning is precisely a system of conditional beliefs about what action he would take at each history. If there is only one co-player, also $\alpha_{i,-i} \in \times_{h \in H}\Delta\left(A_{-i}(h)\right)$ corresponds to a behavioral strategy. With multiple co-players, $\alpha_{i,-i}$ corresponds instead to a "correlated behavioral strategy." Whatever the case, $\alpha_{i,-i}$ gives i's conditional beliefs about others' behavior, and these beliefs may not coincide with the plans of others. We emphasize: a player's plan does not describe actual choices, actions on the path of play are the only actual choices.

A system of conditional probability measures $\alpha_i = \left(\alpha_i(\cdot|Z(h_i))\right)_{h_i \in H_i}$ that satisfies (1)-(2) is a **first-order** belief of i. We let Δ_i^1 denote the space of such first-order beliefs. It can be checked that Δ_i^1 is a compact metric space. Hence, the same holds for $\Delta_{-i}^1 = \times_{j \neq i}\Delta_j^1$, the space of first-order beliefs profiles of the co-players.

Second-order beliefs Players do not only hold beliefs about paths, they also hold beliefs about the beliefs of co-players. In the following analysis, the only co-players' beliefs affecting the values of actions are their first-order beliefs. Therefore, we limit our attention to **second-order** beliefs, *i.e.*, systems of conditional probability measures $\left(\beta_i(\cdot|h_i)\right)_{h_i\in H_i} \in \times_{h_i\in H_i}\Delta\left(Z(h_i)\times\Delta^1_{-i}\right)$ that satisfy properties analogous to (1)-(2).[6] First, if $h_i \prec h'_i$ then

$$\beta_i(h'_i|h_i) > 0 \Rightarrow \beta_i\left(E|h'_i\right) = \frac{\beta_i\left(E|h_i\right)}{\beta_i(h'_i|h_i)} \tag{3}$$

for all $h_i, h'_i \in H_i$ and every event $E \subseteq Z(h'_i)\times\Delta^1_{-i}$. Second, i realizes that his choice cannot influence the first-order beliefs of co-players and their simultaneous choices, so i's beliefs satisfy an independence property:

$$\beta_i\left(Z\left(h,(a_i,a_{-i})\right)\times E_\Delta|(h,a_i)\right) = \beta_i\left(Z\left(h,(a'_i,a_{-i})\right)\times E_\Delta|(h,a'_i)\right), \tag{4}$$

for every $h\in H$, $a_i, a'_i \in A_i(h)$, $a_{-i}\in A_{-i}(h)$, and event $E_\Delta \subseteq \Delta^1_{-i}$ about co-players' first-order beliefs. The space of second-order beliefs of i is denoted Δ^2_{-i}.

It can be checked that starting from $\beta_i \in \Delta^2_i$ and letting $\alpha_i\left(Y|h_i\right) = \beta_i\left(Y\times\Delta^1_{-i}|h_i\right)$ for all $h_i\in H_i$ and $Y\subseteq Z$, we obtain a system α_i satisfying (1)-(2), *i.e.*, an element of Δ^1_i. This α_i is the first-order belief implicit in β_i. Whenever we write in a formula beliefs of different orders for a player, we assume that first-order beliefs are derived from second-order beliefs, otherwise beliefs of different orders would not be mutually consistent. Also, we write initial beliefs omitting the empty history, as in $\beta_i\left(E\right) = \beta_i\left(E|\varnothing\right)$ or $\alpha_i(a) = \alpha_i(a|\varnothing)$, whenever this causes no confusion.

Conditional expectations Let ψ_i be any real-valued measurable function of variables that player i does not know, *e.g.*, the terminal history or the co-players' first-order beliefs. Then i can compute the expected value of ψ_i conditional on any common or personal history $h_i\in H_i$ by means of his belief system β_i. This expected value is denoted $\mathbb{E}[\psi_i|h_i;\beta_i]$. If ψ_i depends only on actions, *i.e.*, on the path z, then $\mathbb{E}[\psi_i|h_i;\beta_i]$ is determined by the first-order belief system α_i derived from β_i, and we can write $\mathbb{E}[\psi_i|h_i;\alpha_i]$. In particular, α_i gives the conditional expected material payoffs:

[6] We use obvious abbreviations, such as writing h for event $Z(h)\times\Delta^1_{-i}$, whenever this causes no confusion.

$$\mathbb{E}[\pi_i|h;\alpha_i] = \sum_{z\in Z(h)} \alpha_i(z|h)\pi_i(z),$$
$$\mathbb{E}[\pi_i|\left(h,a_i\right);\alpha_i] = \sum_{z\in Z(h,a_i)} \alpha_i(z|h,a_i)\pi_i(z)$$

for all $h \in H$, $a_i \in A_i(h)$. $\mathbb{E}[\pi_i|h;\alpha_i]$ is what i expects to get conditional on h given α_i, which also specifies i's plan. $\mathbb{E}[\pi_i|(h,a_i);\alpha_i]$ is i's expected payoff of action a_i. If a_i is what i planned to choose at h, $\alpha_{i,i}(a_i|h) = 1$, and then $\mathbb{E}[\pi_i|h;\alpha_i] = \mathbb{E}[\pi_i|\left(h,a_i\right);\alpha_i]$. For initial beliefs, we omit $h = \varnothing$ from such expressions; in particular, the initially expected payoff is $\mathbb{E}[\pi_i;\alpha_i]$.

3 Frustration

Anger is triggered by frustration. While we focus upon anger as a social phenomenon —frustrated players blame and become angry with and care for the payoffs of *others*— our account of frustration refers to own payoffs only. In Section 7 (in hindsight of definitions to come) we discuss this approach in depth. Here, we define player i's **frustration**, in stage 2, given a^1, as

$$\mathrm{F}_i(a^1;\alpha_i) = \left[\mathbb{E}[\pi_i;\alpha_i] - \max_{a_i^2\in A_i(a^1)} \mathbb{E}[\pi_i|(a^1,a_i^2);\alpha_i]\right]^+,$$

where $[x]^+ = \max\{x,0\}$. In words, frustration is given by the gap, if positive, between i's initially expected payoff and the currently best expected payoff he believes he can obtain. Diminished expectation —$\mathbb{E}[\pi_i|a^1;\alpha_i] < \mathbb{E}[\pi_i;\alpha_i]$— is only a necessary condition for frustration. For i to be frustrated it must also be the case that i cannot close the gap.

$\mathrm{F}_i(a^1;\alpha_i)$ expresses stage-2 frustration. One could define frustration at the root, or at end nodes, but neither would matter for our purposes. At the root nothing has happened, so frustration equals zero. Frustration is possible at the end nodes, but can't influence subsequent choices as the game is over. One might allow the anticipated frustration at end nodes to influence earlier decisions; however, the assumptions we make in the analysis below rule this out. Furthermore, players are influenced by the frustrations of co-players only insofar as their behavior is affected.

Example 2 To illustrate, return to Figure A. Suppose Penny initially expects \$2: $\alpha_p\left((U,L)|\varnothing\right) + \alpha_p\left((D,R)|\varnothing\right) = 1$ and $\mathbb{E}[\pi_p;\alpha_p] = 2$. After $a^1 = (D,L)$ we have

$$\mathrm{F}_p((D,L);\alpha_p) = [\mathbb{E}[\pi_p;\alpha_p] - \max\{\pi_p((D,L),N),\pi_p((D,L),P)\}]^+ = 2-1 = 1.$$

This is independent of her plan, because she is initially certain she will not move. If instead $\alpha_p\left((U,L)|\varnothing\right) = \alpha_p\left((D,L)|\varnothing\right) = \frac{1}{2}$ then

$$\mathrm{F}_p((D,L);\alpha_p) = \frac{1}{2}\cdot 2 + \frac{1}{2}\alpha_p\left(N|(D,L)\right)\cdot 1 - 1 = \frac{1}{2}\alpha_p\left(N|(D,L)\right);$$

Penny's frustration is highest if she initially plans not to punish Bob. ▲

4 Anger

A player's preferences over actions at a given node —his action tendencies— depend on expected material payoffs and frustration. A frustrated player tends to hurt others, if this is not too costly (cf. Dollard *et al.* 1939, Averill 1983, Berkowitz 1989). We consider different versions of this frustration-aggression hypothesis related to different cognitive appraisals of blame. In general, player i moving at history h chooses action a_i to maximize the expected value of a belief-dependent "decision utility" of the form

$$u_i\left(h,a_i;\beta_i\right) = \mathbb{E}\left[\pi_i|\left(h,a_i\right);\alpha_i\right] - \theta_i\sum_{j\neq i}\mathrm{B}_{ij}\left(h;\beta_i\right)\mathbb{E}\left[\pi_j|\left(h,a_i\right);\alpha_i\right], \quad (5)$$

where α_i is the first-order belief system derived from second-order belief β_i, and $\theta_i \geq 0$ is a sensitivity parameter. $\mathrm{B}_{ij}\left(h;\beta_i\right) \geq 0$ measures how much of i's frustration is blamed on co-player j, and the presence of $\mathbb{E}\left[\pi_j|\left(h,a_i\right);\alpha_i\right]$ in the formula translates this into a tendency to hurt j. We assume that $\mathrm{B}_{ij}\left(h;\beta_i\right)$ is positive only if frustration is positive:

$$\mathrm{B}_{ij}(h;\beta_i) \leq \mathrm{F}_i(h;\alpha_i). \quad (6)$$

Therefore, the decision utility of a first-mover coincides with expected material payoff, because there cannot be any frustration in the first stage: $u_i\left(\varnothing,a_i;\beta_i\right) = \mathbb{E}[\pi_i|a_i;\alpha_i]$. When i is the only active player at $h = a^1$, he determines the terminal history with his choice $a_i = a^2$, and decision utility has the form

$$u_i\left(h, a_i; \beta_i\right) = \pi_i\left(h, a_i\right) - \theta_i \sum_{j \neq i} \mathrm{B}_{ij}\left(h; \beta_i\right) \pi_j\left(h, a_i\right).$$

We next consider functional forms that capture different notions of blame.

4.1 Simple Anger (SA)

Our most rudimentary hypothesis, **simple anger (SA)**, is that i's tendency to hurt others is proportional to i's frustration, un-modulated by the cognitive appraisal of blame, so $\mathrm{B}_{ij}\left(h; \beta_i\right) = \mathrm{F}_i(h; \alpha_i)$:

$$u_i^{SA}(h, a_i; \alpha_i) = \mathbb{E}\left[\pi_i \middle| \left(h, a_i\right); \alpha_i\right] - \theta_i \sum_{j \neq i} \mathrm{F}_i(h; \alpha_i) \mathbb{E}\left[\pi_j \middle| \left(h, a_i\right); \alpha_i\right]. \qquad (7)$$

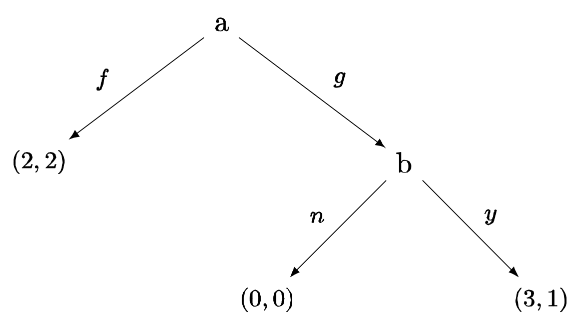

Figure B. Ultimatum Minigame.

Example 3 *(Ultimatum Minigame)* Ann and Bob, a and b in Figure B, negotiate: Ann can make fair offer f, which is automatically accepted, or greedy offer g, which Bob accepts or rejects. His frustration following g is

$$\mathrm{F}_b(g; \alpha_b) = \left[(1 - \alpha_b(g)) \cdot 2 + \alpha_b(g)\alpha_b(y|g) \cdot 1 - 1\right]^+.$$

Therefore

$$u_b^{SA}(g, n; \alpha_b) - u_b^{SA}(g, y; \alpha_b) = 3\theta_b \left[2\left(1 - \alpha_b(g)\right) + \alpha_b(g)\alpha_b(y|g) - 1\right]^+ - 1.$$

For Bob to be frustrated he must not expect g with certainty. If frustrated, the less he expects g, and —interestingly— the less he plans to reject, the more prone he is to reject once offer g materializes. The more resigned Bob is to getting a low payoff, the less frustrated and prone to aggression he is. ▲

4.2 Anger from blaming behavior (ABB)

Action tendencies may depend on a player's cognitive appraisal of how to blame others. When a frustrated player i blames co-players for their behavior, he examines the actions chosen in stage 1, without considering others' intentions. How much i blames j is determined by a continuous function $\mathrm{B}_{ij}(a^1; \alpha_i)$ that depends only on first-order belief α_i such that

$$\mathrm{B}_{ij}(a^1; \alpha_i) = \begin{cases} 0, & \text{if } j \notin I(\varnothing), \\ \mathrm{F}_i(a^1; \alpha_i), & \text{if } \{j\} = I(\varnothing). \end{cases} \tag{8}$$

According to (8), if j is not active in the first stage, he cannot be blamed by i. If instead j is the *only* active player, he is fully blamed.[7] We consider below specific versions of $\mathrm{B}_{ij}(h; \alpha_i)$ that satisfy (6) and (8). With this, i's decision utility with **anger from blaming behavior (ABB)** is

$$u_i^{ABB}(h, a_i; \alpha_i) = \mathbb{E}\left[\pi_i | (h, a_i); \alpha_i\right] - \theta_i \sum_{j \neq i} \mathrm{B}_{ij}(h; \alpha_i) \mathbb{E}\left[\pi_j | (h, a_i); \alpha_i\right].$$

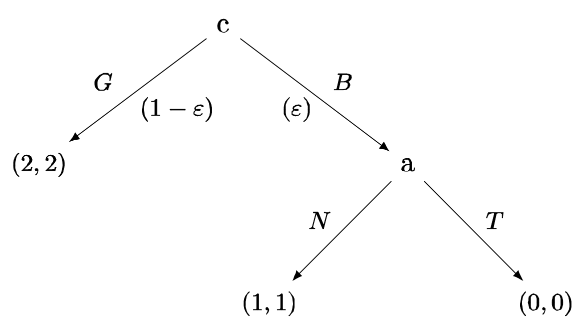

Figure C. Hammering one's thumb.

Example 4 *(Inspired by Frijda, 1993)* To illustrate the difference between SA and ABB, consider Figure C. Andy the handyman (a) uses a hammer. His apprentice, Bob, is inactive. On a bad day (determined by chance) Andy hammers his thumb and can then take it out on Bob or not. If he does, he further disrupts production. Assuming $\alpha_a(B) = \varepsilon < 1/2$, we have

$$\mathrm{F}_a(B; \alpha_a) = (1 - \varepsilon) \cdot 2 + \varepsilon \alpha_a(N|B) \cdot 1 - 1 > 0.$$

With SA and with θ_a sufficiently high, on a bad day Andy chooses T. But, since Bob is passive, with ABB Andy chooses N regardless of θ_a. ▲

[7] Recall that $I(h)$ is the set of active players at h, possibly including chance. For example, $I(\varnothing) = \{c\}$ in the game form of Figure C.

SA and ABB yield the same behavior in the Ultimatum Minigame and similar game forms. Say that a game form is a **leader-followers game** if there is only one active player in the first stage, who does not move in stage two: $I(\varnothing) = \{j\}$ and $I(\varnothing) \cap I(a^1) = \emptyset$ for some $j \in I$ and every a^1. Let us write u_{i,θ_i} to make the dependence of u_i on θ_i explicit; then (8) implies:

Remark 1 *In leader-followers games, SA and ABB coincide, that is,* $u^{SA}_{i,\theta_i} = u^{ABB}_{i,\theta_i}$ *for all* θ_i.

Next, we contrast two specific functional forms for ABB.

Could-have-been blame When frustrated i considers, for each j, what he would have obtained at most, in expectation, had j chosen differently:

$$\max_{a'_j \in A_j(\varnothing)} \mathbb{E}\left[\pi_i | (a^1_{-j}, a'_j); \alpha_i\right].$$

If this could-have-been payoff is more than what i currently expects (that is, $\mathbb{E}[\pi_i|a^1;\alpha_i]$), then i blames j, up to i's frustration (so (6) holds):

$$\mathrm{B}_{ij}(a^1;\alpha_i) = \min\left\{\left[\max_{a'_j \in A_j(\varnothing)} \mathbb{E}\left[\pi_i | (a^1_{-j}, a'_j); \alpha_i\right] - \mathbb{E}[\pi_i|a^1;\alpha_i]\right]^+, \mathrm{F}_i(a^1;\alpha_i)\right\}. \tag{9}$$

Blame function (9) satisfies (8) (cf. Remark 4 below).

Example 5 Consider Penny at $a^1 = (D, L)$ in Figure A. Her could-have-been payoff —with respect to both Ann and Bob— is $2 \geq \mathbb{E}[\pi_p;\alpha_p]$, her updated expected payoff is $\mathbb{E}[\pi_p|(D,L);\alpha_p] \leq 1$, and her frustration is $[\mathbb{E}[\pi_p;\alpha_p] - 1]^+$. Therefore

$$\mathrm{B}_{pa}((D,L);\alpha_p) = \mathrm{B}_{pb}((D,L);\alpha_p) =$$
$$\min\left\{[2 - \mathbb{E}[\pi_p|(D,L);\alpha_p]]^+, [\mathbb{E}[\pi_p;\alpha_p] - 1]^+\right\} = [\mathbb{E}[\pi_p;\alpha_p] - 1]^+,$$

i.e., each of Ann and Bob is fully blamed by Penny for her frustration. ▲

Blaming unexpected deviations When frustrated after a^1, i assesses, for each j, how much he would have obtained had j behaved as expected:

$$\sum_{a'_j \in A_j(\varnothing)} \alpha_{ij}(a'_j)\mathbb{E}\left[\pi_i | (a^1_{-j}, a'_j); \alpha_i\right],$$

where $\alpha_{ij}(a'_j)$ is the marginal probability of action a'_j according to i's belief α_i. With this, the blame formula is

$$\mathrm{B}_{ij}(a^1;\alpha_i) =$$
$$\min\left\{\left[\sum_{a'_j\in A_j(\varnothing)}\alpha_{ij}(a'_j)\mathbb{E}\left[\pi_i|(a^1_{-j},a'_j);\alpha_i\right]-\mathbb{E}[\pi_i|a^1;\alpha_i]\right]^+,\mathrm{F}_i(a^1;\alpha_i)\right\}. \tag{10}$$

If j is not active in the first stage, we get

$$\mathrm{B}_{ij}(a^1;\alpha_i)=\min\left\{\left[\mathbb{E}[\pi_i|a^1;\alpha_i]-\mathbb{E}[\pi_i|a^1;\alpha_i]\right]^+,\mathrm{F}_i(a^1;\alpha_i)\right\}=0;$$

that is, j cannot have deviated and cannot be blamed. If, instead, j is the only active player in the first stage, then

$$\sum_{a'_j\in A_j(\varnothing)}\alpha_{ij}(a'_j)\mathbb{E}\left[\pi_i|(a^1_{-j},a'_j);\alpha_i\right]=\sum_{a'\in A(\varnothing)}\alpha_i(a')\mathbb{E}\left[\pi_i|a';\alpha_i\right]=\mathbb{E}\left[\pi_i;\alpha_i\right],$$

and (10) yields

$$\mathrm{B}_{ij}(a^1;\alpha_i)=\min\left\{\left[\mathbb{E}[\pi_i;\alpha_i]-\mathbb{E}[\pi_i|a^1;\alpha_i]\right]^+,\mathrm{F}_i(a^1;\alpha_i)\right\}=\mathrm{F}_i(a^1;\alpha_i).$$

Therefore, like blame function (9), also (10) satisfies (8).

If a^1_j is what i expected j to do in the first stage ($\alpha_{ij}(a^1_j)=1$) then

$$\mathrm{B}_{ij}(a^1;\alpha_i)=\min\left\{\left[\mathbb{E}[\pi_i|a^1;\alpha_i]-\mathbb{E}[\pi_i|a^1;\alpha_i]\right]^+,\mathrm{F}_i(a^1;\alpha_i)\right\}=0.$$

In other words, j did not deviate from what i expected and j is not blamed by i. This is different from "could-have-been" blame (9).

Example 6 Suppose that, in Figure A, Penny is initially certain of (U,L): $\alpha_p(U,L)=1$ and $\mathbb{E}[\pi_p;\alpha_p]=2$. Upon observing (D,L) her frustration is $\mathrm{F}_p((D,L);\alpha_p)=[\mathbb{E}[\pi_p;\alpha_p]-1]^+=1$. Using Equation (10), at $a^1=(D,L)$, Penny fully blames Ann, who deviated from U to D. Since

$$\sum_{a'_a\in A_a(\varnothing)}\alpha_{pa}(a'_a)\mathbb{E}\left[\pi_p|(a^1_{-a},a'_a);\alpha_p\right]=\pi_p(U,L)=2,$$

we get that Penny's blame of Ann equals Penny's frustration

$$\mathrm{B}_{pa}((D,L);\alpha_p) = \min\left\{\left[2 - \mathbb{E}[\pi_p|a^1;\alpha_p]\right]^+, 1\right\} = 1.$$

On the other hand, Penny does not blame Bob, who played L as expected. To verify this, note that when frustrated after (D,L) Penny assesses how much she would have obtained had Bob behaved as expected:

$$\sum_{a'_b \in A_b(\varnothing)} \alpha_{pb}(a'_b)\mathbb{E}\left[\pi_p|(a^1_{-b}, a'_b);\alpha_p\right] = \mathbb{E}[\pi_p|(D,L);\alpha_p]$$

and

$$\mathrm{B}_{pb}((D,L);\alpha_p) = \min\left\{[\mathbb{E}[\pi_p|(D,L);\alpha_p] - \mathbb{E}[\pi_p|(D,L);\alpha_p]]^+, 1\right\} = 0,$$

in contrast to could-have-been blame (5) under which, as we saw, Penny fully blames Bob (Example 5). ▲

Formulae (9) and (10) each credit the full frustration on the first-mover of a leader-followers game, because each satisfies (8) (see Remark 1).

4.3 Anger from blaming intentions (ABI)

A player i prone to **anger from blaming intentions (ABI)** asks himself, for each co-player j, whether j intended to give him a low expected payoff. Since such intention depends on j's first-order beliefs α_j (which include j's plan, $\alpha_{j,j}$), how much i blames j depends on i's second-order beliefs β_i, and the decision utility function has the form

$$u_i^{ABI}(h, a_i; \beta_i) = \mathbb{E}\left[\pi_i|\,(h, a_i);\alpha_i\right] - \theta_i \sum_{j \neq i} \mathrm{B}_{ij}(h;\beta_i)\,\mathbb{E}\left[\pi_j|\,(h,a_i);\alpha_i\right],$$

where α_i is derived from β_i.

The maximum payoff that j, initially, can expect to give to i is

$$\max_{a^1_j \in A_j(\varnothing)} \sum_{a^1_{-j} \in A_{-j}(\varnothing)} \alpha_{j,-j}(a^1_{-j})\mathbb{E}\left[\pi_i|\left(a^1_j, a^1_{-j}\right);\alpha_j\right].$$

Note that

$$\max_{a_j^1 \in A_j(\varnothing)} \sum_{a_{-j}^1 \in A_{-j}(\varnothing)} \alpha_{j,-j}(a_{-j}^1)\mathbb{E}\left[\pi_i | \left(a_j^1, a_{-j}^1\right); \alpha_j\right]$$
$$\geq \sum_{a^1 \in A(\varnothing)} \alpha_j(a^1)\mathbb{E}\left[\pi_i | a^1; \alpha_j\right] = \mathbb{E}\left[\pi_i | \alpha_j\right],$$

where the inequality holds by definition, and the equality is implied by the chain rule (3). Note also that $\alpha_j(\cdot|a^1)$ is kept fixed under the maximization; we focus on what j initially believes he could achieve, taking the view that at the root he cannot control a_j^2 but predicts how he will choose in stage 2. We assume that i's blame on j at a^1 equals i's expectation, given second-order belief β_i and conditional on a^1, of the difference between the maximum payoff that j can expect to give to i and what j actually plans/expects to give to i, capped by i's frustration:

$$\mathrm{B}_{ij}(a^1; \beta_i) = \tag{11}$$
$$\min\left\{\mathbb{E}\left[\max_{a_j^1} \sum_{a_{-j}^1} \alpha_{j,-j}(a_{-j}^1)\mathbb{E}\left[\pi_i | \left(a_j^1, a_{-j}^1\right); \alpha_j\right] - \mathbb{E}[\pi_i; \alpha_j] \middle| a^1; \beta_i\right], \mathrm{F}_i(a^1; \alpha_i)\right\},$$

where α_i is derived from β_i. The expression is non-negative as per the previously highlighted inequality. Now, i's decision utility after $h = a^1$ is

$$u_i^{ABI}(h, a_i; \beta_i) = \mathbb{E}\left[\pi_i | (h, a_i); \alpha_i\right] - \theta_i \sum_{j \neq i} \mathrm{B}_{ij}(h; \beta_i)\mathbb{E}\left[\pi_j | (h, a_i); \alpha_i\right].$$

Example 7 In the Ultimatum Minigame (Figure B), the maximum payoff Ann can expect give to Bob is 2, independently of α_a. Suppose that Bob, upon observing g, is certain that Ann "randomized" and planned to offer g with probability p, *i.e.*, $\beta_b(\alpha_a(g) = p|g) = 1$, with $p < 1$. Also, Bob is certain after g that Ann expected him to accept that offer with probability q, *i.e.*, $\beta_b(\alpha_a(y|g) = q|g) = 1$. Finally, suppose Bob initially expected to get the *fair* offer ($\alpha_b(f) = 1$), so that his frustration after g is $F_b(a^1; \alpha_b) = 2 - 1 = 1$. Bob's blame of Ann's intentions is

$$\mathrm{B}_{ba}(g; \beta_b) = \min\left\{2 - [2(1-p) + qp], 1\right\} = \min\left\{p(2-q), 1\right\}.$$

If p is low enough, or q high enough, Bob does not blame all his frustration on Ann. He gives her some credit for the initial intention to make the fair offer with probability $1 - p > 0$, and the degree of credit depends on q. ▲

5 Equilibrium analysis

We depart from traditional game-theoretic analysis in that we use belief-dependent decision-utility functions. Our equilibrium analysis is otherwise quite traditional. We interpret an equilibrium as a profile of strategies and beliefs representing a "commonly understood" way to play the game by rational (utility maximizing) agents. This is a choice of focus rather than a full endorsement of traditional equilibrium analysis.[8]

We consider two equilibrium notions. The first is Battigalli & Dufwenberg's (2009) sequential equilibrium (SE) concept,[9] extending Kreps & Wilson's (1982) classic notion to psychological games. In a complete information framework like the one we adopt here for simplicity,[10] SE requires that each player i is certain and never changes his mind about the true beliefs and plans, hence intentions, of his co-players. We find this feature questionable; therefore, we also explore a generalization — "polymorphic sequential equilibrium" (PSE)— that allows for meaningful updating about others' intentions.

The SE concept gives equilibrium conditions for infinite hierarchies of conditional probability systems. In our particular application, utility functions only depend on first- or second-order beliefs, so we define SEs for assessments comprising beliefs up to only the second order. Since, technically, first-order beliefs are features of second-order beliefs (see 2.2), we provide definitions that depend only on second-order beliefs, which gives SEs for games where psychological utility functions depend only of first-order beliefs as a special case. Finally, although we so far restrict our analysis of frustration and anger

[8]As stressed by Battigalli & Dufwenberg (2009), with belief-dependent preferences alternative solution concepts like rationalizability and self-confirming equilibrium are even more plausible than with standard preferences (and in their Section 5 they analyze rationalizability and psychological forward-induction reasoning). Battigalli, Charness & Dufwenberg (2013) apply a notion of incomplete-information rationalizability to show that observed patterns of deceptions in a cheap-talk, sender-receiver game can be explained by guilt aversion. To our knowledge, self-confirming equilibrium has not yet been used in the analysis of psychological games.

[9]We consider the version for preferences with own-plan dependence and "local" psychological utility functions (see Battigalli & Dufwenberg 2009, Section 6).

[10]Recall that complete information means that the rules of the game and players' (psychological) preferences are common knowledge. For an illustration of incomplete-information equilibrium analysis of psychological games see, *e,g.*, Attanasi, Battigalli & Manzoni (2015).

to two-stage game forms, our abstract definitions of equilibrium for games with belief-dependent preferences (and the associated existence theorem) apply to all multistage game forms.

5.1 Sequential equilibrium (SE)

Fix a game form and decision-utility functions $u_i(h,\cdot;\cdot) : A_i(h) \times \Delta_i^2 \to \mathbb{R}$ ($i \in I$, $h \in H$). This gives a **psychological game** in the sense of Battigalli & Dufwenberg (2009). An **assessment** is a profile of behavioral strategies and beliefs $(\sigma_i, \beta_i)_{i\in I} \in \times_{i\in I}\Sigma_i \times \Delta_i^2$ such that $\Sigma_i = \times_{h\in H}\Delta\left(A_i(h)\right)$ and for each $i \in I$, σ_i is the plan $\alpha_{i,i}$ entailed by second-order belief β_i:

$$\sigma_i(a_i|h) = \alpha_{i,i}(a_i|h) = \beta_i\left(Z(h,a_i) \times \Delta_{-i}^1|h\right) \tag{12}$$

for all $i \in I$, $h \in H$, $a_i \in A_i(h)$. Eq. (12) implies that the behavioral strategies contained in an assessment are implicitly determined by players' beliefs about paths; therefore, they could be dispensed with. We follow Battigalli & Dufwenberg (2009) and make behavioral strategies explicit in assessments only to facilitate comparisons with the equilibrium refinements literature.

Definition 1 *An assessment $(\sigma_i, \beta_i)_{i\in I}$ is **consistent** if, for all $i \in I$, $h \in H$, and $a = (a_j)_{j\in I(h)} \in A(h)$,*
(a) $\alpha_i(a|h) = \times_{j\in I(h)}\sigma_j(a_j|h)$,
(b) $marg_{\Delta_{-i}^1}\beta_i(\cdot|h) = \delta_{\alpha_{-i}}$,
where α_j is derived from β_j for each $j \in I$, and $\delta_{\alpha_{-i}}$ is the Dirac probability measure that assigns probability one to the singleton $\{\alpha_{-i}\} \subseteq \Delta_{-i}^1$.

Condition (a) requires that players' beliefs about actions satisfy independence across co-players (on top of own-action independence), and — conditional on each h— each i expects each j to behave in the continuation as specified by j's plan $\sigma_j = \alpha_{j,j}$, even though j has previously deviated from $\sigma_{j,j}$. All players thus have the same first-order beliefs. Condition (b) requires that players' beliefs about co-players' first-order beliefs (hence their plans) are correct and never change, on or off the path. Thus all players, essentially, have the same second-order beliefs (considering that they are introspective and therefore know their own first-order beliefs).

Definition 2 *An assessment* $(\sigma_i, \beta_i)_{i\in I}$ *is a* ***sequential equilibrium (SE)*** *if it is consistent and satisfies the following sequential rationality condition: for all* $h \in H$ *and* $i \in I(h)$, $\mathrm{Supp}\sigma_i(\cdot|h) \subseteq \arg\max_{a_i\in A_i(h)} u_i(h, a_i; \beta_i)$.

It can be checked that this definition of SE is equivalent to the traditional one when players have standard preferences, *i.e.*, when there is a profile of utility functions $(v_i : Z \to \mathbb{R})_{i\in I}$ such that $u_i(h, a_i; \beta_i) = \mathbb{E}[v_i|(h, a_i); \alpha_i]$.[11] A special case is the **material-payoff game**, where $v_i = \pi_i$ for each $i \in I$.

Theorem 1 *If* $u_i(h, a_i; \cdot)$ *is continuous for all* $i \in I$, $h \in H$ *and* $a_i \in A_i(h)$, *then there is at least one SE.*

Battigalli & Dufwenberg (2009) prove a version of this existence result where first-order beliefs are modeled as belief systems over pure strategies profiles rather than paths. But their "trembling-hand" technique can be used here with straightforward adaptations. We omit the details.[12]

What we said so far about equilibrium does not assume specific functional forms. From now on, we focus on u_i^{SA}, u_i^{ABB}, and u_i^{ABI}. Since frustration and blame are continuous in beliefs, decision-utility is also continuous, and we obtain existence in all cases of interest:

Corollary 1 *Every game with SA, ABB, or ABI has at least one SE.*

Remark 2 *Let* $(\sigma_i, \beta_i)_{i\in I}$ *be a SE assessment of a game with SA, ABB, or ABI; if a history* $h \in H$ *has probability one under profile* $(\sigma_i)_{i\in I}$, *then*

$$\mathrm{F}_i(h'; \alpha_i) = 0,\ \mathrm{Supp}\sigma_i(\cdot|h') \subseteq \arg\max_{a_i'\in A_i(h')} \mathbb{E}[\pi_i|h'; \alpha_i]$$

for all $h' \preceq h$ *and* $i \in I$, *where* α_i *is derived from* β_i. *Therefore, a SE strategy profile of a game with SA, ABB, or ABI with randomization (if any) only in the last stage is also a Nash equilibrium of the agent form of the corresponding material-payoff game.*

[11] According to the standard definition of SE, sequential rationality is given by global maximization over (continuation) strategies at each $h \in H$. By the One-Shot-Deviation principle, this is equivalent to "local" maximization over actions at each $h \in H$.

[12] A similar technique is used in the first part of the proof of Proposition 1 in the online appendix.

To illustrate, in the Ultimatum Minigame (Figure B), (f, n) can be a SE under ABB, and is a Nash equilibrium of the agent form with material-payoff utilities. With (counterfactual) anger, n becomes a credible threat. Corollary 1 and Remark 2 also hold for the multistage extension of Section 6.

We say that two assessments are **realization-equivalent** if the corresponding strategy profiles yield the same probability distribution over terminal histories.

Proposition 1 *In every perfect-information (two-stage) game form with no chance moves and a unique SE of the material-payoff game, this unique material-payoff equilibrium is realization-equivalent to a SE of the psychological game with ABI, ABB, or —with only two players— SA.*

Intuitively, if unique, the material-payoff SE of a perfect-information game must be in pure strategies. By Remark 2, players must maximize their material payoff on the equilibrium path even if they are potentially prone to anger. As for off-equilibrium path decision nodes, deviations from the material-payoff SE strategies can only be due to the desire to hurt the first-mover, which can only increase his incentive to stick to the material-payoff SE action.

The assumption of a unique material-payoff SE holds generically in game forms with perfect information. It is quite easy to show by example that without perfect information, or with chance moves, a material-payoff SE need not be a SE with frustration and anger. The same holds for some multistage game forms (cf. Section 6). Disregarding chance moves, randomization, and ties, the common feature of material-payoff equilibria that are not realization-equivalent to equilibria with frustration and anger is the following (see Figure A and Example 9 below): Start with a material-payoff equilibrium and add anger to decision utility; now, at an off-path node after a deviation by Ann, frustrated Penny wants to hurt Bob, which implies rewarding Ann; this makes it impossible to incentivize both Ann not to deviate and Penny to punish Bob after Ann's deviation.

We close this section with three examples, which combine to illustrate how the SE works (including a weakness) and how SA, ABB (both versions), and ABI may alter material incentives and produce different predictions.

Example 8 Consider Figure C. With u_a^{ABB} (either version), or u_a^{ABI}, Andy will not blame Bob so his SE-choice is the material-payoff equilibrium, N.

But with u_a^{SA} Andy may take it out on Bob (*i.e.,* choose T). Recall that $\mathrm{F}_a(B;\alpha_a) = 2(1-\varepsilon) + \varepsilon\alpha_a(N|B) - 1$, so the more likely Andy believes it to be that he will take it out on Bob, the less he expects initially and the less frustrated he is when B happens. Yet, in SE, the more prone to get angry he is (as measured by θ_a) the more likely that he will take it out on Bob: Andy's utility from N and T is

$$u_a^{SA}(B, N; \alpha_a) = 1 - \theta_a[2(1-\varepsilon) + \varepsilon\alpha_a(N|B) - 1] \cdot 1,$$
$$u_a^{SA}(B, T; \alpha_a) = 0 - \theta_a[2(1-\varepsilon) + \varepsilon\alpha_a(N|B) - 1] \cdot 0 = 0.$$

Sequential rationality of SE implies that one possibility is $\alpha_a(N|B) = 1$ and $u_a^{SA}(B, N; \alpha_a) \geq u_a^{SA}(B, T; \alpha_a)$, implying $\theta_a \leq \frac{1}{1-\varepsilon}$. Another possibility is $\alpha_a(N|B) = 0$ and $u_a^{SA}(B, N; \alpha_a) \leq u_a^{SA}(B, T; \alpha_a)$, implying $\theta_a \geq \frac{1}{1-2\varepsilon}$. That is, if Andy is sufficiently susceptible to simple anger, on bad days he will always take out his frustration on Bob. If $\theta_a \in (\frac{1}{1-\varepsilon}, \frac{1}{1-2\varepsilon})$, we can solve for a SE where $u_a^{SA}(B, N; \alpha_a) = u_a^{SA}(B, T; \alpha_a)$ and $\alpha_a(N|B) = \frac{1}{\varepsilon\theta_a} - \frac{1-2\varepsilon}{\varepsilon} \in (0, 1)$. ▲

The final case, where $\theta_a \in (\frac{1}{1-\varepsilon}, \frac{1}{1-2\varepsilon})$, illustrates how we cannot take for granted the existence of a SE in which players use deterministic plans (a point relevant also for u_i^{ABB} or u_i^{ABI} in other games). Here this happens in a game form with a single active player, highlighting that we deal with a psychological game, as this could not be the case in a standard game.

Example 9 Consider Figure A. Can material-payoff equilibrium outcome (U, L) be part of a SE with frustration and anger? The answer is yes under ABI and the blaming-unexpected-deviations version of ABB. To see this note that Ann and Bob act as-if selfish (as they are not frustrated). Hence they would deviate if they could gain material payoff. In the SE, they would expect 5 if not deviating, making Ann the sole deviation candidate (she would get $6 > 5$ were Penny to choose P; for Bob, 5 is the best he could hope for). Ann deviating can be dismissed though, since if (D, L) were reached Penny would not blame Bob (the only co-player she can punish) under either relevant blame function, and so she would choose N (regardless of θ_p). Under SA and the could-have-been version of ABB, however, it may be impossible to sustain a SE with (U, L); at (D, L) Penny would blame each of Ann and Bob (as explained earlier). By choosing P she hurts Bob more than she helps Ann and would do so if

$$u_p^{ABB}((D, L), P; \alpha_p) > u_p^{ABB}((D, L), N; \alpha_p)$$

$$\Longleftrightarrow$$

$$0-6\theta_p \mathrm{B}_{pa}((D,L);\alpha_p) > 1-8\theta_p \mathrm{B}_{pa}((D,L);\alpha_p).$$

The rhs of the last inequality uses $\mathrm{B}_{pb}((D,L);\alpha_p) = \mathrm{B}_{pa}((D,L);\alpha_p)$. Since $\mathrm{B}_{pa}((D,L);\alpha_p) = \mathrm{F}_p((D,L);\alpha_p) = 1 > 0$, Penny would choose P if $-6\theta_p > 1-8\theta_p \Longleftrightarrow \theta_p > 1/2$, so Ann would want to deviate and choose D. ▲

Example 10 Consider Figure B. By Proposition 1, every utility function discussed admits (g,y) as a SE, regardless of anger sensitivity. To check this directly, just note that, if Bob expects g, he cannot be frustrated, so —when asked to play— he maximizes his material payoff. Under SA and ABB (both versions), (f,n) qualifies as another SE if $\theta_b \geq 1/3$; following g, Bob would be frustrated and choose n, so Ann chooses f. Under ABI (f,n) cannot be an SE. To verify, assume it were, so $\alpha_a(f)=1$. Since the SE concept does not allow for players revising beliefs about beliefs, we get $\beta_b(\alpha_a(f)=1|g)=1$ and $\mathrm{B}_{ba}(g;\beta_b)=0$; Bob maintains his belief that Ann planned to choose f, hence she intended to maximize Bob's payoff. Hence, Bob would choose y, contradicting that (f,n) is a SE. Next, note that (g,n) is not a SE under any concept: Given SE beliefs Bob would not be frustrated and so he would choose y. The only way to observe rejected offers with positive probability in a SE is with non-deterministic plans. To find such a SE, note that we need $\alpha_a(g) \in (0,1)$; if $\alpha_a(g)=0$ Bob would not be reached and if $\alpha_a(g)=1$ he would not be frustrated, and hence, he would choose y. Since Ann uses a non-degenerate plan she must be indifferent, so $\alpha_b(y)=2/3$, implying that Bob is indifferent too. In SE, Bob's frustration is $\left[2\left(1-\alpha_a(g)\right)+\frac{2}{3}\alpha_a(g)-1\right]^+ = \left[1-\frac{4}{3}\alpha_a(g)\right]^+$, which equals his blame of Ann under SA and ABB. Hence we get the indifference condition

$$\begin{aligned}
1-\theta_b\left[1-\frac{4}{3}\alpha_a(g)\right]^+\cdot 3 \quad &= \quad 0-\theta_b\left[1-\frac{4}{3}\alpha_a(g)\right]^+\cdot 0 \\
&\Longleftrightarrow \\
\alpha_a(g) \quad &= \quad \frac{3}{4}-\frac{1}{4\theta_b},
\end{aligned}$$

where $\theta_b \geq 1/3$. The more prone to anger Bob is the more likely he is to get the low offer, so Bob's initial expectations, and hence his frustration and blame, is kept low. Under ABI we get another indifference condition:

$$\begin{aligned}
1-\theta_b \mathrm{B}_{ba}(g;\beta_b)\cdot 3 \quad &= \quad 0-\theta_b \mathrm{B}_{ba}(g;\beta_b)\cdot 0 \\
&\Longleftrightarrow
\end{aligned}$$

$$1-\theta_b \min\left\{1-\frac{4}{3}\alpha_a(g), \frac{4}{3}\alpha_a(g)\right\}\cdot 3 = 0.$$

The left term in braces is Bob's frustration, while

$$\frac{4}{3}\alpha_a(g) = 2-\left[2(1-\alpha_a(g))+\frac{2}{3}\alpha_a(g)\right]$$

is the difference between the maximum payoff Ann could plan for Bob and her actually planned one. The first term is lower if $\alpha_a(g) \geq 3/8$; so, if we can solve the equation for such a number, we duplicate the SA/ABB-solution; again, this is doable if $\theta_b > 1/3$. If $\theta_b \geq 2/3$, with ABI, there is second non-degenerate equilibrium plan with $\alpha_a(g) \in (0, \frac{3}{8})$ such that $\alpha_a(g) = 1/4\theta_b$; to see this, solve the ABI indifference condition assuming that $\frac{4}{3}\alpha_a(g) \leq 1-\frac{4}{3}\alpha_a(g)$. This SE exhibits starkly different comparative statics: The more prone to anger that Bob is, the less likely he is to get a low offer and the less he blames Ann following g in light of her intention to choose f with higher probability. ▲

In the last example we explained why, with ABI, (f, n) cannot be an SE. We find the interpretation unappealing. If Bob initially expects Ann to choose f, and she doesn't, so that Bob is frustrated, then he would rate her choice a mistake and not blame her! It may seem more plausible for Bob not to be so gullible, and instead revise his beliefs of Ann's intentions. The SE concept rules that out. Because of this, and because it makes sense regardless, we next define an alternative concept that to a degree overcomes the issue.

5.2 Polymorphic sequential equilibrium (PSE)

Suppose a game is played by agents drawn at random and independently from large populations, one for each player role $i \in I$. Different agents in the same population i have the same belief-dependent preferences,[13] but they may have different plans, hence different beliefs about paths, even if their beliefs agree about the behavior and beliefs of co-players $-i$. In this case, we say that the population is "polymorphic." Once an agent observes some moves of co-players, he makes inferences about their intentions.

[13] Recall that we are not modelling incomplete information.

Let λ_i be a finite support distribution over $\Sigma_i \times \Delta_i^2$, with $\text{Supp}\lambda_i = \{(\sigma_{t_i^1}, \beta_{t_i^1}), (\sigma_{t_i^2}, \beta_{t_i^2}), ...\}$, where $t_i = t_i^1, t_i^2, ...$ is an index we refer to as "type" of i.[14] We interpret λ_i as a statistical distribution of plans and beliefs of agents playing in role i.[15] With a slight abuse of notation, we let $\lambda_i(t_i)$ denote the fraction of agents in population i with plan and beliefs $(\sigma_{t_i}, \beta_{t_i})$. Also, we denote by

$$T_i(\lambda_i) = \left\{t_i : (\sigma_{t_i}, \beta_{t_i}) \in \text{Supp}\lambda_i\right\}$$

the set of possible types of i in distribution λ_i, and we write $T_{-i}(\lambda_{-i}) = \times_{j \neq i} T_j(\lambda_j)$ for the set of profiles of co-players' types.

Let us take the perspective of an agent of type t_i who knows that the distribution over co-players' types is $\lambda_{-i} = \prod_{j \neq i} \lambda_j$ and believes that the behavior of each t_j is indeed described by t_j's plan σ_{t_j} (in principle, t_i may otherwise believe that t_j behaves differently from his plan). Then it is possible to derive the conditional probability of a type profile t_{-i} given history h. Given that beliefs satisfy independence across players (everybody knows there is independent random matching), the distribution is independent of t_i and can be factorized. In the current two-stage setting we have $\lambda_{-i}(t_{-i}|\varnothing) = \prod_{j \neq i} \lambda_j(t_j)$ and

$$\begin{aligned}
\lambda_{-i}(t_{-i}|a^1) &= \frac{\prod_{j \neq i} \sigma_{t_j}(a_j^1)\lambda_j(t_j)}{\sum_{t'_{-i} \in T_{-i}(\lambda_{-i})} \prod_{j \neq i} \sigma_{t'_j}(a_j^1)\lambda_j(t'_j)} \\
&= \frac{\prod_{j \neq i} \sigma_{t_j}(a_j^1)\lambda_j(t_j)}{\prod_{j \neq i} \sum_{t'_j \in T_j(\lambda_j)} \sigma_{t'_j}(a_j^1)\lambda_j(t'_j)} \\
&= \prod_{j \neq i} \frac{\sigma_{t_j}(a_j^1)\lambda_j(t_j)}{\sum_{t'_j \in T_j(\lambda_j)} \sigma_{t'_j}(a_j^1)\lambda_j(t'_j)}.
\end{aligned}$$

for all t_{-i} and a^1, provided that $\sum_{t'_j} \sigma_{t'_j}(a_j^1)\lambda_j(t'_j) > 0$ for each $j \neq i$. Letting

$$\lambda_j(t_j|a^1) = \frac{\sigma_{t_j}(a_j^1)\lambda_j(t_j)}{\sum_{t'_j \in T_j(\lambda_j)} \sigma_{t'_j}(a_j^1)\lambda_j(t'_j)},$$

[14] These are "types" in the sense of epistemic game theory (*e.g.*, Battigalli, Di Tillio & Samet 2013).

[15] The marginal of λ_i on Σ_i is a behavior strategy mixture (see Selten 1975).

we get

$$\lambda_{-i}(t_{-i}|a^1) = \prod_{j \neq i} \lambda_j(t_j|a^1).$$

We say that λ_j is **fully randomized** if σ_{t_j} is strictly positive for every type $t_j \in T_j(\lambda_j)$. If each λ_j is fully randomized, then, for all $h \in H$, $\lambda_{-i}(\cdot|h)$ is well defined, with $\lambda_{-i}(t_{-i}|h) = \prod_{j \neq i} \lambda_j(t_j|h)$ for all $t_{-i} \in T_{-i}(\lambda_{-i})$.

Definition 3 *A* ***polymorphic assessment*** *is a profile of finite support probability measures $\lambda = (\lambda_i)_{i \in I} \in \times_{i \in I} \Delta(\Sigma_i \times \Delta_i^2)$ such that, for every $i \in I$ and $t_i \in T_i(\lambda_i)$, σ_{i,t_i} is the behavior strategy obtained from β_{t_i} as per (12). A polymorphic assessment λ is* ***consistent*** *if there is a sequence $(\lambda^n)_{n=1}^{\infty}$ of polymorphic assessments converging to λ such that, for all $j \in I$ and $n \in \mathbb{N}$, λ_j^n is fully randomized, and*
(a-p) *for all $h \in H$, $a \in A(h)$, and $t_i \in T_i(\lambda_i^n)$,*

$$\alpha_{t_i,-i}^n(a_{-i}|h) = \prod_{j \neq i} \sum_{t_j \in T_j(\lambda_j^n)} \sigma_{t_j}^n(a_j|h)\lambda_j^n(t_j|h),$$

(b-p) *for all $h \in H$ and $t_i \in T_i(\lambda_i^n)$,*

$$\text{marg}_{\Delta_{-i}^1} \beta_{t_i}^n(\cdot|h) = \sum_{t_{-i} \in T_{-i}(\lambda_{-i}^n)} \lambda_{-i}^n(t_{-i}|h)\delta_{\alpha_{t_{-i}}^n},$$

where, for all $j \in I$, $t_j \in T_j(\lambda_j^n)$ and $n \in \mathbb{N}$, $\alpha_{t_j}^n$ is the first-order belief system derived from $\beta_{t_j}^n$.

Condition (a-p) extends independence condition (a) of Definition 1 to the multiple-types setting. Condition (b-p) implies that, conditional on the co-players' types, everyone has correct beliefs about the others' beliefs, including their plans. Yet uncertainty about co-players' types allows for uncertainty and meaningful updating about such beliefs. Conditions (a-p) and (b-p) imply that different types of the same player share the same beliefs about co-players, but may have different plans. Definition 3 thus is a minimal departure from the notion of consistent assessment, allowing for uncertainty and meaningful updating about the plans, hence intentions, of co-players.

Definition 4 *A polymorphic assessment λ is a* ***polymorphic sequential equilibrium (PSE)*** *if it is consistent and satisfies the following sequential rationality condition: for all $h \in H$, $i \in I$, and $t_i \in T_i(\lambda_i)$,*

$$\text{Supp}\sigma_{t_i}(\cdot|h) \subseteq \arg\max_{a_i \in A_i(h)} u_i(h, a_i; \beta_{t_i}).$$

Remark 3 *Every SE is a degenerate (or monomorphic) PSE. Therefore, Theorem 1 implies that, if every decision-utility function* $u_i(h,\cdot;\cdot)$ *(*$i \in I$ *,* $h \in H$*) is continuous, then there is at least one PSE. In particular, every game with SA, ABB, or ABI has at least one PSE.*

Finally, we demonstrate how the PSE alters predictions in the Ultimatum Minigame and in leader-followers games more generally.

Example 11 Consider again Figure B. If $|\mathrm{Supp}\lambda_i| = 1$ for all i, then our results for the SE analysis still hold as a special case of the more general PSE analysis. Interesting new possibilities arise if $|\mathrm{Supp}\lambda_a| = 2$. Recall that, in the SE with SA/ABB utility functions and non-degenerate plans (Example 10) we had $\alpha_a(g) = \frac{3}{4} - \frac{1}{4\theta_b}$ (with $\theta_b > 1/3$) to keep Bob indifferent. Suppose instead there are two types of Ann, a fraction of $\frac{3}{4} - \frac{1}{4\theta_b}$ of them planning to choose g while the others plan for f. There is a corresponding PSE where (naming Ann's types by planned choice) $\mathrm{Supp}\lambda_a = \{(f, \beta_f), (g, \beta_g)\}$, $\alpha_f(y|g) = \alpha_g(y|g) = \alpha_b(y|g) = 2/3$, and this holds for also for ABI, not only SA and ABB. The first-order belief of type f of Ann, α_f, is derived from β_f, etc. Bob initially believes Ann is either an f- or a g-type, assigning probability $\lambda_g = \frac{3}{4} - \frac{1}{4\theta_b}$ to the latter possibility. After action g he ceases to assign positive probability to being matched with an f-type, assigning instead probability 1 to the g-type, a form of updating about Ann's intentions implied by consistency (Def. 3). This inference makes ABI work as ABB (and SA). Bob's frustration is as in Example 10, so equal to his blame of Ann for each blaming function. Again Bob is indifferent between y and n, and sequentially rational if $\alpha_b(y|g) = 2/3$. Condition $\alpha_f(y|g) = \alpha_g(y|g) = 2/3$ implies both types of Ann are indifferent, hence sequentially rational. Thus, starting with the non-degenerate SE under ABB (and SA) we obtain a PSE, under every blaming function, where Ann's plan is purified. ▲

The insight of the previous example can be generalized:[16]

Proposition 2 *Consider a leader-followers game and arbitrary parameter profile* $(\theta_i)_{i\in I}$*. Every SE with decision-utility functions* $(u^{ABB}_{i,\theta_i})_{i\in I}$ *[or* $(u^{SA}_{i,\theta_i})_{i\in I}$*] where the behavioral strategy of the leader has full support corresponds to a PSE with decision-utility functions* $(u^{ABB}_{i,\theta_i})_{i\in I}$ *[or* $(u^{SA}_{i,\theta_i})_{i\in I}$*] and also* $(u^{ABI}_{i,\theta_i})_{i\in I}$ *where the leader is purified.*

[16] Recall that, by Remark 1, in leader-followers games SA is equivalent to both versions of ABB.

6 Multistage extension

In a multistage game form, a (non-empty) non-terminal history is a sequence of action profiles, $h = (a^1, ..., a^t)$ where $t \geq 1$. As in the two-stage case, we assume that actions are observable; hence, every non-terminal history is public. Our notation for the multistage setting is essentially the same as before. The set of sequences observable by player i also includes personal histories of the form (h, a_i): $H_i = H \cup \{(h, a_i) : h \in H, a_i \in A_i(h)\}$.

A belief system for player i over paths and beliefs of others is an array of probability measures $\beta_i = (\beta_i(\cdot|h_i))_{h_i \in H_i}$ satisfying conditions (3) and (4), which apply to the multistage setting as well. Also the notation on first- and second-order beliefs is the same as before: $\alpha_i \in \Delta_i^1$, $\beta_i \in \Delta_i^2$, and α_i is the first-order belief system derived from β_i when they appear in the same formula. The definitions of SE and PSE concepts can be applied to the multistage setting without modifications.

We distinguish two extreme scenarios according to the behaviorally relevant periodization: In the **slow-play** scenario, stages correspond to periods, and the reference belief of player i to determine frustration at the beginning of period (stage) $t + 1$ is given by his belief at the beginning of period t. In the **fast-play** scenario, the different stages of the game occur in the same period and the relevant reference belief of player i in each stage t is given by his initial belief, that is, his belief at the root $\varnothing$.[17] In either case, we maintain the assumption that blame is continuous in beliefs, capped by frustration, and equal to frustration in the case of simple anger.

6.1 Slow play

We start with this scenario because it allows for a relatively simple extension of the two-stage setting, with initial beliefs replaced by one-period-lagged beliefs: For any non-terminal history of the form $h = (\bar{h}, a)$ the frustration of i conditional on h given α_i is

$$\mathrm{F}_i(h; \alpha_i) = \left[\mathbb{E}[\pi_i|\bar{h}; \alpha_i] - \max_{a_i \in A_i(h)} \mathbb{E}[\pi_i|(h, a_i); \alpha_i]\right]^+.$$

(When $\bar{h} = \varnothing$ and $h = a^1$, we are back to the two-period formula.) The decision-utility of action $a_i \in A_i(h^t)$ has the general form (5), where the

[17] Applications may involve intermediate cases, as in alternating-offer bargaining models where a period comprises two stages. The two extremes convey the main ideas.

blame functions $\mathrm{B}_{ij}(h;\beta_i)$ are of the SA, ABB, or ABI type. Specifically: $\mathrm{B}_{ij}(h;\beta_i) = \mathrm{F}_i(h;\alpha_i)$ for SA, whereas the could-have-been blame, blaming deviations, and blaming intentions can be defined with straightforward adaptations of (9), (10), and (11) respectively; therefore we omit the details.

This extension of the two-stage setting has the stark feature that past frustrations do not affect current behavior. (A more nuanced version features a decaying effect of past frustrations.)

A detail in modeling game forms becomes relevant in the slow play scenario: We have to explicitly allow for non-terminal histories after which no player (not even chance) is active, such as history g in Figure D.

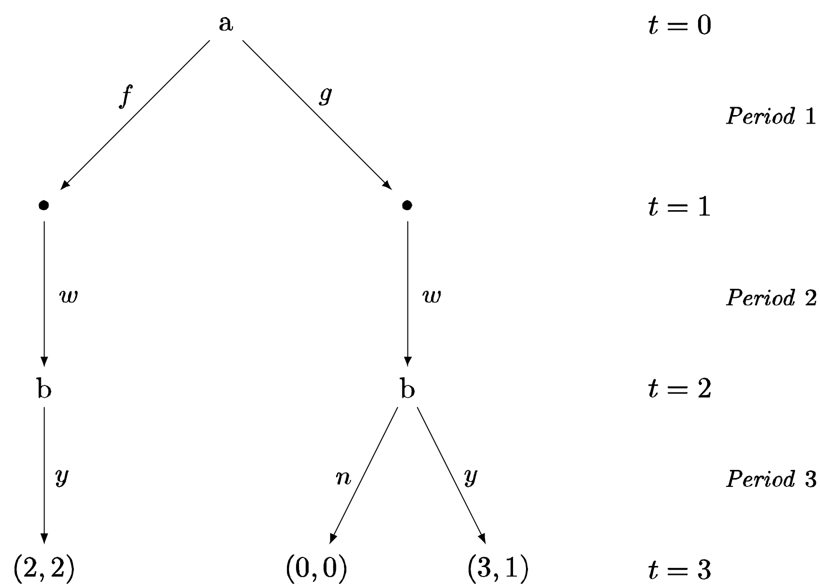

Figure D. Ultimatum Minigame with delayed reply.

At such histories there is only one feasible action profile, as each player has only one feasible action, to wait. In the two-periods setting this detail is irrelevant: If nobody is active at the root, play effectively starts (and ends) in the second period; if nobody is active at a^1, it can be modeled as a terminal history. With more than two periods, having to wait may affect behavior.

Example 12 Consider Figure D. Suppose that Bob initially expects f with positive probability. Then in period 2 after g he is frustrated, but cannot hurt Ann because he has to wait. In period 3, Bob's lagged expectation has fully adapted downward, hence there is no "incremental frustration" in this period. According to our slow-play model, the frustration experienced by Bob in period 2 does not affect his decision utility in period 3: Bob fully "cools off" and behaves as-if selfish. Therefore the unique (polymorphic) SE outcome of the game is (g, w, y), where w denotes waiting. ▲

6.2 Fast play

When play is fast, all stages belong to the same period, therefore the reference belief that determines player i's frustration conditional on any history is i's initially expected monetary payoff. Thus, i's frustration at h given α_i is

$$\mathrm{F}_i(h; \alpha_i) = \left[\mathbb{E}[\pi_i; \alpha_i] - \max_{a_i \in A_i(h)} \mathbb{E}[\pi_i | (h, a_i); \alpha_i]\right]^+ .$$

This implies that *there cannot be any "cooling off"* due to reference-point acclimatization. Formally, histories where nobody (not even chance) is active play no role and can be deleted from the game form without affecting the analysis. For example, in the fast-play scenario, the ultimatum game form of Figure D is equivalent to the one of Figure B.

The fast-play frustration formula can be plugged into the SA decision-utility function (7). As for the ABB decision-utility, property (8) of B_{ij} extends to the multistage setting as follows:

$$\mathrm{B}_{ij}(h; \alpha_i) = \begin{cases} 0, & \text{if } j \notin I(h') \text{ for all } h' \prec h, \\ \mathrm{F}_i(a^1; \alpha_i), & \text{if } \{j\} = I(h') \text{ for all } h' \prec h. \end{cases} \tag{13}$$

In words, co-player j cannot be blamed if he was never active in the past, and he is fully blamed if instead he was the only active player. A relatively simple extension of could-have-been blame satisfies this property:

$$\mathrm{B}_{ij}(h; \alpha_i) = \min \left\{ \left[\max_{h' \prec h, a'_j \in A_j(h')} \mathbb{E}\left[\pi_i | (h', a'_j); \alpha_i\right] - \mathbb{E}[\pi_i | h; \alpha_i] \right]^+ , \mathrm{F}_i(h; \alpha_i) \right\}. \tag{14}$$

We can follow a similar logic to extend ABI.

Remark 4 *If* B_{ij} *is defined by (14), then it satisfies (13).*

We now illustrate our definition, elucidating a modeling choice:

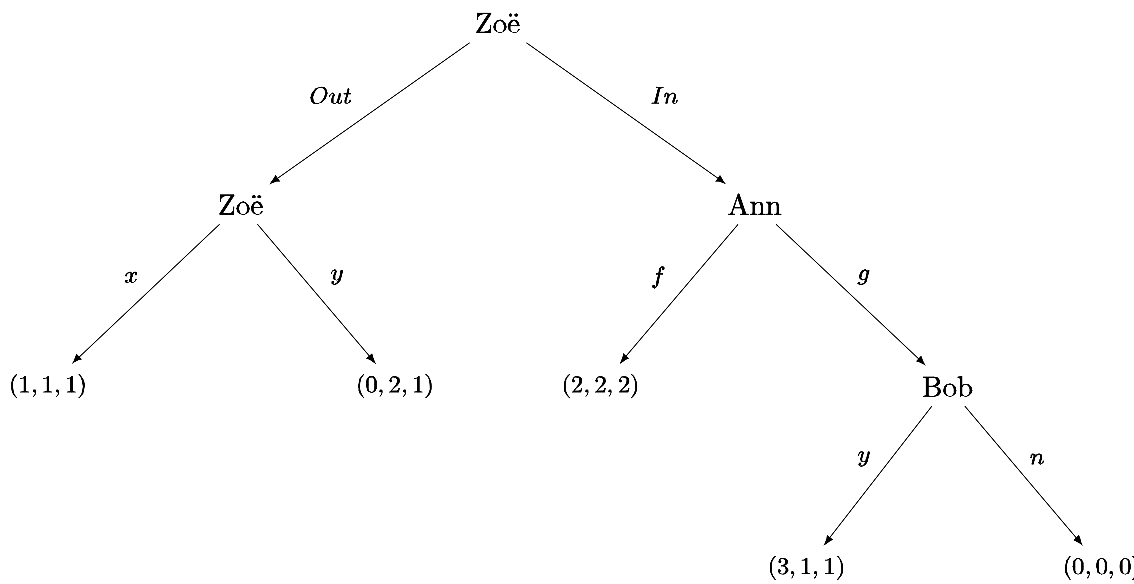

Figure E. Multistage Ultimatum featuring Zoë.

Example 13 Consider the game form in Figure E (material payoffs are in alphabetical order). If Zoë chooses In, then Ann and Bob interact in an ultimatum minigame, but Zoë may instead exercise outside options and play (Out, x) or (Out, y). Zoë's payoffs equal Bob's, except following (Out, y) where a payoff transfer from Ann to Bob occurs, relative to (Out, x). Can strategy profile $(In\text{-}x, f, n)$ be a SE under ABB? Given equilibrium beliefs, this is the case if $0-\theta_b \cdot 1 \cdot 0 \geq 1-\theta_b \cdot 1 \cdot 3$, or $\theta_b \geq 1/3$. The calculation involves Bob blaming Ann, not Bob blaming Zoë, because if Zoë switched from In to Out (thus implementing (Out, x) instead of In) this would not improve Bob's payoff. This reflects a non-obvious modeling choice: Our definition assesses blame on the basis of single-agent deviations from the realized path, but if Bob alternatively assessed blame on the basis of multi-agent deviations, including off-realized-path deviations, he would consider that Zoë could have played (Out, y). She would then have increased Bob's payoff from 1 to 2, preventing his frustration of 1. If Bob's blame of Zoë were thus 1, then $(In\text{-}x, f, n)$ would be a SE under ABB if $0 - \theta_b \cdot 1 \cdot 0 \geq 1 - \theta_b \cdot 1 \cdot 3 - \theta_b \cdot 1 \cdot 1$, or $\theta_b \geq 1/4 \neq 1/3$. (This also shows that SE under ABB is not invariant with respect to coalescing sequential moves.) Finally, note that also $(In\text{-}y, f, n)$ is a SE under ABB in the fast-play scenario for $\theta_b \geq 1/4$, because at (In, g)

Zoë would be blamed for not switching to Out (implementing (Out, y)); but it is a SE under ABB in the slow-play scenario for larger parameter values, $\theta_b \geq 1/3$, because Bob would be frustrated only in the third period, after (In, g), and Zoë —who played in the first— could not be blamed. ▲

The single- vs. multi-agent deviation issue illustrated here can arise also in two-stage games (with simultaneous moves), but the point is clearer, and perhaps more relevant, in games with more than two stages. We defend our chosen formulation thrice: It harmonizes well with how we define rational play, where players optimize only locally (although in equilibrium they predict correctly and choose as planned). The (hinted at) alternative definition would be formally convoluted. It is an open issue which formulation is empirically more relevant, so we stick with what is simpler.

6.3 Counterfactual anger and unique SE in hold-up

It is important to emphasize, worth a separate section, that anger (and in fact emotions more generally) can shape behavior without occurring. If anger is anticipated, this may steer behavior down alternative paths (cf. Remark 2). We already saw examples, *e.g.*, (f, n) is a SE in the Ultimatum Minigame, alongside (g, y). Our next example highlights how there may be circumstances where the SE is *unique* and has that property. It also illustrates a difference between fast and slow play.

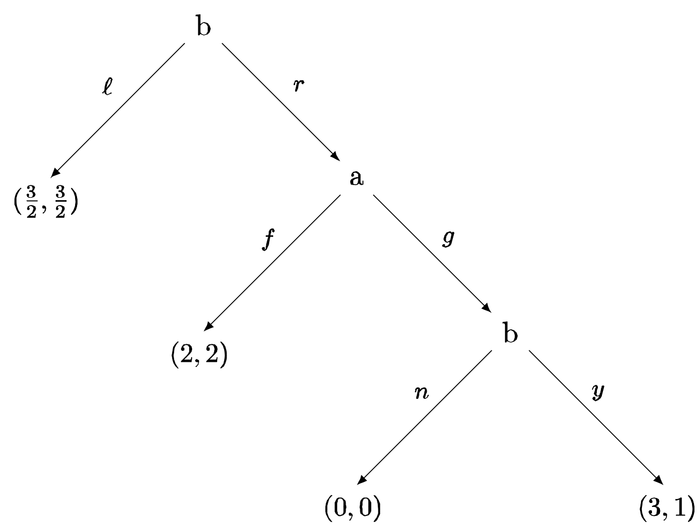

Figure F. Hold-up.

Example 14 Modify the Ultimatum Minigame by adding an initial move for Bob, as in Figure F, to get an illustration of a hold-up problem (cf. Dufwenberg, Smith & Van Essen 2013).[18] Under fast play, for each utility function seen so far,[19] if $\theta_b > 2/3$, there is a unique SE: Bob uses plan $(r\text{-}n)$, Ann plans for f. To verify this, the key step is to check that if Bob plans for (ℓ, y) and Ann for g this is *not* an SE; if Bob initially expects \$1.5, off-path at (r, g), he would be frustrated and deviate to n. ▲

With slow play, by contrast, with $\theta_b > 2/3$, there are multiple SE, exactly as in the Ultimatum Minigame. In particular, both $(r\text{-}n, f)$ and $(\ell\text{-}y, g)$ are SE; in the latter, Bob's updated expected payoff after (counterfactual) action r is only \$1, hence he cannot be frustrated by g.

7 Discussion

Incorporating the effects of emotions in economic analysis is a balancing act. One wants to focus on sentiments that make empirical sense, but human psychology is multi-faceted and there is no unambiguous yardstick. Our chosen formulation provides a starting point for exploring how anger shapes interaction, and experimental or other evidence will help to assess empirical relevance and suggest revised formulas. We conclude by discussing topics that may be helpful for gaining perspective on, building on, or further developing our work. We mix commentary on chosen concepts, comparisons with related notions, and remarks about empirical tests.

Frustration Consider substituting $\mathbb{E}[\pi_i; \alpha_i] - \mathbb{E}[\pi_i | a^1; \alpha_i]$ for frustration $\mathrm{F}_i(a^1; \alpha_i)$ of Section 3. This alternative would measure i's *actual* diminished expectations at a^1, unlike $\mathrm{F}_i(a^1; \alpha_i)$ which captures diminished expectations relative to what i believes is the most he can get (which we think of as the adequate way to capture goal-blockage). To appreciate how dramatically this change would impact implied behavior, consider a two-player common-interest game: Ann chooses *Out* or *In*; in the former case the game ends with

[18] Bob and Ann face a joint business opportunity worth $(2, 2)$ via path (r, f); however, r involves partnership-specific investment by Bob, which Ann can exploit choosing g (reneging), etc. As always, we list payoffs by alphabetical order of players: (π_a, π_b).

[19] Except the blaming-unexpected-deviations version of ABB, which we did not define explicitly for fast play.

payoffs $(1, 1)$, in the latter case Bob chooses between $(0, 0)$ and $(2, 2)$. *Mutatis mutandis*, for high enough θ_b, with the alternative formulation, under SA and ABB, there is an SE where Ann chooses *Out* anticipating that Bob would go for $(0, 0)$. Following *In*, Bob would be frustrated because he (so-to-say) sees himself as locked-in with his stage-2 planned action. Our formulation of $\mathrm{F}_i(a^1; \alpha_i)$ rules that out.

Consider a binary gamble where with probability $p > 0$ Ann wins $\$x > 0$, and otherwise gets \$0. Her frustration, using our definition, equals her initial expectation: $p \cdot x$. This embodies strong implications for how frustrations compare across contexts, *e.g.* the frustration of a highly expected failure to win the state lottery versus that of some unlikely small loss. We are agnostic as regards empirical relevance, but alert the reader to the issue.[20]

The psychological evidence (cited in Section 1) says a player becomes frustrated when his goals are unexpectedly thwarted. We addressed but one aspect: own material rewards. Cases 1-3 indicate the broad applied potential. Yet our focus is restrictive, as one may imagine other sources of frustration. To see this, consider two more cases:

> **Case 4:** In 2007 Apple launched its iPhone at \$499. Two months later they introduced a new version at \$399, re-priced the old model at \$299, and caused outrage among early adopters. Apple paid back the difference. Did this help long run profit?
>
> **Case 5:** The 2008 TARP bank bail-out infuriated some US voters. Did this ignite the Tea Party/Occupy-Wall Street movements?

In case 4, an early adopter is frustrated because he regrets he already bought, not because new information implies that his expected rewards drop. In case 5, even an activist who is materially unaffected personally may be frustrated because of unexpected perceived unfairness. These examples are not exhaustive; further sources of frustration may *e.g.* involve shocks to self-esteem.[21] We suggest that techniques analogous to those we have developed

[20] The example involves one-player with a dummy-choice only to facilitate the frustration-calculation; interesting testable implications obviously arise more generally, *e.g.* in modified versions of the hammering one's thumb game.

[21] See Baumeister, Smart & Boden (1996) for an interesting discussion linking (threatened) self-esteem and violence.

in this paper may be applicable in all these cases, but details exploring these meaningful ways to get frustrated are left for future research.

As regards the effects of frustration, we considered changes to a player's utility function, but we neglected other plausible adjustments. Gneezy & Imas (2014) report data from an intriguing experiment involving two-player zero-sum material payoff games. In one game players gain if they are strong, in the other they are rewarded for being smart. Gneezy & Imas explore an added game-feature: before play starts, one subject may anger his opponent and force him to stay in the lab to do boring tasks after play ends. A thus frustrated player's performance is enhanced when strength is beneficial (possibly from increased adrenaline flow), but reduced when cool logic is called for (as if an angered player becomes cognitively impaired). Our model can capture the first consideration, but not the second. Specifically, to capture the first effect, we can let the consequences of actions depend also on beliefs, *e.g.*, because emotions affect strength or speed (cf. Rauh & Seccia 2006); this ultimately translates into belief-dependent utility (or cost) of actions. However, to capture the second effect, we would need a theory of endogenous cognitive abilities of boundedly rational players.

Valence and action-tendency Psychologists classify emotions in multiple ways. Two prominent aspects are **valence**, the intrinsic pleasantness or aversiveness of an emotion, and **action-tendency**, or how behavior is shaped as the emotion occurs. Both notions have bearing on anger. For example, most psychologists believe anger has negative valence (see, *e.g.*, Harmon-Jones & Sigelman 2001, p. 978). Perhaps such considerations steer people to avoid frustrations, say by not investing in the stock market. That said, the distinguishing feature of anger that psychologists stress concerns its action-tendency of aggression, not its valence. In developing our theory, we have exaggerated this, abstracting away from frustration avoidance, while emphasizing frustration-induced aggression. This is reflected in the decision utility functions, which are shaped by current frustration, but not by the anticipation of the negative valence of future frustrations.[22]

[22] In previous work we modeled another emotion: guilt; see, *e.g.*, Battigalli & Dufwenberg (2007), Chang, Smith, Dufwenberg & Sanfey (2011). To gain perspective note that in that work our approach to anticipation of valence and action-tendency was reversed. Guilt may have valence (negative!) as well as action-tendency (say to engage in "repair behavior"; see, *e.g.*, Silfver 2007). In modeling guilt we highlighted the anticipation of its negative valence while neglecting action-tendencies."

Blame We explored various ways a player may blame the frustration he experiences on others, but many more notions are conceivable. For example, with anger from blaming behavior i's blame of j depends on what i believes he would truly get at counterfactual histories, rather than the most he could get there. We view this modeling choice as reflecting local agency; i's current agent views other agents of i as uncontrollable, and he has no direct care for their frustrations.

Another example relates to how we model anger from blaming intentions: i's blame of j depends on β_i, his second-order beliefs. Recall that the interpretation concerns beliefs about beliefs about material payoffs. It does not concern beliefs about beliefs about frustration, which would be third- rather than second-order beliefs. Battigalli & Dufwenberg (2007), in a context which concerned guilt rather than anger, worked with such a third-order belief based notion of blame.

Our blame notions one way or another assess the *marginal* impact of other players. For example, consider a game where i exits a building while all $j \in I \backslash \{i\}$, unexpectedly to i, simultaneously hurl buckets of water at i, who thus gets soaked. According to our blame notions, i cannot blame any j as long as there are at least two hurlers. One could imagine alternatives where i blames, say, all the hurlers on the grounds that *collectively* they could thwart i's misery.

People may blame others in unfair ways, *e.g.* nominating scapegoats. Although our notions of SA and ABB may perhaps be interpreted to embody related notions to some degree, it has not been our intention to address such concerns systematically.

Several recent experiments explore interesting aspects of blame (Bartling & Fischbacher 2012, Gurdal *et al.*, Celen, Schotter & Blanco 2014). We wish to emphasize that our focus on blame is restricted to its relation to frustration only. Our paper is not about blame more generally, as there are reasons besides frustration that may lead people to blame each other.[23]

Kőszegi & Rabin Card & Dahl (2011) show that reports of domestic abuse go up when football home teams favored to win lose. They argue that

[23] For example, Celen *et al.* present a model where i asks how he would have behaved had he been in j's position and had j's beliefs. Then i blames j if j appears to be less generous to i than i would have been, and may blame j even if i is not surprised/frustrated. Or imagine a model where players blame those considered unkind, as defined in reciprocity theory (cf. subsection below), independently of frustration.

this is in line with Kőszegi & Rabin's (2006, 2007) theory of expectations-dependent reference points. Kőszegi & Rabin model the loss felt when a player gets less than he expected, which one may think of as a form of disappointment with negative valence (cf. Bell 1985, Loomes & Sugden 1986). Unlike in our models, anticipation of the negative valence of future frustrations then influences decision utility directly. That account per se does not imply that aggression follows, though it is natural to consider such an angle. Our model of simple anger does focus on the connection between frustration and aggression, capturing Card & Dahl's result. Modeling details distinguish how we define frustration and how Kőszegi & Rabin define loss (*e.g.*, how we cap frustration using the highest attainable payoff).

Anger Management People aware of their inclination to be angry may attempt to manage or contain their anger. Our players anticipate how frustrations shape behavior, and they may avoid or seek certain subgames because of that. However, there are interesting related phenomena we do not address: Can i somehow adjust θ_i say by taking an "anger management class?" If so, would rational individuals want to raise, or to lower, their θ_i? How might that depend on the game forms they play? These are potentially relevant questions related to how we have modeled action-tendency. Further issues would arise if we were to consider aspects involving anticipated negative valence of future frustrations, or bursts of anger.

Rotemberg's approach In a series of intriguing papers Rotemberg explores how consumer anger shapes firms' pricing (2005, 2011), as well as interaction in ultimatum games (2008). He proposes (versions of) a theory in which players are slightly altruistic, and consumers/responders also care about their co-players' degrees of altruism. Namely, they abruptly become very angry and punish a co-player whom they come to believe has an altruism parameter lower than some (already low) threshold. "One can thus think of individual i as acting as a classical statistician who has a null hypothesis that people's altruism parameter is at least as large as some cutoff value. If a person acts so that i is able to reject this hypothesis, individual i gains ill-will towards this person" (Rotemberg 2008, p. 464).

On the one hand, as a literal statement of what makes people upset, this assumption does not match well our reading of the relevant psychology. Recall that frustration is anchored in goal-blockage, where individuals are unexpectedly denied things they care about. Matters like "own payoff," which is our focus, and "fairness" or "quality of past decisions," which we have mentioned, come to mind; a co-player's altruism being λ rather than $\lambda - \varepsilon$, where both λ and ε are tiny numbers, hardly does. On the other hand, perhaps one may think of the approach as capturing some reasonable notion of scapegoating. Moreover, it is impressive how well Rotemberg's model captures the action in his data sets. It is natural to wonder whether our models could achieve that too. As regards behavior in ultimatum (and some other) games, there is already some existing evidence that is consistent with our modeling efforts; see the discussion in the final subsection below. Regarding pricing, we leave for empirical economists the task of exploring how our models will fare in applications.

Negative reciprocity Negative reciprocity (cf. Rabin 1993, Dufwenberg & Kirchsteiger 2004, Falk & Fischbacher 2006) joins anger as a motivation that can trigger hostile action. In some cases implications may be similar, but anger and negative reciprocity differ in key ways. The following sketched comparison is with Dufwenberg & Kirchsteiger's notion of sequential reciprocity equilibrium (SRE; refer to their article for formal definitions).

In the Hammering-One's-Thumb game of Figure C, Andy may take it out on Bob if he is motivated by simple anger. If he were motivated by reciprocity, this could never happen: Bob's kindness, since he is a dummy-player, equals 0, implying that Andy chooses as-if selfish. In this example reciprocity captures intuitions similar to the ABI concept, as perceived kindness assesses intentions similarly to how blame is apportioned.

That analogy only carries so far though. A player may be perceived as unkind even if he fails to hurt another, whereas under all anger notions frustration is a prerequisite for hostility. We exemplify:

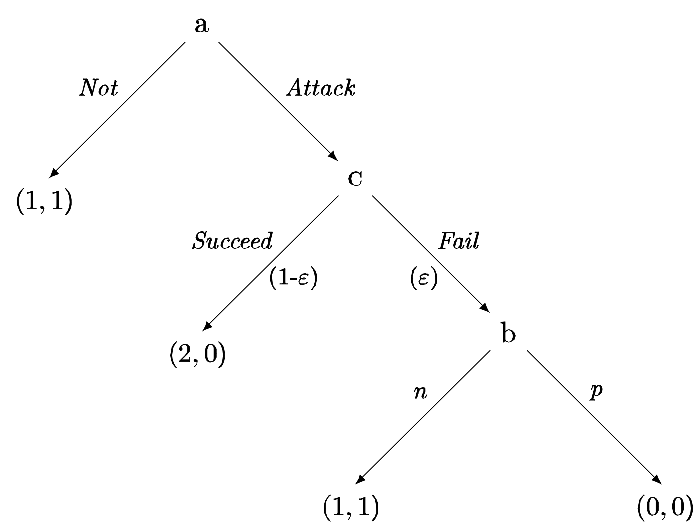

Figure G. Failed attack.

In the game form depicted in Figure G, if b is asked to play, then a's attack failed. Under reciprocity (augmented to allow incorporating a chance move; cf. Sebald 2010), b would deem a unkind, and —if sufficiently motivated by reciprocity— choose p in response. By contrast, under our anger concepts (SA, ABB, ABI) b would not be frustrated, and since frustration is a prerequisite for hostility b chooses n.

Reciprocity allows for so-called "miserable equilibria," where a player reciprocates expected unkindness before it occurs. For example, in the mini-ultimatum game of Figure B, (g, n) may be a SRE. Ann makes offer g despite believing that Bob will reject; given her beliefs about Bob's beliefs, Ann perceives Bob as seeing this coming, which makes him unkind, so she punishes by choosing g. Such self-fulfilling prophecies of destructive behavior have no counterpart under either anger notions. Since Ann moves at the root, she cannot be frustrated, and hence, regardless of how prone to anger she may be, she chooses as-if selfish.[24]

With reference to our discussion of cooling-off effects in Section 6, these have no counterpart in the reciprocity theory, which makes the same predic-

[24] Another example is the hold-up game of Figure F. We gave conditions where $(r\text{-}n, f)$ was the unique SE. If Ann and Bob were reciprocal, $(\ell\text{-}n, g)$ and $(r\text{-}n, g)$ could be SRE, with miserable interaction, respectively, off and on the equilibrium path.

tion in the games of Figures B and D. Reciprocal players do not cool off. "*La vengeance est un plat qui se mange froid.*"

Experimental testing Our models tell stories of what happens when players prone to anger interact. It is natural to wonder about empirical relevance, and here experiments may be helpful.

Several existing studies provide support for the notion that emotions drive behavior, and that many of them, and anger in particular, are generated from comparisons of outcomes with expectations. There is evidence that anger is a key driving force behind costly punishment. A few papers rely on emotion-self reports: Pillutla & Murnighan (1996) find that reported anger predicted rejections better than perceived unfairness in ultimatum games. Fehr & Gächter (2002) elicit self-reports of the level of anger towards free riders in a public goods game, concluding that negative emotions including anger are the proximate cause of costly punishment. Other studies directly connect unfulfilled expectations and costly punishment in ultimatum games. Schotter & Sopher (2007) measure second-mover expectations, concluding that unfulfilled expectations drive rejections of low offers. Similarly, Sanfey (2009) finds that psychology students who are told that a typical offer in the ultimatum game is \$4-\$5 reject low offers more frequently than students who are told that a typical offer is \$1-\$2.

A series of papers by Frans van Winden (with several coauthors) records both emotions and expectations in the power-to-take game (which resembles ultimatum games, but allows for partial rejections).[25] Second-mover expectations about first-mover "take rates" are a key factor in the decision to destroy income. Furthermore, anger-like emotions are triggered by the difference between expected and actual take rates. The difference between the actual and reported "fair" take rate is not significant in determining anger-like emotions, suggesting that deviations from expectations, rather than from fairness benchmarks, drive both anger and the destruction of endowments in the games.

Apropos the cooling off effects discussed in Section 6, Grimm & Mengel (2011) run ultimatum games that force some responders to wait ten minutes before making their choice. Without delay, less than 20% of low offers were accepted while 60–80% were accepted if the acceptance decision were delayed.

[25] Bosman & van Winden (2002), Bosman, Sutter & van Winden (2005), Reuben & van Winden (2008).

A literature in neuroscience connects expectations with social norms to study the neural underpinnings of emotional behavior. In Xiang, Lohrenz & Montague (2013), subjects respond to a sequence of ultimatum game offers whilst undergoing fMRI imaging. Unbeknownst to subjects, the experimenter controls the distribution of offers in order to manipulate beliefs. Rejections occur more often when subjects expect high rather than low offers. They make a connection between norm violations and reward prediction errors from reinforcement learning, which are known to be the computations instantiated by the dopaminergic reward system. Xiang *et al.* note that "when the expectation (norm) is violated, these error signals serve as control signals to guide choices. They may also serve as the progenitor of subjective feelings."

Going forward, it would be useful to develop tests specifically designed to target key features of our theory. For example, which version —SA, ABB, ABI— seems more empirically relevant, and how does the answer depend on context (*e.g.*, whether is SA perhaps more relevant for tired subjects)? Some insights may again be gleaned from existing studies. For example, Gurdal *et al.* (2014) study games where an agent invests on behalf of a principal, choosing between a safe outside option and a risky alternative. If the latter is chosen, then it turns out that many principals punish the agent if and only if by chance a poor outcome is realized. This seems to indicate some empirical relevance of our ABB solution (relative to ABI). That said, Gurdal *et al.*'s intriguing design is not tailored to specifically test our theory (and beliefs and frustrations are not measured), so more work seems needed to draw clearer conclusions.

Our models are abstractions. We theorize about the consequences of anger while neglecting myriad other obviously important aspects of human motivation (say altruism, warm glow, inequity aversion, reciprocity, social status, or emotions like guilt, disappointment, regret, or anxiety). Our models are not intended to explain every data pattern, but rather to highlight the would-be consequences of anger, if anger were the only form of motivation at play (in addition to care for material payoff). This statement may seem trivially obvious, but it has subtle implications for how to evaluate experimental work. To illustrate, consider again the Failed Attack game form in Figure G and suppose that in an experiment many subjects in player b's position chose to punish (p). Taken at face value, this would constitute a rejection of our theory (which predicts n rather than p). However, what may obviously be going on is that one of the forms of motivation that our theory abstracts away from affects subjects' choices (presumably negative reciprocity, in this case). It would seem more relevant to ask whether those choices of p were in fact driven by anger (as might be measured by, *e.g.*, emotion self-reports, physiological activity, or both, as in Chang *et al.*); if they were that could indicate that our theory could benefit from revision.

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Frustration and Anger in Games.
Online Appendix

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Abstract

This document contains the proofs of the results stated in "Frustration and Anger in Games."

1 Preliminaries

For each topological space X, we let $\Delta(X)$ denote the space of Borel probability measures on X endowed with the topology of weak convergence of measures. Every Cartesian product of topological spaces is endowed with the product topology. A topological space X is metrizable if there is a metric that induces its topology. A Cartesian product of a countable (finite, or denumerable) collection of metrizable spaces is metrizable.

To ease exposition, we report below some key definitions and equations contained in "Frustration and Anger in Games" (equation numbers may differ from those of the paper).

$\Delta_i^1 \subseteq \times_{h_i \in H_i} \Delta\left(Z(h_i)\right)$ is the set of first-order beliefs, that is, the set of $\alpha_i = \left(\alpha_i\left(\cdot|Z(h_i)\right)\right)_{h_i \in H_i}$ such that:

- for all $h_i, h_i' \in H_i$, if $h_i \prec h_i'$ then for every $Y \subseteq Z(h_i')$

$$\alpha_i(Z(h_i')|Z(h_i)) > 0 \Rightarrow \alpha_i\left(Y|Z(h_i')\right) = \frac{\alpha_i(Y|Z(h_i))}{\alpha_i(Z(h_i')|Z(h_i))}; \qquad (1)$$

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- for all $h \in H$, $a_i \in A_i(h)$, $a_{-i} \in A_{-i}(h)$ (using obvious abbreviations)

$$\alpha_{i,-i}(a_{-i}|h) = \alpha_{i,-i}(a_{-i}|h, a_i). \tag{2}$$

$\Delta_i^2 \subseteq \times_{h_i \in H_i} \Delta\left(Z(h_i) \times \Delta_{-i}^1\right)$ —where $\Delta_{-i}^1 = \times_{j \neq i} \Delta_j^1$— is the set of second-order beliefs, that is, the set of $\beta_i = (\beta_i(\cdot|h_i))_{h_i \in H_i}$ such that:

- if $h_i \prec h_i'$ then

$$\beta_i(h_i'|h_i) > 0 \Rightarrow \beta_i\left(E|h_i'\right) = \frac{\beta_i\left(E|h_i\right)}{\beta_i(h_i'|h_i)} \tag{3}$$

  for all $h_i, h_i' \in H_i$ and every event $E \subseteq Z(h_i') \times \Delta_{-i}^1$;

- i's beliefs satisfy an own-action independence property:

$$\beta_i\left(Z\left(h, (a_i, a_{-i})\right) \times E_\Delta|(h, a_i)\right) = \beta_i\left(Z\left(h, (a_i', a_{-i})\right) \times E_\Delta|(h, a_i')\right), \tag{4}$$

  for every $h \in H$, $a_i, a_i' \in A_i(h)$, $a_{-i} \in A_{-i}(h)$, and (measurable) $E_\Delta \subseteq \Delta_{-i}^1$. The space of second-order beliefs of i is denoted Δ_{-i}^2.

Note that (1) and (4) are given by equalities between marginal measures (on $A_{-i}(h)$ and $A_{-i}(h) \times \Delta_{-i}^1$ respectively):$\alpha_{i,-i}(a_{-i}|h) = \alpha_{i,-i}(a_{-i}|h, a_i)$.

Lemma 1 *For each player $i \in I$, Δ_i^2 is a compact metrizable space.*

Proof Let Θ be a non-empty, compact metrizable space. Lemma 1 in Battigalli & Siniscalchi (1999) (B&S) establishes that the set of arrays of probability measures $(\mu(\cdot|h_i))_{h_i \in H_i} \in \times_{h_i \in H_i} \Delta\left(Z(h_i) \times \Theta\right)$ such that

$$h_i \prec h_i' \wedge \mu(h_i'|h_i) > 0 \Rightarrow \mu\left(E|h_i'\right) = \frac{\mu\left(E|h_i\right)}{\mu(h_i'|h_i)}$$

is closed. Note that, in the special case where Θ is a singleton, each $\Delta\left(Z(h_i) \times \Theta\right)$ is isomorphic to $\Delta\left(Z(h_i)\right)$; hence, the set of first-order beliefs satisfying (1) is closed. Letting $\Theta = \Delta_{-i}^1$, we obtain that the set of second-order beliefs satisfying (3) is closed.

Since $\times_{h_i \in H_i} \Delta\left(Z(h_i)\right)$ is a compact subset of a Euclidean space and eq. (2) is a closed condition (equalities between marginal measures are preserved in the limit), Lemma 1 in B&S implies that Δ_i^1 is a closed subset of a compact metrizable space. Hence, Δ_i^1 is a compact metrizable space.

It is well known that if $X_1, ..., X_K$ are compact metrizable, so is $\times_{k=1}^K \Delta(X_k)$ (see Aliprantis & Border 2006, Theorem 15.11). Hence, by Lemma 1 in B&S, the set of second-order beliefs satisfying (3) is a closed subset of a compact metrizable space. Since eq. (4) is a closed condition (equalities between marginal measures are preserved in the limit), this implies that Δ_i^2 is compact metrizable. ■

Lemma 2 *For each profile of behavioral strategies $\sigma = (\sigma_i)_{i \in I}$ there is a unique profile of second-order beliefs $\beta^\sigma = (\beta_i^\sigma)_{i \in I}$ such that (σ, β^σ) is a consistent assessment. The map $\sigma \mapsto \beta^\sigma$ is continuous.*

Proof Write $\mathbb{P}^\sigma(h'|h)$ for the probability of reaching h' from h, *e.g.*,

$$\mathbb{P}^\sigma(a^1, a^2|\varnothing) = \left(\prod_{j \in I} \sigma_j(a_j^1|\varnothing)\right)\left(\prod_{j \in I} \sigma_j(a_j^2|a^1)\right).$$

Define α_i^σ as $\alpha_i^\sigma(z|h) = \mathbb{P}^\sigma(z|h)$ for all $i \in I$, $h \in H$, and $z \in Z$. Define β^σ as $\beta_i^\sigma(\cdot|h) = \alpha_i^\sigma(\cdot|h) \times \delta_{\alpha_{-i}^\sigma}$ for all $i \in I$, $h \in H$. It can be checked that (1) $\beta^\sigma \in \Delta_i^2$ for each $i \in I$, (2) (σ, β^σ) is a consistent assessment, and (3) if $\beta \neq \beta^\sigma$, then either (a) or (b) of the definition of consistency is violated. It is also apparent from the construction that the map $\sigma \mapsto \beta^\sigma$ is continuous, because $\sigma \mapsto \alpha^\sigma$ is obviously continuous, and the Dirac-measure map $\alpha_{-i} \mapsto \delta_{\alpha_{-i}}$ is continuous. ■

Lemma 3 *The set of consistent assessments is compact.*

Proof Lemma 1 implies that $\times_{i \in I}(\Sigma_i \times \Delta_i^2)$ is a compact metrizable space that contains the set of consistent assessments. Therefore, it is enough to show that the latter is closed. Let $(\sigma^n, \beta^n)_{n \in \mathbb{N}}$ be a converging sequence of consistent assessments with limit $(\sigma^\infty, \beta^\infty)$. For each $i \in I$, let α_i^n be the first-order belief derived from β_i^n ($n \in \mathbb{N} \cup \{\infty\}$), that is,

$$\alpha_i^n(Y|h) = \beta_i^n(Y \times \Delta_{-i}^1|h)$$

for all $h \in H$ and $Y \subseteq Z(h)$. By consistency, for all $n \in \mathbb{N}$, $i \in I$, $h \in H$, $a \in A(h)$, and $E_{-i} \subseteq \Delta_{-i}$ it holds that

- (a.n) $\alpha_i^n(a|h) = \beta_i^n\left(Z(h,a) \times \Delta_{-i}^1|h\right) = \prod_{j\in I}\sigma_j^n(a_j|h)$,
- (b.n) $\mathrm{marg}_{\Delta_{-i}^1}\beta_i^n(\cdot|h) = \delta_{\alpha_{-i}^n}$, where each α_j^n is determined as in (a.n).

Then,

$$\alpha_i^\infty(a|h) = \beta_i^\infty\left(Z(h,a) \times \Delta_{-i}^1|h\right) = \prod_{j\in I}\sigma_j^\infty(a_j|h)$$

for all $i \in I$, $h \in H$, $a \in A(h)$. Furthermore, $\mathrm{marg}_{\Delta_{-i}^1}\beta_i^\infty(\cdot|h) = \delta_{\alpha_{-i}^\infty}$ for all $i \in I$ and $h \in H$, because $\alpha_{-i}^n \to \alpha_{-i}^\infty$ and the marginalization and Dirac maps $\beta_i \mapsto \mathrm{marg}_{\Delta_{-i}^1}\beta_i$ and $\alpha_{-i} \mapsto \delta_{\alpha_{-i}}$ are continuous. ■

2 Proofs

2.1 Proof of Remark 2

Fix $i \in I$ arbitrarily. First-order belief α_i is derived from β_i and, by consistency, gives the behavioral strategies profile σ. Therefore, by assumption each $h' \preceq h$ has probability one under α_i, which implies that $\mathbb{E}[\pi_i|h';\alpha_i] = \mathbb{E}[\pi_i;\alpha_i]$, hence $\mathrm{F}_i(h';\alpha_i) = 0$. Since blame is capped by frustration, $u_i(h', a_i'; \beta_i) = \mathbb{E}[\pi_i|h';\alpha_i]$. Therefore, sequential rationality of the equilibrium assessment implies that $\mathrm{Supp}\sigma_i(\cdot|h') \subseteq \arg\max_{a_i'\in A_i(h')}\mathbb{E}[\pi_i|h';\alpha_i]$. If there is randomization only in the last stage (or none at all), then players maximize locally their expected material payoff on the equilibrium path. Hence, the second claim follows by inspection of the definitions of agent form of the material-payoff game and Nash equilibrium. ■

2.2 Proof of Proposition 1

Let $(\bar{\sigma}, \bar{\beta}) = \left(\bar{\sigma}_i, \bar{\beta}_i\right)_{i\in I}$ be the SE of the material payoff game, which is in pure strategies by the perfect information assumption. Fix decision-utility functions $u_i(h, a_i; \cdot)$ of the ABI, or ABB kind, and a sequence of real numbers $(\varepsilon_n)_{n\in\mathbb{N}}$, with $\varepsilon_n \to 0$ and $0 < \varepsilon_n < \frac{1}{\max_{i\in I, h\in H}|A_i(h)|}$ for all $n \in \mathbb{N}$. Consider the constrained psychological game where players can choose mixed actions in the following sets:

$$\Sigma_i^n(h) = \{\sigma_i(\cdot|h) \in \Delta(A_i(h)) : \|\sigma_i(\cdot|h) - \bar{\sigma}_i(\cdot|h)\| \leq \varepsilon_n\}$$

if h is on the $\bar{\sigma}$-path, and

$$\Sigma_i^n(h) = \{\sigma_i(\cdot|h) \in \Delta(A_i(h)) : \forall a_i \in A_i(h), \sigma_i(a_i|h) \geq \varepsilon_n\}$$

if h is off the $\bar{\sigma}$-path. By construction, these sets are non-empty, convex, and compact. Since the decision-utility functions are continuous, and the consistent assessment map $\sigma \longmapsto \beta^\sigma$ is continuous (Lemma 2), correspondence

$$\sigma \longmapsto \times_{h\in H} \times_{i\in I} \arg \max_{\sigma_i'(\cdot|h)\in\Sigma_i^n(h)} \sum_{a_i\in A_i(h)} \sigma_i'(a_i|h) u_i(h, a_i; \beta_i^\sigma)$$

is upper-hemicontinuous, non-empty, convex, and compact valued; therefore (by Kakutani's theorem), it has a fixed point σ^n. By Lemma 3, the sequence of consistent assessments $\left(\sigma^n, \beta^{\sigma^n}\right)_{n=1}^{\infty}$ has a limit point (σ^*, β^*), which is consistent too. By construction, $\bar{\sigma}(\cdot|h) = \sigma^*(\cdot|h)$ for h on the $\bar{\sigma}$-path, therefore $(\bar{\sigma}, \bar{\beta})$ and (σ^*, β^*) are realization-equivalent. We let $\bar{\alpha}_i$ (respectively, α_i^*) denote the first-order beliefs of i implied by $(\bar{\sigma}, \bar{\beta})$ (respectively, (σ^*, β^*)).

We claim that the consistent assessment (σ^*, β^*) is a SE of the psychological game with decision-utility functions $u_i(h, a_i; \cdot)$. We must show that (σ^*, β^*) satisfies sequential rationality. If h is off the $\bar{\sigma}$-path, sequential rationality is satisfied by construction. Since $\bar{\sigma}$ is deterministic and there are no chance moves, if h is on the $\bar{\sigma}$-path (*i.e.* on the σ^*-path) it must have unconditional probability one according to each player's beliefs and there cannot be any frustration; hence, $u_i(h, a_i; \beta_i^*) = \mathbb{E}[\pi_i|h, a_i; \alpha_i^*]$ $(i \in I)$ where α_i^* is determined by σ^*. If, furthermore, it is the second stage ($h = \bar{a}^1$, with $\bar{\sigma}(\bar{a}^1|\varnothing) = 1$), then —by construction— $\mathbb{E}[\pi_i|h, a_i; \alpha_i^*] = \mathbb{E}[\pi_i|h, a_i; \bar{\alpha}_i]$, where $\bar{\alpha}_i$ is determined by $\bar{\sigma}$. Since $\bar{\sigma}$ is a SE of the material-payoff game, sequential rationality is satisfied at h. Finally, we claim that (σ^*, β^*) satisfies sequential rationality also at the root $h = \varnothing$. Let $\iota(h)$ denote the active player at h. Since $\iota(\varnothing)$ cannot be frustrated at $\varnothing$, we must show that action $\bar{a}^1$ with $\bar{\sigma}(\bar{a}^1|\varnothing) = 1$ maximizes his expected material payoff given belief $\alpha_{\iota(\varnothing)}$. According to ABB and ABI, player $\iota(a^1)$ can only blame the first mover $\iota(\varnothing)$ and possibly hurt him, if he is frustrated. Therefore, in assessment (σ^*, β^*) at node a^1, either $\iota(a^1)$ plans to choose his (unique) payoff maximizing action, or he blames $\iota(\varnothing)$ strongly enough to give up some material payoff in order to bring down the payoff of $\iota(\varnothing)$. Hence, $\mathbb{E}[\pi_{\iota(\varnothing)}|a^1; \alpha^*_{\iota(a^1)}] \leq \mathbb{E}[\pi_{\iota(\varnothing)}|a^1; \bar{\alpha}_{\iota(a^1)}]$ (anger). By consistency of (σ^*, β^*) and $(\bar{\sigma}, \bar{\beta})$, $\alpha^*_{\iota(a^1)} = \alpha^*_{\iota(\varnothing)}$ and $\bar{\alpha}_{\iota(a^1)} = \bar{\alpha}_{\iota(\varnothing)}$ (cons.). Since (σ^*, β^*) is realization-equivalent to $(\bar{\sigma}, \bar{\beta})$ (r.e.), which is the material-payoff equilibrium (m.eq.), for each $a^1 \in A(\varnothing)$,

$$\begin{aligned}
&\mathbb{E}[\pi_{\iota(\varnothing)}|\bar{a}^1;\alpha^*_{\iota(\varnothing)}] \overset{\text{(r.e.)}}{=} \mathbb{E}[\pi_{\iota(\varnothing)}|\bar{a}^1;\bar{\alpha}_{\iota(\varnothing)}] \overset{\text{(m.eq.)}}{\geq} \\
&\mathbb{E}[\pi_{\iota(\varnothing)}|a^1;\bar{\alpha}_{\iota(\varnothing)}] \overset{\text{(cons.)}}{=} \mathbb{E}[\pi_{\iota(\varnothing)}|a^1;\bar{\alpha}_{\iota(a^1)}] \overset{\text{(anger)}}{\geq} \\
&\mathbb{E}[\pi_{\iota(\varnothing)}|a^1;\alpha^*_{\iota(a^1)}] \overset{\text{(cons.)}}{=} \mathbb{E}[\pi_{\iota(\varnothing)}|a^1;\alpha^*_{\iota(\varnothing)}].
\end{aligned}$$

This completes the proof for the ABB and ABI cases. If there are only two players, then we have a leader-follower game and SA is equivalent to ABB (Remark 1 of "Frustration and Anger in Games"), so (σ^*, β^*) is a SE in this case too. ■

2.3 Proof of Proposition 2

We denote the leader by $\iota(\varnothing)$. Let $(\sigma_i, \beta_i)_{i\in I}$ be a SE under ABB/SA with parameter profile $(\theta_i)_{i\in I}$, and suppose that the leader's strategy has full support: $\text{Supp}\sigma_{\iota(\varnothing)}(\cdot|\varnothing) = A_{\iota(\varnothing)}(\varnothing)$. Construct a polymorphic consistent assessment $\bar{\lambda}$ as follows: For each follower i, $T_i(\bar{\lambda}_i) = \{t_i\}$ (a singleton) and $\left(\bar{\sigma}_{t_i}, \bar{\beta}_{t_i}\right) = (\sigma_i, \beta_i)$. For the leader $\iota(\varnothing)$, $T_{\iota(\varnothing)}(\bar{\lambda}_{\iota(\varnothing)}) = A_{\iota(\varnothing)}(\varnothing)$, and, for each type $a_{\iota(\varnothing)}$, $\bar{\sigma}_{a_{\iota(\varnothing)}}(a_{\iota(\varnothing)}|\varnothing) = 1$ and $\bar{\alpha}_{a_{\iota(\varnothing)}}(\cdot|a^1) = \prod_{i\in I}\sigma_i(\cdot|a^1)$ for all non-terminal a^1, where $\bar{\alpha}_{a_{\iota(\varnothing)}}$ is the first-order belief derived from $\bar{\beta}_{a_{\iota(\varnothing)}}$. By construction, each type of leader is indifferent, because the leader (who acts as-if selfish) is indifferent in the original assessment $(\sigma_i, \beta_i)_{i\in I}$. As for the followers, they have the same first-order beliefs, hence the same second-stage frustrations as in $(\sigma_i, \beta_i)_{i\in I}$. Under ABB/SA, blame always equals frustration in leader-followers games. As for ABI, Bayes' rule implies that, after observing $a^1 = a_{\iota(\varnothing)}$, each follower becomes certain that the leader indeed planned to choose $a_{\iota(\varnothing)}$ with probability one, and blame equals frustration in this case too. Therefore, the incentive conditions of the followers hold in $\bar{\lambda}$ as in $(\sigma_i, \beta_i)_{i\in I}$ for all kinds of decision utility (ABI, ABB, SA) under the same parameter profile $(\theta_i)_{i\in I}$. ■

2.4 Proof of Remark 4

Fix $h \in H$. We consider the following simple extension of could-have-been blame in multistage games under fast play:

$$\mathrm{B}_{ij}(h;\alpha_i) = \min\left\{\left[\max_{h' \prec h, a'_j \in A_j(h')} \mathbb{E}\left[\pi_i|(h', a'_j);\alpha_i\right] - \mathbb{E}[\pi_i|h;\alpha_i]\right]^+, \mathrm{F}_i(h;\alpha_i)\right\}. \tag{5}$$

We must show that $\mathrm{B}_{ij}(h;\alpha_i) = 0$ if j is not active at any $h' \prec h$, and $\mathrm{B}_{ij}(h;\alpha_i) = \mathrm{F}_i(h;\alpha_i)$ if j is the only active player at each $h' \prec h$.

First note that if j was never active before, then $A_j(h')$ is a singleton for each $h' \prec h$, hence the term in brackets of (5) is zero. Next suppose that i is frustrated at h and j was the only active player in the past. Then there must be some $\bar{h} \prec h$ such that j deviated from i's expectations $\alpha_i(\cdot|\bar{h})$ for the first time, that is, $\bar{h}$ is the shortest predecessor $h' \prec h$ such that $\alpha_j(a'_j|h') < 1$ for $(h', a'_j) \preceq h$. Such $\bar{h}$ must have probability one according to the initial belief $\alpha_i(\cdot|\varnothing)$, thus $\mathbb{E}[\pi_i|\bar{h};\alpha_i] = \mathbb{E}[\pi_i;\alpha_i]$. Since $\max_{a'_j \in A_j(\bar{h})} \mathbb{E}\left[\pi_i|(\bar{h}, a'_j);\alpha_i\right] \geq \mathbb{E}[\pi_i|\bar{h};\alpha_i]$, we have $\max_{a'_j \in A_j(\bar{h})} \mathbb{E}\left[\pi_i|(\bar{h}, a'_j);\alpha_i\right] \geq \mathbb{E}[\pi_i;\alpha_i]$. Therefore

$$\begin{aligned}
& \max_{h' \prec h, a'_j \in A_j(h')} \mathbb{E}\left[\pi_i|(h', a'_j);\alpha_i\right] - \mathbb{E}[\pi_i|h;\alpha_i] \\
\geq\ & \max_{a'_j \in A_j(\bar{h})} \mathbb{E}\left[\pi_i|(\bar{h}, a'_j);\alpha_i\right] - \mathbb{E}[\pi_i|h;\alpha_i] \\
\geq\ & \mathbb{E}[\pi_i;\alpha_i] - \mathbb{E}[\pi_i|h;\alpha_i] \\
\geq\ & \mathbb{E}[\pi_i;\alpha_i] - \max_{a_i \in A_i(h)} \mathbb{E}[\pi_i|(h, a_i);\alpha_i] = \mathrm{F}_i(h;\alpha_i),
\end{aligned}$$

which implies $\mathrm{B}_{ij}(h;\alpha_i) = \mathrm{F}_i(h;\alpha_i)$ according to (5). ■

References

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