The Probability Premium and Other Approaches to Higher-degree Comparative Risk Aversion

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Abstract: In the framework of expected utility, *n*th-degree risk aversion/loving is unequivocally characterized by the sign of the *n*th-order derivative of the utility function, but there exist different notions of one decision maker being *n*th-degree more risk averse than another. This paper first reformulates Pratt's (1964) probability premium approach to comparative (2^{nd}-degree) risk aversion with a nonrandom starting wealth, and then shows that the reformulated probability premium approach can be easily extended to deal with random starting wealth and comparative *n*th-degree risk aversion. The paper shows that interpersonal comparisons of various versions of probability premia for *n*th-degree risk aversion are characterized by the (n/m)th-degree Ross more risk aversion of Liu and Meyer (2013), where $n > m \geq 1$. Besides the original Pratt setting, the same comparative *n*th-degree risk aversion extends to probability premia derived from the risk apportionment setting of Eeckhoudt and Schlesinger (2006) and the comparative statics setting of Jindapon and Neilson (2007).

Key Words: Risk aversion; Comparative risk aversion; Probability premium; Downside risk aversion; Risk apportionment

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1. Introduction

Consider the following scenario. An individual's wealth is currently given by the random variable $\tilde{w}$. He has the opportunity to improve his wealth, but only if the good state *G* occurs; otherwise, his wealth worsens. With probability *p* the good state occurs and his wealth is $\tilde{w}_G$ where $Eu(\tilde{w}_G) > Eu(\tilde{w})$, and with probability 1 – *p* the bad state occurs and his wealth is $\tilde{w}_B$ where $Eu(\tilde{w}_B) < Eu(\tilde{w})$.

Examples of such a setting abound. An individual could choose to pre-pay for a hotel room at the beach, making him better off unless it rains. An individual could consider changing companies for a higher-paying job which will increase his wealth unless the destination company hits troubled times and must lay him off. An individual could consider purchasing an illiquid asset that would improve his wealth position unless an emergency occurs and he needs the funds for something else. An individual could also hire a lawyer to recover a financial loss in court which will improve his financial position only if the case is won.

A famous exploration of this setting comes in Pratt (1964) in his definition of the probability premium, which is intended to measure the strength of risk aversion. In this case initial wealth $\tilde{w}$ is non-stochastic and fixed at *w*, the good lottery $\tilde{w}_G$ is $w+\varepsilon$ for certain, where $\varepsilon > 0$, and the bad lottery $\tilde{w}_B$ is $w-\varepsilon$ for sure. He defines the probability premium *q* as the probability of the good event *G* that makes the individual indifferent between initial wealth *w* and the event-dependent lottery.[1] His central theorem establishes that one individual always

[1] Variations on the Pratt probability premium have been recently employed in experimental work to quantify risk aversion, notably the "uncertainty premium" experiments of Andreoni and Sprenger (2012) and Callen et al. (2014). In their case the initial lottery $\tilde{w}$ pays the amount $x > 0$ with probability p_x and the higher outcome $y > x$ otherwise. The good lottery $\tilde{w}_G$ is *y* for sure, while the bad lottery $\tilde{w}_B$ is 0 with certainty. They define the

requires a higher probability premium than another if and only if the first individual is Arrow-Pratt more risk averse than the second one.

The purpose of this paper is to identify $\tilde{w}_G$ and $\tilde{w}_B$ in relation to initial wealth $\tilde{w}$, for which the required probability of the good state that ensures equivalence in expected utility between $\tilde{w}$ and the state-dependent lottery – the probability premium – provides a measure of *n*th-degree risk aversion, and to characterize the interpersonal comparison of the probability premium. Doing this generalizes Pratt's probability premium approach to comparative risk aversion to random starting wealth and to risk aversion of higher degrees.

Specifically, following Ekern (1980), a general *n*th-degree increase in risk is one that makes worse off every expected utility maximizer whose utility function has an *n*th derivative satisfying $(-1)^{n+1}u^{(n)}(x) > 0$, or equivalently, an *n*th-degree stochastically *dominated* change that holds the first $n - 1$ moments of the distribution constant. The familiar Rothschild-Stiglitz (1970) mean-preserving spread is a 2nd-degree increase in risk, and a 1st-degree increase in risk would be a first-order stochastically *dominated* shift. In compound binary lotteries considered in this paper, i.e., $\tilde{w}_G$ with probability *p* and $\tilde{w}_B$ with probability $1 - p$, the bad-state lottery $\tilde{w}_B$ differs from $\tilde{w}$ by an *n*th-degree increase in risk and the good-state lottery $\tilde{w}_G$ differs from $\tilde{w}$ by an *m*th-degree *reduction* in risk, with $n > m \geq 1$. Then the value of *p* that equalizes the expected utility from $\tilde{w}$ and the expected utility from the binary compound lottery provides a probability-like measure of *n*th-degree risk aversion in terms of compensation by an *m*th-degree

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uncertainty premium *q* as the probability of *G* that makes the individual indifferent between the initial lottery $\tilde{w}$ and the event-dependent lottery yielding *y* with probability *q* and 0 with probability $1 - q$. Holt and Laury (2002) uses a similar but slightly different probability-like measure to quantify the strength of risk aversion in their experiments.

reduction is risk. The central result here is that $u(x)$ always has a higher probability premium than $v(x)$ if and only if u is (n/m)th-degree Ross more risk averse than v as defined by Liu and Meyer (2013), a notion that includes Ross more risk aversion as a special case (Ross 1981).

We then extend the probability premium approach to the risk apportionment literature begun by Eeckhoudt and Schlesinger (2006) and Eeckhoudt et al. (2009). These works show that nth-degree risk aversion, i.e., aversion to nth-degree risk increases, can be characterized by preferences over 50-50 lotteries that display a preference for risk apportionment: combining "good" with "bad" is preferred to combining "good" with "good" and "bad" with "bad". While risk apportionment has proven useful for characterizing higher-degree risk attitudes, it has yielded only limited success for comparing those attitudes across individuals with the one example being Jindapon's (2010) examination of comparative downside risk aversion.[2] Extending the probability premium approach to risk apportionment allows for a comparison of nth-degree risk aversion across individuals. Interestingly, the approach also yields multiple versions of the probability premium for measuring nth-degree risk aversion (there is a unique version when $n = 2$), and when $n = 3$ these alternatives provide insight into comparative downside risk aversion or prudence.

We go on to extend the probability premium to the comparative statics setting introduced by Jindapon and Neilson (2007). Unlike the risk apportionment approach, the comparative statics approach was designed to facilitate interpersonal comparisons. The results here match and extend those in Jindapon and Neilson (2007), confirming the robustness of the probability premium approach to comparative risk attitudes. The comparative statics problem itself involves

[2] See also Watt (2011).

an individual facing a compound binary lottery choosing how to allocate resources between an *n*th-degree risk reduction in one state and an *m*th-degree risk reduction in the other, with $n > m \geq 1$. The result states that (n/m)th-degree Ross more risk aversion is both sufficient and necessary for devoting more effort to the *n*th-degree risk reduction.

All told, this paper uses the probability premium approach to explore three different problems involving compound binary lotteries where the individual faces lottery $\tilde{w}_1$ when event 1 occurs and lottery $\tilde{w}_2$ when event 2 occurs. The generalized Pratt probability premium setting looks at assigning probabilities to the two events to leave the individual indifferent between the binary compound lottery and the status quo, the risk apportionment setting examines assigning additional risks to the two events, and the comparative statics setting explores devoting resources to the two events. In all of these settings behavior is governed by (n/m)th-degree Ross more risk aversion as characterized by Liu and Meyer (2013), where $n > m \geq 1$.

Recent experimental studies have demonstrated, in various contexts, a salient aversion to risk increases of 3rd and even higher degrees.[3] At the same time, experimentalists have developed tools designed to measure the strength of 2nd-degree risk aversion in the lab.[4] The results of this paper can be used to construct new *n*th-degree risk aversion measures that are compatible with common notions of comparative risk preferences, and therefore do not require specific functional forms to induce increased risk aversion.[5] In the future, economists and other social scientists may want to investigate the determining factors of the strength of 3rd- and higher-degree risk aversion, just as they have extensively done so for the 2nd-degree risk

[3] For example, see Deck and Schlesinger (2010, 2014), Ebert and Wiesen (2011), Grossman and Eckel (2012), Maier and Ruger (2011) and Noussair et al. (2014).
[4] For example, see Holt and Laury (2002), and Eckel and Grossman (2002).
[5] Both the Holt-Laury and the Eckel-Grossman papers assume constant relative risk aversion.

aversion. It is our hope that the results in this paper will deepen the understanding of, and help in creating alternative measures for, the intensity of *n*th-degree risk aversion.

The rest of the paper is organized as follows. We first reformulate the original probability premium approach of Pratt (1964). Section 2 shows that Pratt's result still holds for the reformulated probability premium: $u(x)$ always requires a larger reformulated probability premium for a risk introduction than $v(x)$ if and only if $u(x)$ is Arrow-Pratt more risk averse than $v(x)$. Then, in Section 3, we demonstrate that the reformulated probability premium approach can be easily extended to deal with random starting wealth and comparative *n*th-degree risk aversion. Specifically, for $n = 2$ with random starting wealth, the paper establishes that $u(x)$ always requires a larger reformulated probability premium than $v(x)$ if and only if $u(x)$ is Ross more risk averse than $v(x)$; for $n \geq 3$, the paper shows that there exist multiple versions of the probability premium for measuring *n*th-degree risk aversion, each corresponding to the notion of the (n/m)th-degree Ross more risk aversion of Liu and Meyer (2013), for an appropriately chosen m such that $n > m \geq 1$. Section 4 extends the probability premium to the risk apportionment approach of Eeckhoudt and Schlesinger (2006), and Section 5 extends it to the comparative statics approach of Jindapon and Neilson (2007). Section 6 offers brief conclusions.

2. A Reformulation of Pratt's Probability Premium Approach

In this section, we propose a reformulation of Pratt's probability premium approach to measuring risk aversion, which is then extended in the next section for measuring *n*th-degree risk aversion. Pratt's approach is briefly reviewed first. For decision maker $u(x)$ with $u'(x) > 0$ and $u''(x) < 0$,

$$u(w) > \frac{1}{2}u(w+\varepsilon) + \frac{1}{2}u(w-\varepsilon) \tag{1}$$

for all w and all $\varepsilon > 0$. Pratt (1964) defines the probability premium q_u, a measure of risk aversion, as the additional probability shifted to the favorite outcome – which is $w+\varepsilon$ – that is needed to make the decision maker indifferent between the certain outcome and the uncertain outcome. That is, the probability premium q_u solves

$$u(w) = \left(\frac{1}{2} + q_u\right)u(w+\varepsilon) + \left(\frac{1}{2} - q_u\right)u(w-\varepsilon)\,. \tag{2}$$

It is easy to see that for any given w and $\varepsilon > 0$, there is a unique $q_u \in (0, \tfrac{1}{2})$ such that (2) is satisfied. Pratt goes on to demonstrate that $q_u \geq q_v$ for all w and all $\varepsilon > 0$ if and only if $u(x)$ is Arrow-Pratt more risk averse than $v(x)$, i.e., $-u''(x)/u'(x) \geq -v''(x)/v'(x)$ for all x.

The starting point of the reformulated probability premium approach is to observe that, for decision maker $u(x)$ with $u'(x) > 0$ and $u''(x) < 0$ and for any positive constant k and non-degenerate zero-mean random variable $\tilde{\varepsilon}$, $w + k$ is strictly preferred to w which is in turn strictly preferred to $w + \tilde{\varepsilon}$. Then, consider a comparison between the certain outcome w and a (compound) binary lottery with $w + k$ and $w + \tilde{\varepsilon}$ as its outcomes in, respectively, the good and bad states:

$$w \qquad \text{vs.} \qquad \begin{cases} w+k & \text{with probability } p \\ w+\tilde{\varepsilon} & \text{with probability } 1-p \end{cases} \tag{3}$$

Obviously, the larger the probability of the good state p, the more attractive the binary lottery. It can be easily shown that there is a unique $p_u \in (0,1)$ – for any given w, k and $\tilde{\varepsilon}$ – such that $u(x)$ is indifferent between the two sides in (3), or

$$u(w) = p_u u(w+k) + (1-p_u) Eu(w+\tilde{\varepsilon}) \; . \tag{4}$$

Definition 1. Given w, k > 0 and non-degenerate $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, the (reformulated) probability premium for decision maker $u(x)$, denoted p_u, is determined by (4).

The probability premium p_u is larger than zero only because $u(x)$ is risk averse, and a larger p_u implies a greater compensation for the introduction of the pure risk $\tilde{\varepsilon}$. So p_u is a natural measure of risk aversion for $u(x)$. Note also that p_u and Pratt's q_u are closely related. Let $k = \varepsilon$ and $\tilde{\varepsilon}$ have two outcomes, ε and $-\varepsilon$, with equal probability ½ in (4). Then it is easy to see that $p_u = 2q_u$.

Similarly, one can define p_v for $v(x)$. The following theorem shows that, like both the risk premium and Pratt's probability premium q_u, interpersonal comparison of the reformulated probability premium p_u is characterized by the Arrow-Pratt more risk aversion.

Theorem 1. $p_u \geq p_v$ for all w, k and $\tilde{\varepsilon}$ if and only if $u(x)$ is Arrow-Pratt more risk averse than $v(x)$.

Proof: "if"

Suppose $-u''(x)/u'(x) \geq -v''(x)/v'(x)$ at every x. Then according to Pratt (1964), there exists a transformation function $g(\cdot)$ with $g' > 0$ and $g'' \leq 0$ such that $u(x) \equiv g\left(v(x)\right)$. Because p_v satisfies $v(w) = p_v v(w+k) + (1-p_v) Ev(w+\tilde{\varepsilon})$, we have

$$
\begin{aligned}
u(w) &= g\left(v(w)\right) = g\left(p_v v(w+k) + (1-p_v)Ev(w+\tilde{\varepsilon})\right) \\
&\geq p_v g\left(v(w+k)\right) + (1-p_v) g\left(Ev(w+\tilde{\varepsilon})\right) \\
&\geq p_v u(w+k) + (1-p_v) Eu(w+\tilde{\varepsilon}),
\end{aligned}
$$

which implies $p_u \geq p_v$.

"only if"

Use proof by contradiction. Suppose that at some x_0, $-u''(x_0)/u'(x_0) < -v''(x_0)/v'(x_0)$. Then continuity would imply $-u''(x)/u'(x) < -v''(x)/v'(x)$ for all x in a small neighborhood of x_0. With similar steps to those taken in the "if" part above, we can show that $p_u < p_v$ for w that is close enough to x_0, and k and $\tilde{\varepsilon}$ that are sufficiently small. This contradicts the condition that $p_u \geq p_v$ for all w, k and $\tilde{\varepsilon}$. Q.E.D.

3. Extensions to Random Starting Wealth and to Comparative *n*th-Degree Risk Aversion

3.1 Random Starting Wealth

To allow for random starting wealth levels as Ross (1981) does, consider the following comparison:

$$
\tilde{w} \qquad \text{vs.} \qquad \begin{cases} \tilde{z} & \text{with probability } p \\ \tilde{y} & \text{with probability } 1-p \end{cases}, \tag{5}
$$

where $\tilde{w}$ is a random starting wealth, $\tilde{y}$ is a Rothschild and Stiglitz (1970) risk increase from $\tilde{w}$, and $\tilde{z}$ 1st-degree stochastically dominates $\tilde{w}$. Note that the change from (3) to (5) is two-fold. First, the certain starting wealth w is replaced with the random starting wealth $\tilde{w}$, and as a result, adding $\tilde{\varepsilon}$ to w is replaced with increasing the risk from $\tilde{w}$ to $\tilde{y}$. Second, adding $k > 0$ to

w is replaced with improving $\tilde{w}$ to $\tilde{z}$, a 1st-degree stochastically dominant change that includes adding $k > 0$ to $\tilde{w}$ as a special case.

Obviously, for any $u(x)$ with $u'(x) > 0$ and $u''(x) < 0$, there is a unique $p_u \in (0,1)$ – for any given $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$ described above – such that $u(x)$ is indifferent between the two sides of (5), or

$$Eu(\tilde{w}) = p_u Eu(\tilde{z}) + (1 - p_u) Eu(\tilde{y}) . \tag{6}$$

Similarly, one can define p_v for $v(x)$. Comparing p_u and p_v leads to the following theorem that is a special case of Theorem 3, which is proved in the next subsection.

Theorem 2. $p_u \geq p_v$ for all $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$ such that $\tilde{y}$ is more risky than $\tilde{w}$ and $\tilde{z}$ 1st-degree stochastically dominates $\tilde{w}$ if and only if $u(x)$ is Ross more risk averse than $v(x)$.

3.2 The Probability Premium for Measuring *n*th-Degree Risk Aversion ($n \geq 2$)

The comparison in (5) and the probability premium defined in (6) for 2nd-degree risk aversion can be easily generalized to provide a measuring stick for *n*th-degree risk aversion. We begin with a review of the definitions of *i*th-degree risk aversion and *i*th-degree risk increases, where $i \geq 1$.

Let $F(x)$ and $G(x)$ represent the cumulative distribution functions (CDFs) of two random variables whose supports are contained in a finite interval denoted $[a, b]$ with no probability mass at point a. This implies that $F(a) = G(a) = 0$ and $F(b) = G(b) = 1$. Letting $F^{[1]}(x)$ denote $F(x)$, higher order cumulative functions are defined according to $F^{[k]}(x) = \int_a^x F^{[k-1]}(y)dy$, $k =$

2,3,…. Similar notation applies to $G(x)$ and other CDFs. For any utility function $u(x)$: $[a, b] \rightarrow R^1$, assume that $u \in C^{\infty}$. Denote by $u^{(k)}(x)$ the kth derivative of $u(x)$, $k = 1, 2, 3...$.

For any integer $i \geq 1$, Ekern (1980) gives the following definition.

<u>Definition 2.</u>

(i) Decision maker $u(x)$ is ith-degree risk averse if

$$(-1)^{i+1}u^{(i)}(x) > 0 \qquad \text{for all } x \text{ in } [a, b]. \tag{7}$$

(ii) $G(x)$ has more ith-degree risk than $F(x)$ if

$$G^{[k]}(b) = F^{[k]}(b) \qquad \text{for } k = 1, 2, \ldots, i, \text{ and} \tag{8}$$

$$G^{[i]}(x) \geq F^{[i]}(x) \qquad \text{for all } x \text{ in } [a, b] \text{ with "}>\text{" holding for some } x \text{ in } (a, b). \tag{9}$$

Note that $u(x)$ is said to be ith-degree *weakly* risk averse when the strict inequality in (7) is replaced with a weak one. Note also that the 1st-degree risk increase is the same as the 1st-degree stochastically dominated change, and for $i \geq 2$, the ith-degree risk increase is a special case of the ith-degree stochastically dominated change where the kth moment is kept the same for all $1 \leq k < i$.[6] The relationship between the two concepts in Definition 2 is given in Lemma 1 below that is proved by Ekern.

<u>Lemma 1.</u> $G(x)$ has more ith-degree risk than $F(x)$ if and only if every ith-degree risk averse decision maker prefers $F(x)$ to $G(x)$.

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[6] Ekern's definition of ith-degree risk increase includes the Rothschild and Stiglitz risk increase (corresponding to $i = 2$) and the downside risk increase due to Menezes et al. (1980) (corresponding to $i = 3$) as special cases.

Now assume that *m* and *n* are any two positive integers such that $n > m \geq 1$, and consider the following comparison:

$$\tilde{w} \qquad \text{vs.} \qquad \begin{cases} \tilde{z} & \text{with probability } p \\ \tilde{y} & \text{with probability } 1-p \end{cases}, \tag{5'}$$

where $\tilde{w}$ is a random starting wealth, $\tilde{y}$ is an *n*th-degree risk increase from $\tilde{w}$, and $\tilde{z}$ is an *m*th-degree risk *decrease* from $\tilde{w}$. Obviously, for any $u(x)$ that is both *m*th-degree risk averse and *n*th-degree risk averse, there is a unique $p_u \in (0,1)$ – for any given $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$ such that $\tilde{y}$ is an *n*th-degree risk increase from $\tilde{w}$, and $\tilde{z}$ is an *m*th-degree risk decrease from $\tilde{w}$ – so that $u(x)$ is indifferent between the two sides of (5'), or

$$Eu(\tilde{w}) = p_u Eu(\tilde{z}) + (1 - p_u) Eu(\tilde{y}) . \tag{6'}$$

Similarly, one can define p_v for $v(x)$. Comparing p_u and p_v leads to Theorem 3 below. Before stating the theorem, we first give the following definition and lemma.

Continuing to assume that *m* and *n* are any two positive integers such that $n > m \geq 1$, let the two utility functions $u(x)$ and $v(x)$ each be both *n*th-degree and *m*th-degree risk averse on [*a*, *b*]. The following definition of (*n/m*)th-degree Ross more risk aversion from Liu and Meyer (2013) includes the existing definition of *n*th-degree Ross more risk aversion as a special case in which *m* = 1.[7]

<u>Definition 3</u>. $u(x)$ is (*n/m*)th-degree Ross more risk averse than $v(x)$ on [*a*, *b*] if

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[7] The *n*th-degree Ross more risk aversion studied in Jindapon and Neilson (2007), Li (2009) and Denuit and Eeckhoudt (2010) generalizes the well-known Ross more risk aversion (*n* = 2) in Ross (1981) and the Ross more downside risk aversion (*n* = 3) in Modica and Scarsini (2005).

$$\frac{(-1)^{n-1}u^{(n)}(x)}{(-1)^{m-1}u^{(m)}(y)} \geq \frac{(-1)^{n-1}v^{(n)}(x)}{(-1)^{m-1}v^{(m)}(y)} \qquad \text{for all } x, y \in [a, b], \tag{10}$$

or equivalently, if there exists $\lambda > 0$, such that $\frac{u^{(n)}(x)}{v^{(n)}(x)} \geq \lambda \geq \frac{u^{(m)}(y)}{v^{(m)}(y)}$ for all $x, y \in [a, b]$.

The following lemmas regarding the (*n*/*m*)th-degree Ross more risk averse condition will be used in proving the main results in the paper.

Lemma 2. $u(x)$ is (*n*/*m*)th-degree Ross more risk averse than $v(x)$ on $[a, b]$ if and only if there exist $\lambda > 0$ and $\phi(x)$ with $(-1)^{m+1}\phi^{(m)}(x) \leq 0$ and $(-1)^{n+1}\phi^{(n)}(x) \geq 0$ for all x in $[a, b]$ such that $u(x) \equiv \lambda v(x) + \phi(x)$.

Lemma 3. If $u(x)$ is NOT (*n*/*m*)th-degree Ross more risk averse than $v(x)$ on $[a, b]$, then there exist $\mu > 0$, $[a_1, b_1] \subset (a, b)$ and $[a_2, b_2] \subset (a, b)$ such that $\phi(x) \equiv u(x) - \mu v(x)$ satisfies

$$(-1)^{n+1}\phi^{(n)}(x) < 0 \qquad \text{for all x} \in [a_1, b_1]$$
$$(-1)^{m+1}\phi^{(m)}(x) > 0 \qquad \text{for all x} \in [a_2, b_2]$$

Proof: The proof of Lemma 2 is straightforward, and the proof of Lemma 3 is as follows. If $u(x)$ is NOT (*n*/*m*)th-degree Ross more risk averse than $v(x)$ on $[a, b]$, then there exist some y and z $\in [a, b]$ and $\mu > 0$, such that

$$\frac{u^{(n)}(y)}{v^{(n)}(y)} < \mu < \frac{u^{(m)}(z)}{v^{(m)}(z)} ,$$

which implies, due to continuity, that there exist $[a_1, b_1] \subset (a, b)$ and $[a_2, b_2] \subset (a, b)$ such that

$$\frac{u^{(n)}(y)}{v^{(n)}(y)} < \mu < \frac{u^{(m)}(z)}{v^{(m)}(z)} \quad \text{for all } y \in [a_1, b_1] \text{ and all } z \in [a_2, b_2].$$

Define $\phi(x) \equiv u(x) - \mu v(x)$. Differentiating yields

$$(-1)^{n+1}\phi^{(n)}(x) = (-1)^{n+1}u^{(n)}(x) - \mu(-1)^{n+1}v^{(n)}(x) < 0 \qquad \text{for all x} \in [a_1, b_1]$$
$$(-1)^{m+1}\phi^{(m)}(x) = (-1)^{m+1}u^{(m)}(x) - \mu(-1)^{m+1}v^{(m)}(x) > 0 \qquad \text{for all x} \in [a_2, b_2]$$

Q.E.D.

We are now ready to state and prove the following theorem that includes Theorem 2 as a special case in which *n* = 2 and *m* = 1.

Theorem 3. $p_u \geq p_v$ for all $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$ such that $\tilde{y}$ is *n*th-degree more risky than $\tilde{w}$ and $\tilde{z}$ is *m*th-degree less risky than $\tilde{w}$ if and only if *u*(*x*) is (*n*/*m*)th-degree Ross more risk averse than *v*(*x*).

Proof: For any given $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$ such that $\tilde{y}$ is *n*th-degree more risky than $\tilde{w}$ and $\tilde{z}$ is *m*th-degree less risky than $\tilde{w}$, define

$$U(p) \equiv Eu(\tilde{w}) - [pEu(\tilde{z}) + (1-p)Eu(\tilde{y})]. \qquad (11)$$

Clearly, $U'(p) = Eu(\tilde{y}) - Eu(\tilde{z}) < 0$ because u(x) is both *m*th-degree risk averse and *n*th-degree risk averse. By construction, $U(p_u) = 0$, where $p_u \in (0,1)$ is given by (6'). $V(p)$ for v(x) can be similarly defined, and $V(p_v) = 0$.

"if"

Suppose that *u*(*x*) is (*n*/*m*)th-degree Ross more risk averse than *v*(*x*). Then, from Lemma 2, there exist $\lambda > 0$ and $\phi(x)$ with $(-1)^{m+1}\phi^{(m)}(x) \leq 0$ and $(-1)^{n+1}\phi^{(n)}(x) \geq 0$ for all *x* in [*a*, *b*] such that $u(x) \equiv \lambda v(x) + \phi(x)$. Note that $\phi(x)$ is *m*th-degree weakly risk tolerant and *n*th-degree weakly risk averse.

Evaluating $U(p)$ at p_v, we have

$$\begin{aligned} U(p_v) &= \lambda V(p_v) + E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\ &= E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\ &= p_v[E\phi(\tilde{w}) - E\phi(\tilde{z})] + (1-p_v)[E\phi(\tilde{w}) - E\phi(\tilde{y})] \\ &\geq 0. \end{aligned} \tag{12}$$

The inequality in (12) is from that $\phi(x)$ is weakly *m*th-degree risk tolerant and weakly *n*th-degree risk averse. Because $U(p)$ is strictly decreasing in *p*, we have $p_u \geq p_v$.

"only if"

Suppose that $p_u \geq p_v$ for all $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$ such that $\tilde{y}$ is *n*th-degree more risky than $\tilde{w}$ and $\tilde{z}$ is *m*th-degree less risky than $\tilde{w}$. To prove that *u*(*x*) is (*n*/*m*)th-degree Ross more risk averse than *v*(*x*), assume otherwise. Then, according to Lemma 3, there exist $\mu > 0$, $[a_1, b_1] \subset (a, b)$ and $[a_2, b_2] \subset (a, b)$, such that $\phi(x) \equiv u(x) - \mu v(x)$ satisfies

$$\begin{aligned} &(-1)^{n+1}\phi^{(n)}(x) < 0 \qquad \text{for all } x \in [a_1, b_1] \\ &(-1)^{m+1}\phi^{(m)}(x) > 0 \qquad \text{for all } x \in [a_2, b_2] \end{aligned} \tag{13}$$

Now denote the CDFs for $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$ as *F*(*x*), *G*(*x*) and *H*(*x*), respectively, and choose *F*(*x*), *G*(*x*) and *H*(*x*) such that

$$\begin{cases} G^{[n]} - F^{[n]} > 0 & x \in (a_1, b_1) \\ G^{[n]} - F^{[n]} = 0 & x \notin (a_1, b_1) \end{cases} \qquad \begin{cases} F^{[m]} - H^{[m]} > 0 & x \in (a_2, b_2) \\ F^{[m]} - H^{[m]} = 0 & x \notin (a_2, b_2) \end{cases}. \tag{14}$$

Evaluating $U(p)$ at p_v, we have

$$
\begin{aligned}
U(p_v) &= \mu V(p_v) + E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\
&= E\phi(\tilde{w}) - [p_v E\phi(\tilde{z}) + (1-p_v)E\phi(\tilde{y})] \\
&= p_v[E\phi(\tilde{w}) - E\phi(\tilde{z})] + (1-p_v)[E\phi(\tilde{w}) - E\phi(\tilde{y})] \\
&= p_v \int_a^b \phi(x) d[F(x) - H(x)] + (1-p_v)\int_a^b \phi(x) d[F(x) - G(x)] \\
&= p_v \int_a^b (-1)^{m+1}\phi^{(m)}(x)[H^{[m]}(x) - F^{[m]}(x)]dx + (1-p_v)\int_a^b (-1)^{n+1}\phi^{(n)}(x)[G^{[n]}(x) - F^{[n]}(x)]dx \\
&= p_v \int_{a_2}^{b_2} (-1)^{m+1}\phi^{(m)}(x)[H^{[m]}(x) - F^{[m]}(x)]dx + (1-p_v)\int_{a_1}^{b_1} (-1)^{n+1}\phi^{(n)}(x)[G^{[n]}(x) - F^{[n]}(x)]dx \\
&< 0.
\end{aligned} \tag{15}
$$

The inequality in (15) is from (13) and (14). Because $U(p)$ is strictly decreasing in *p*, we have $p_u < p_v$, a contradiction. Therefore, *u*(*x*) must be (*n*/*m*)th-degree Ross more risk averse than *v*(*x*).

Q.E.D.

The thought experiment behind Theorem 3 is the comparison between an initial wealth $\tilde{w}$ and a compound binary lottery that results in $\tilde{z}$ with probability *p* and $\tilde{y}$ with probability 1 – *p*, where the bad-state lottery $\tilde{y}$ differs from $\tilde{w}$ by an *n*th-degree increase in risk and the good-state lottery $\tilde{z}$ differs from $\tilde{w}$ by an *m*th-degree *reduction* in risk, with $n > m \geq 1$. An inquiry arises as to what would happen if the good-state lottery $\tilde{z}$ differs from $\tilde{w}$ by a (more general) *m*th-degree stochastically dominant change.[8] Following the steps in the proof of Theorem 3, one can readily prove the following result.

[8] As is well known from Pratt (1964) and Ross (1981), risk aversion is about the attitudes towards (2nd-degree) increases in risk, not about the attitudes towards 1st- or 2nd-degree stochastically dominated changes. So, in order to have a meaningful analysis of comparative *n*th-degree risk aversion, we must maintain that the bad-state lottery $\tilde{y}$ differs from $\tilde{w}$ by an *n*th-degree increase in risk.

Corollary 1. $p_u \geq p_v$ for all $\tilde{w}$, $\tilde{y}$ and $\tilde{z}$ such that $\tilde{y}$ is *n*th-degree more risky than $\tilde{w}$ and $\tilde{z}$ *m*th-degree stochastically dominates $\tilde{w}$ if and only if *u*(*x*) is (*n*/*k*)th-degree Ross more risk averse than *v*(*x*) for all integer *k* such that $m \geq k \geq 1$.

4. An Alternative Probability Premium Approach Based on Risk Apportionment

In the framework of expected utility, *n*th-degree risk aversion, i.e., aversion to *n*th-degree risk increases, is characterized by a positive $(-1)^{n+1}u^{(n)}(x)$ for all *x*, where $u^{(n)}(x)$ is the *n*th derivative of utility function *u*(*x*) (Ekern 1980). Eeckhoudt and Schlesinger (2006) and Eeckhoudt et al. (2009) further establish that $(-1)^{n+1}u^{(n)}(x)$ being positive for all *x* can be characterized by preferences over 50-50 lotteries that display a preference for combining "good" with "bad" over combining "good" with "good" and "bad" with "bad," where "good" is a risk reduction of various degrees from "bad." In other words, a decision maker's "direction" in *n*th-degree risk attitude can be explained by risk apportionment.

In this section we examine an alternative probability premium approach to comparative risk aversion that is based on risk apportionment. We define a probability premium as the "strength" measure of *n*th-degree risk aversion, and show that interpersonal comparison of this probability premium is also governed by (*n*/*m*)th-degree Ross more risk aversion.[9]

[9] The risk apportionment approach of Eeckhoudt and Schlesinger (2006) to *characterizing* *n*th-degree risk aversion has been extended to characterizing *n*th-degree risk aversion for multiattribute preferences (Tsetlin and Winkler 2009), to characterizing *n*th-degree risk aversion in non-EU models (Eeckhoudt et al. 2015), and to characterizing specific changes in mutual risk aggravation (Ebert et al. 2015). In contrast, our paper provides the first systemic approach to *measuring* *n*th-degree risk aversion that is based on the risk apportionment framework.

4.1. The General Results

Again, let n and m be two integers such that $n > m \geq 1$. Suppose that $\tilde{y}_j$ has more jth-degree risk than $\tilde{x}_j$ ($j = m$, n-m), and that the random variables with subscript m are independent of those with subscript n-m. For individuals who are jth-degree risk averse, therefore, $\tilde{x}_j$ is "relatively good" and $\tilde{y}_j$ is "relatively bad".

In the following theorem that generalizes the results of Eeckhoudt and Schlesinger (2006), Eeckhoudt et al. (2009) show that all nth-degree risk averse individuals would prefer combining the relatively good with the relatively bad in a 50-50 lottery to combining the relatively good with the relatively good and the relatively bad with the relatively bad.[10]

Theorem 4. (Eeckhoudt et al. 2009). Suppose that $\tilde{y}_j$ has more jth-degree risk than $\tilde{x}_j$ for $j = m$, n-m. The 50-50 lottery [$\tilde{x}_m + \tilde{x}_{n-m}$, $\tilde{y}_m + \tilde{y}_{n-m}$] has more nth-degree risk than the 50-50 lottery [$\tilde{x}_m + \tilde{y}_{n-m}$, $\tilde{y}_m + \tilde{x}_{n-m}$].

According to Theorem 4, $u(x)$ being nth-degree risk averse implies that, for any given m such that $n > m \geq 1$,

$$\frac{1}{2}Eu(\tilde{x}_m + \tilde{y}_{n-m}) + \frac{1}{2}Eu(\tilde{y}_m + \tilde{x}_{n-m}) > \frac{1}{2}Eu(\tilde{x}_m + \tilde{x}_{n-m}) + \frac{1}{2}Eu(\tilde{y}_m + \tilde{y}_{n-m}) \tag{16}$$

for all $\tilde{x}_j$ and $\tilde{y}_j$ ($j = m$, n-m) such that $\tilde{y}_j$ has more jth-degree risk than $\tilde{x}_j$.

[10] Eeckhoudt et al. (2009) present theorems both for the case where the relatively bad is an nth-degree risk increase from the relatively good, and for the case where the relatively bad is nth-degree stochastically dominated by the relatively good. For the purpose of the present paper, we only need to consider the case of risk increases.

Based on (16), we can define the "*m*th probability premium of *n*th-degree risk aversion" for $u(x)$, denoted $q_u^{n/m}$. In addition to assuming that $u(x)$ is *n*th-degree risk averse so that inequality (16) holds, it is also assumed that $u(x)$ is both *m*th-degree risk averse and (*n*-*m*)th-degree risk averse.[11] These assumptions imply that $\tilde{x}_m + \tilde{x}_{n-m}$ is preferred to $\tilde{y}_m + \tilde{x}_{n-m}$ and $\tilde{x}_m + \tilde{y}_{n-m}$, both of which are in turn preferred to $\tilde{y}_m + \tilde{y}_{n-m}$.

<u>Definition 4</u>. The *m*th probability premium of *n*th-degree risk aversion for $u(x)$, denoted $q_u^{n/m}$, is the solution to

$$\begin{aligned}&\left(\tfrac{1}{2} - q\right) Eu(\tilde{x}_m + \tilde{y}_{n-m}) + \left(\tfrac{1}{2} + q\right) Eu(\tilde{y}_m + \tilde{x}_{n-m}) \\ &= \left(\tfrac{1}{2} + q\right) Eu(\tilde{x}_m + \tilde{x}_{n-m}) + \left(\tfrac{1}{2} - q\right) Eu(\tilde{y}_m + \tilde{y}_{n-m})\end{aligned} \tag{17}$$

It can be easily seen that there is a unique $q_u^{n/m} \in (0, \tfrac{1}{2})$ satisfying (17), given $\tilde{x}_j$ and $\tilde{y}_j$ ($j = m, n-m$) such that $\tilde{y}_j$ has more *j*th-degree risk than $\tilde{x}_j$.

A question immediately arises about whether $q_u^{n/m}$ is the only way to think about a probability premium as there seem to be alternative ways to define one based on (16) by using, instead of (17), the following equations:

$$\begin{aligned}&\left(\tfrac{1}{2} + q\right) Eu(\tilde{x}_m + \tilde{y}_{n-m}) + \left(\tfrac{1}{2} - q\right) Eu(\tilde{y}_m + \tilde{x}_{n-m}) \\ &= \left(\tfrac{1}{2} + q\right) Eu(\tilde{x}_m + \tilde{x}_{n-m}) + \left(\tfrac{1}{2} - q\right) Eu(\tilde{y}_m + \tilde{y}_{n-m})\end{aligned} \tag{17'}$$

or

[11] This parallels the assumption of $u'(x) > 0$ in defining the probability premium for the regular 2nd-degree risk aversion.

$$\frac{1}{2}Eu(\tilde{x}_m+\tilde{y}_{n-m})+\frac{1}{2}Eu(\tilde{y}_m+\tilde{x}_{n-m}) \\ =\left(\tfrac{1}{2}+q\right)Eu(\tilde{x}_m+\tilde{x}_{n-m})+\left(\tfrac{1}{2}-q\right)Eu(\tilde{y}_m+\tilde{y}_{n-m}) \tag{17”}$$

Indeed, we have the following Definition 5 based on (17”). On the other hand, the *q* that is determined by (17’) is simply $q_u^{n/(n-m)}$, a concept already defined in Definition 4.

<u>Definition 5.</u> The consolidated *m*th probability premium of *n*th-degree risk aversion for *u*(*x*), denoted $\overline{q}_u^{n/m}$, is the solution to (17”).

It can be easily seen that there is a unique $\overline{q}_u^{n/m} \in (0,\frac{1}{2})$ satisfying (17”), given $\tilde{x}_j$ and $\tilde{y}_j$ (*j* = *m*, *n*-*m*) such that $\tilde{y}_j$ has more *j*th-degree risk than $\tilde{x}_j$.

The following theorem shows that interpersonal comparison of $q_u^{n/m}$ is governed by (*n*/*m*)th-degree Ross more risk aversion.

<u>Theorem 5.</u> $q_u^{n/m} \geq q_v^{n/m}$ for all $\tilde{x}_j$ and $\tilde{y}_j$ (*j* = *m*, *n*-*m*) such that $\tilde{y}_j$ has more *j*th-degree risk than $\tilde{x}_j$ if and only if *u*(*x*) is (*n/m*)th-degree Ross more risk averse than *v*(*x*).

<u>Proof:</u> For any given $\tilde{x}_j$ and $\tilde{y}_j$ (*j* = *m*, *n*-*m*) such that $\tilde{y}_j$ has more *j*th-degree risk than $\tilde{x}_j$, define

$$\Gamma(q;u)\equiv\left(\tfrac{1}{2}-q\right)Eu(\tilde{x}_m+\tilde{y}_{n-m})+\left(\tfrac{1}{2}+q\right)Eu(\tilde{y}_m+\tilde{x}_{n-m}) \\ -\left\{\left(\tfrac{1}{2}+q\right)Eu(\tilde{x}_m+\tilde{x}_{n-m})+\left(\tfrac{1}{2}-q\right)Eu(\tilde{y}_m+\tilde{y}_{n-m})\right\}$$

Differentiating with respect to *q* yields

$$\begin{aligned}\Gamma'(q;u) &= \left[Eu\left(\tilde{y}_m+\tilde{x}_{n-m}\right)-Eu\left(\tilde{x}_m+\tilde{x}_{n-m}\right)\right] \\ &\quad +\left[Eu\left(\tilde{y}_m+\tilde{y}_{n-m}\right)-Eu\left(\tilde{x}_m+\tilde{y}_{n-m}\right)\right] \\ &< 0.\end{aligned}$$

Both bracketed terms above are negative because $\tilde{y}_m$ has more *m*th-degree risk than $\tilde{x}_m$ and the individual is *m*th-degree risk averse. Similarly, $\Gamma(q;v)$ can be defined and it can be showed that $\Gamma'(q;v)<0$. By construction, $\Gamma(q_u^{n/m};u)=0$ and $\Gamma(q_v^{n/m};v)=0$. According to Theorem 4, $\Gamma(0;u)>0$ and $\Gamma(0;v)>0$. Therefore, $q_u^{n/m}>0$ and $q_v^{n/m}>0$.

“if”

According to Lemma 2, if *u*(*x*) is (*n*/*m*)th-degree Ross more risk averse than *v*(*x*), then there exist $\lambda>0$ and $\phi(x)$ with $(-1)^{m+1}\phi^{(m)}(x)\le 0$ and $(-1)^{n+1}\phi^{(n)}(x)\ge 0$ for all x in [a, b] such that $u(x)\equiv\lambda v(x)+\phi(x)$. Note that $\phi(x)$ is *m*th-degree weakly risk tolerant and *n*th-degree weakly risk averse.

Evaluating at $q_v^{n/m}$ yields

$$\begin{aligned}&\Gamma(q_v^{n/m};u)=\lambda\Gamma(q_v^{n/m};v)\\ &+\left(\tfrac{1}{2}-q_v^{n/m}\right)E\phi\left(\tilde{x}_m+\tilde{y}_{n-m}\right)+\left(\tfrac{1}{2}+q_v^{n/m}\right)E\phi\left(\tilde{y}_m+\tilde{x}_{n-m}\right)-\left[\left(\tfrac{1}{2}+q_v^{n/m}\right)E\phi\left(\tilde{x}_m+\tilde{x}_{n-m}\right)+\left(\tfrac{1}{2}-q_v^{n/m}\right)E\phi\left(\tilde{y}_m+\tilde{y}_{n-m}\right)\right]\\ &\qquad=\left(\tfrac{1}{2}-q_v^{n/m}\right)E\phi\left(\tilde{x}_m+\tilde{y}_{n-m}\right)+\left(\tfrac{1}{2}+q_v^{n/m}\right)E\phi\left(\tilde{y}_m+\tilde{x}_{n-m}\right)-\left[\left(\tfrac{1}{2}+q_v^{n/m}\right)E\phi\left(\tilde{x}_m+\tilde{x}_{n-m}\right)+\left(\tfrac{1}{2}-q_v^{n/m}\right)E\phi\left(\tilde{y}_m+\tilde{y}_{n-m}\right)\right]\\ &\qquad=\left\{\tfrac{1}{2}E\phi\left(\tilde{x}_m+\tilde{y}_{n-m}\right)+\tfrac{1}{2}E\phi\left(\tilde{y}_m+\tilde{x}_{n-m}\right)-\left[\tfrac{1}{2}E\phi\left(\tilde{x}_m+\tilde{x}_{n-m}\right)+\tfrac{1}{2}E\phi\left(\tilde{y}_m+\tilde{y}_{n-m}\right)\right]\right\}\\ &+q_v^{n/m}\left\{[E\phi\left(\tilde{y}_m+\tilde{x}_{n-m}\right)-E\phi\left(\tilde{x}_m+\tilde{x}_{n-m}\right)]+[E\phi\left(\tilde{y}_m+\tilde{y}_{n-m}\right)-E\phi\left(\tilde{x}_m+\tilde{y}_{n-m}\right)]\right\}\\ &\qquad\ge 0\end{aligned}$$

The week inequality above holds because both braced terms are nonnegative. The first braced term is nonnegative because $\phi(x)$ is *n*th-degree weakly risk averse (Theorem 4), and the second braced term is nonnegative because $\phi(x)$ is *m*th-degree weakly risk tolerant.

Due to $\Gamma'(q;u) < 0$, therefore, we have $q_u^{n/m} \geq q_v^{n/m}$.

"only if"

Suppose that $q_u^{n/m} \geq q_v^{n/m}$ for all $\tilde{x}_j$ and $\tilde{y}_j$ ($j = m$, n-m) such that $\tilde{y}_j$ has more jth-degree risk than $\tilde{x}_j$ but $u(x)$ is NOT (n/m)th-degree Ross more risk averse than $v(x)$. Then, according to Lemma 3, there exist $\mu > 0$, $[a_1, b_1] \subset (a, b)$ and $[a_2, b_2] \subset (a, b)$ such that $\phi(x) \equiv u(x) - \mu v(x)$ satisfies

$$\begin{aligned} &(-1)^{n+1}\phi^{(n)}(x) < 0 \qquad \text{for all x} \in [a_1, b_1] \\ &(-1)^{m+1}\phi^{(m)}(x) > 0 \qquad \text{for all x} \in [a_2, b_2] \end{aligned} \tag{18}$$

Evaluating at $q_v^{n/m}$, we have

$$\begin{aligned} \Gamma(q_v^{n/m};u) = \mu\Gamma(q_v^{n/m};v) &+ \left(\tfrac{1}{2} - q_v^{n/m}\right)E\phi\left(\tilde{x}_m + \tilde{y}_{n-m}\right) + \left(\tfrac{1}{2} + q_v^{n/m}\right)E\phi\left(\tilde{y}_m + \tilde{x}_{n-m}\right) \\ &- \left[\left(\tfrac{1}{2} + q_v^{n/m}\right)E\phi\left(\tilde{x}_m + \tilde{x}_{n-m}\right) + \left(\tfrac{1}{2} - q_v^{n/m}\right)E\phi\left(\tilde{y}_m + \tilde{y}_{n-m}\right)\right] \\ &= \left(\tfrac{1}{2} - q_v^{n/m}\right)E\phi\left(\tilde{x}_m + \tilde{y}_{n-m}\right) + \left(\tfrac{1}{2} + q_v^{n/m}\right)E\phi\left(\tilde{y}_m + \tilde{x}_{n-m}\right) \\ &- \left[\left(\tfrac{1}{2} + q_v^{n/m}\right)E\phi\left(\tilde{x}_m + \tilde{x}_{n-m}\right) + \left(\tfrac{1}{2} - q_v^{n/m}\right)E\phi\left(\tilde{y}_m + \tilde{y}_{n-m}\right)\right] \\ &= \left\{\tfrac{1}{2}E\phi\left(\tilde{x}_m + \tilde{y}_{n-m}\right) + \tfrac{1}{2}E\phi\left(\tilde{y}_m + \tilde{x}_{n-m}\right) - \left[\tfrac{1}{2}E\phi\left(\tilde{x}_m + \tilde{x}_{n-m}\right) + \tfrac{1}{2}E\phi\left(\tilde{y}_m + \tilde{y}_{n-m}\right)\right]\right\} \\ &+ q_v^{n/m}\left\{[E\phi\left(\tilde{y}_m + \tilde{x}_{n-m}\right) - E\phi\left(\tilde{x}_m + \tilde{x}_{n-m}\right)] + [E\phi\left(\tilde{y}_m + \tilde{y}_{n-m}\right) - E\phi\left(\tilde{x}_m + \tilde{y}_{n-m}\right)]\right\}, \end{aligned} \tag{19}$$

where $\phi(x)$ satisfies (18).

Applying the same technique used in the proof of Theorem 3, choose $\tilde{x}_j$ and $\tilde{y}_j$ ($j = m$, n-m) appropriately so that the first braced term in the last expression of (19) is negative due to $(-1)^{n+1}\phi^{(n)}(x) < 0$ for all x $\in [a_1, b_1]$, and the second braced term is also negative due to $(-1)^{m+1}\phi^{(m)}(x) > 0$ for all x $\in [a_2, b_2]$.

Therefore, for such specially chosen $\tilde{x}_j$ and $\tilde{y}_j$ (j = m, n-m) , we have $\Gamma(q_v^{n/m};u) < 0$, which, together with $\Gamma'(q;u) < 0$, implies $q_u^{n/m} < q_v^{n/m}$. This contradicts the original condition that $q_u^{n/m} \geq q_v^{n/m}$ for all $\tilde{x}_j$ and $\tilde{y}_j$ (j = m, n-m) such that $\tilde{y}_j$ has more jth-degree risk than $\tilde{x}_j$. Therefore, $u(x)$ must be (n/m)th-degree Ross more risk averse than $v(x)$.

Q.E.D.

A similar result holds for $\overline{q}_u^{n/m}$ as stated in Theorem 6, which can be similarly proved following the steps in the proof of Theorem 5. Just note that Lemma 2 can be extended so that "$u(x)$ is both (n/m)th-degree and ($n/(n$-$m)$)th-degree Ross more risk averse than $v(x)$ on [a, b] if and only if there exist λ > 0 and $\phi(x)$ with $(-1)^{m+1}\phi^{(m)}(x) \leq 0$, $(-1)^{n-m+1}\phi^{(n-m)}(x) \leq 0$ and $(-1)^{n+1}\phi^{(n)}(x) \geq 0$ for all x in $[a, b]$ such that $u(x) \equiv \lambda v(x) + \phi(x)$".

Theorem 6. $\overline{q}_u^{n/m} \geq \overline{q}_v^{n/m}$ for all $\tilde{x}_j$ and $\tilde{y}_j$ (j = m, n-m) such that $\tilde{y}_j$ has more jth-degree risk than $\tilde{x}_j$ if and only if $u(x)$ is both (n/m)th-degree and ($n/(n$-$m)$)th-degree Ross more risk averse than $v(x)$.

4.2. Measuring Downside Risk Aversion Using Various Versions of Probability Premiums

As an interesting special case, consider various versions of probability premia for downside risk aversion in the risk apportionment framework of Eeckhoudt and Schlesinger (2006). Compare two 50-50 lotteries $X = [w-k, w+\tilde{\varepsilon}]$ and $Y = [w, w-k+\tilde{\varepsilon}]$, where $k > 0$ and

$\tilde{\varepsilon}$ is a nondegenerate zero-mean risk. Eeckhoudt and Schlesinger (2006) establish that *X* is preferred to *Y* by *u*(*x*), or

$$\frac{1}{2}u(w-k)+\frac{1}{2}Eu(w+\tilde{\varepsilon})>\frac{1}{2}u(w)+\frac{1}{2}Eu(w-k+\tilde{\varepsilon}), \tag{20}$$

for all $w, k,$ and $\tilde{\varepsilon}$ if and only if $u'''>0$. According to Eeckhoudt and Schlesinger, inequality (20) holds because in lottery *X*, the two "bads," (–*k*) and $\tilde{\varepsilon}$, are disaggregated into two separate states of nature, whereas in lottery *Y* they are combined into a single state of nature. Note that *Y* has more downside risk than *X* because the pure risk $\tilde{\varepsilon}$ is placed at *w* in *X* and at *w* – *k* in *Y*. So inequality (20) is also a statement of downside risk aversion.

Based on (20), we can naturally define three alternative versions of probability premia for downside risk aversion.

<u>Definition 6</u>. The type-*i* probability premium for downside risk aversion, q_u^i, *i* = I, II or III, is defined by the respective equation in the following:

$$\left(\frac{1}{2}+q_u^I\right)u(w-k)+\left(\frac{1}{2}-q_u^I\right)Eu(w+\tilde{\varepsilon})=\left(\frac{1}{2}+q_u^I\right)u(w)+\left(\frac{1}{2}-q_u^I\right)Eu(w-k+\tilde{\varepsilon}) \tag{21.1}$$

$$\left(\frac{1}{2}-q_u^{II}\right)u(w-k)+\left(\frac{1}{2}+q_u^{II}\right)Eu(w+\tilde{\varepsilon})=\left(\frac{1}{2}+q_u^{II}\right)u(w)+\left(\frac{1}{2}-q_u^{II}\right)Eu(w-k+\tilde{\varepsilon}) \tag{21.2}$$

$$\frac{1}{2}u(w-k)+\frac{1}{2}Eu(w+\tilde{\varepsilon})=\left(\frac{1}{2}+q_u^{III}\right)u(w)+\left(\frac{1}{2}-q_u^{III}\right)Eu(w-k+\tilde{\varepsilon}) \tag{21.3}$$

Given *u*(*x*) with $u'>0$, $u''<0$ and $u'''>0$, it is easy to see that for all *w*, *k* and $\tilde{\varepsilon}$, there is a unique $q_u^i \in (0,1/2)$, *i* = I, II or III, such that the respective equality holds.

Intuitively, $q_u^i \geq q_v^i$, i = I, II or III, for all $w, k,$ and $\tilde{\varepsilon}$ can be used to characterize u being more downside risk averse than v. q_u^{II} and q_u^{III} have precedents in the literature. q_u^{III} is first proposed by Jindapon (2010), and q_u^{II} by Watt (2011) as an alternative to q_u^{III}. Jindapon (2010) identifies a sufficient condition for $q_u^{III} \geq q_v^{III}$, but his sufficient condition is not on the utility functions alone because it also depends on $\tilde{\varepsilon}$. Watt's (2011) sufficient condition for $q_u^{II} \geq q_v^{II}$ is incomplete for the same reason.

By appropriately identifying the terms in (21.1) – (21.3) according to Definitions 4 and 5, it can be easily seen that q_u^I q_u^{II} and q_u^{III} are, respectively, $q_u^{3/1}$ $q_u^{3/2}$ and $\overline{q}_u^{3/1}$. Then, according to Theorems 5 and 6, we immediately obtain the following corollary indicating that various 3rd-degree Ross more risk aversion conditions are both sufficient and necessary for the interpersonal comparison of these alternative probability premiums of downside risk aversion.

Corollary 2.

(i) $q_u^I \geq q_v^I$ for all $w, k,$ and $\tilde{\varepsilon}$ if and only if $u(x)$ is (3/1)rd-degree Ross more risk averse than $v(x)$;

(ii) $q_u^{II} \geq q_v^{II}$ for all $w, k,$ and $\tilde{\varepsilon}$ if and only if $u(x)$ is (3/2)rd-degree Ross more risk averse than $v(x)$;

(iii) $q_u^{III} \geq q_v^{III}$ for all $w, k,$ and $\tilde{\varepsilon}$ if and only if $u(x)$ is both (3/1)rd- and (3/2)rd-degree Ross more risk averse than $v(x)$.

5. A Comparative Statics Approach

The third approach to characterizing (*n*/*m*)th-degree Ross more risk aversion comes from the comparative statics approach of Jindapon and Neilson (2007). Suppose that nature rolls the dice to reveal two states, with probability p for state 1 and $1-p$ for state 2. Also suppose that $\tilde{w}_i$ is the random wealth obtained in state i (i = 1, 2). Ex ante, an individual can expend some resource to improve the wealth distribution in each state. The individual optimally allocates a given total amount of resource between improving $\tilde{w}_1$ and $\tilde{w}_2$. Specifically, the individual with utility function $u(x)$ solves the following problem

$$\begin{aligned} &\max_{e_1,e_2} \quad pEu\left[\tilde{w}_1(e_1)\right]+(1-p)Eu\left[\tilde{w}_2(e_2)\right] \\ &s.t. \quad e_1+e_2=\overline{e} \end{aligned} \tag{22}$$

where $\tilde{w}_1(e_1)$ has less *m*th-degree risk as e_1 increases, and $\tilde{w}_2(e_2)$ has less *n*th-degree risk as e_2 increases. To make this more precise, let $F_i(x,e_i)$ be the distribution function for $\tilde{w}_i(e_i)$, with support in [*a*,*b*]. Let $F_{i,e}^{[k]}(x,e_i)$ be the partial derivative of $F_i^{[k]}(x,e_i)$ with respect to e_i for k = 1,2,… An increase in e_1 entails an *m*th-degree reduction in risk if $F_{1,e}^{[k]}(b,e_1)=0$ for $k = 1,2,\ldots,m$ and $F_{1,e}^{[m]}(x,e_1)\le 0$ for all $x\in\left[a,b\right]$. An increase in e_2 entails an *n*th-degree reduction in risk if $F_{2,e}^{[k]}(b,e_2)=0$ for $k = 1,2,\ldots,n$ and $F_{2,e}^{[n]}(x,e_2)\le 0$ for all $x\in\left[a,b\right]$.

Optimization problem (22) can be looked upon as a typical consumer choice problem over goods e_1 and e_2, both of which have a price of unity, or it could be looked at as a typical time-allocation or any other fixed resource-allocation problem. Substituting $e_2 = \bar{e} - e_1$ into the maximization problem yields

$$\max_{e_1} U(e_1) \equiv p\int_a^b u(x)dF_1(x, e_1) + (1-p)\int_a^b u(x)dF_2(x, \bar{e} - e_1)$$

The first-order condition is

$$U'(e_1) \equiv p\int_a^b u(x)dF_{1,e}(x, e_1) - (1-p)\int_a^b u(x)dF_{2,e}(x, \bar{e} - e_1) = 0.$$

Assume that $U''(e_1) < 0$ so that the maximization problem has a unique solution, and similarly that *v*'s objective function, $V(e_1)$, has $V''(e_1) < 0$.

We are interested in how individuals with different risk preferences would allocate the limited risk-reducing resource differently. The relevant comparative statics result is given in the following theorem, which says that a decision-maker will always exert less effort to reduce the *m*th-degree risk of the random wealth in state 1 and (hence) more effort to reduce the *n*th-degree risk of the random wealth in state 2 than another decision maker if and only if the former is (*n*/*m*)th-degree Ross more risk averse than the latter.

Theorem 7. Suppose that the unique optimal allocation for $u(x)$ is $\left(e_1^u, e_2^u\right)$ and the unique optimal allocation for $v(x)$ is $\left(e_1^v, e_2^v\right)$. Then $e_1^u \leq e_1^v$ and $e_2^u \geq e_2^v$ if and only if

$u(x)$ is (n/m)th-degree Ross more risk averse than $v(x)$.

Proof.

"if"

By Lemma 2 there exists $\lambda > 0$ and $\phi(x)$ such that $u(x) \equiv \lambda v(x) + \phi(x)$, where $(-1)^{m+1}\phi^{(m)}(x) \le 0$ and $(-1)^{n+1}\phi^{(n)}(x) \ge 0$ for all x. Evaluating $U'(e_1)$ at e_1^v yields

$$
\begin{aligned}
U'(e_1^v) &= p\int_a^b u(x)dF_{1,e}(x,e_1^v) - (1-p)\int_a^b u(x)dF_{2,e}(x,\overline{e}-e_1^v) \\
&= p\left[(-1)^m\int_a^b u^{(m)}(x)F_{1,e}^{[m]}(x,e_1^v)dx\right] - (1-p)\left[(-1)^n\int_a^b u^{(n)}(x)F_{2,e}^{[n]}(x,\overline{e}-e_1^v)dx\right] \\
&= \lambda\left\{p\left[(-1)^m\int_a^b v^{(m)}(x)F_{1,e}^{[m]}(x,e_1^v)dx\right] - (1-p)\left[(-1)^n\int_a^b v^{(n)}(x)F_{2,e}^{[n]}(x,\overline{e}-e_1^v)dx\right]\right\} \\
&\quad + p\left[(-1)^m\int_a^b \phi^{(m)}(x)F_{1,e}^{[m]}(x,e_1^v)dx\right] - (1-p)\left[(-1)^n\int_a^b \phi^{(n)}(x)F_{2,e}^{[n]}(x,\overline{e}-e_1^v)dx\right] \\
&\le \lambda\left\{p\left[(-1)^m\int_a^b v^{(m)}(x)F_{1,e}^{[m]}(x,e_1^v)dx\right] - (1-p)\left[(-1)^n\int_a^b v^{(n)}(x)F_{2,e}^{[n]}(x,\overline{e}-e_1^v)dx\right]\right\} \\
&= \lambda\left\{p\int_a^b v(x)dF_{1,e}(x,e_1^v) - (1-p)\int_a^b v(x)dF_{2,e}(x,\overline{e}-e_1^v)\right\} \\
&= \lambda V'(e_1^v) = 0
\end{aligned}
$$

where the inequality comes from the fact that $(-1)^m\phi^{(m)}(x) \ge 0$ and $(-1)^n\phi^{(n)}(x) \le 0$ coupled with $F_{1,e}^{[m]}(x) \le 0$ and $F_{2,e}^{[n]}(x) \le 0$. The second-order condition guarantees that $U'(e_1)$ is declining in e_1, and therefore $e_1^u \le e_1^v$.

"only if"

Suppose that $e_1^u \le e_1^v$ is always the case, but *u*(*x*) is NOT (*n*/*m*)th-degree Ross more risk averse than *v*(*x*). Then, according to Lemma 3, there exist $\mu > 0$, $[a_1, b_1] \subset (a, b)$ and $[a_2, b_2] \subset (a, b)$ such that $\phi(x) \equiv u(x) - \mu v(x)$ satisfies

$$\begin{aligned} &(-1)^{n+1}\phi^{(n)}(x) < 0 \qquad \text{for all x} \in [a_1, b_1] \\ &(-1)^{m+1}\phi^{(m)}(x) > 0 \qquad \text{for all x} \in [a_2, b_2] \end{aligned} \tag{23}$$

Choose $F_2(x,e_2)$ and $F_1(x,e_1)$ such that $F_{2,e}^{[k]}(b,e_2) = 0$ for $k = 1,2,\ldots,n$, $F_{1,e}^{[k]}(b,e_1) = 0$ for $k = 1,2,\ldots,m$, and

$$\begin{cases} F_{2,e}^{[n]}(x,e_2) < 0 & x \in (a_1, b_1) \\ F_{2,e}^{[n]}(x,e_2) = 0 & x \notin (a_1, b_1) \end{cases} \qquad \begin{cases} F_{1,e}^{[m]}(x,e_1) < 0 & x \in (a_2, b_2) \\ F_{1,e}^{[m]}(x,e_1) = 0 & x \notin (a_2, b_2) \end{cases}. \tag{24}$$

Evaluating $U'(e_1)$ at e_1^v, we have

$$\begin{aligned} U'(e_1^v) &= \mu V'(e_1^v) + p\int_a^b \phi(x) dF_{1,e}(x, e_1^v) - (1-p)\int_a^b \phi(x) dF_{2,e}(x, \bar{e} - e_1^v) \\ &= p\int_a^b \phi(x) dF_{1,e}(x, e_1^v) - (1-p)\int_a^b \phi(x) dF_{2,e}(x, \bar{e} - e_1^v) \\ &= p\int_a^b (-1)^m \phi^{(m)}(x) F_{1,e}^{[m]}(x, e_1^v) dx - (1-p)\int_a^b (-1)^n \phi^{(n)}(x) F_{2,e}^{[n]}(x, \bar{e} - e_1^v) dx \\ &= p\int_{a_2}^{b_2} (-1)^m \phi^{(m)}(x) F_{1,e}^{[m]}(x, e_1^v) dx - (1-p)\int_{a_1}^{b_1} (-1)^n \phi^{(n)}(x) F_{2,e}^{[n]}(x, \bar{e} - e_1^v) dx \\ &> 0. \end{aligned}$$

Therefore $e_1^u > e_1^v$, a contradiction. So *u*(*x*) must be (*n*/*m*)th-degree Ross more risk averse than *v*(*x*).

Q.E.D.

To get an idea of how Theorem 7 matters, consider the case of a manager facing two states of the world, bankruptcy and solvency. The manager can devote time to protecting the company's assets, resulting in an ith-degree improvement in company payoffs if it goes bankrupt, or to improving sales, resulting in a jth-degree improvement in profit if the company remains solvent. If $i < j$, a (j/i)th-degree Ross more risk averse manager will devote more effort to improving the performance in the solvency state. Taking the special case of $i = 1$ and $j = 2$, the Ross more risk averse manager would devote more time to reducing the variability of profits conditional on solvency and less time to improving the mean of payoffs conditional on bankruptcy. This suggests that (n/m)th-degree Ross more risk averse individuals devote more energy to tasks to make higher-order distributional improvements (like variance) and less to lower-order distributional improvements (like mean), as long as those tasks affect mutually exclusive states of the world.

6. Conclusion

Pratt (1964) proposes two measures of risk aversion, the risk premium (the reduction in the nonrandom initial wealth the decision maker is willing to pay to avoid a zero-mean gamble) and the probability premium (the probability of winning the positive outcome of a zero-mean binary gamble that makes the decision maker indifferent between the gamble and the status quo), and shows that interpersonal comparisons of both measures are characterized by the Arrow-Pratt more risk aversion. Since then, the risk premium approach to comparative risk aversion has been generalized to deal with random initial wealth and risk aversion of higher degrees.[12] By

[12] For example, see Ross (1981), Machina and Neilson (1987), Kimball (1990), Modica and Scarsini (2005), Jindapon and Neilson (2007), Crainich and Eeckhoudt (2008), Li (2009), and Denuit and Eeckhoudt (2010).

comparison, the probability premium approach to comparative risk aversion has largely been abandoned.

In this paper, Pratt's original probability premium is reformulated according to the following basic idea: the individual makes a decision involving trading in his current wealth distribution for a new, state-dependent one. If event *G* (good) occurs then an *m*th-degree risk decrease in wealth takes place, but if event *B* (bad) occurs then an *n*th-degree risk increase in wealth takes place. The required probability of event *G* is defined as the *m*th probability premium of *n*th-degree risk aversion. It is shown that the interpersonal comparison of the *m*th probability premium of *n*th-degree risk aversion is characterized by the (*n*/*m*)th-degree Ross more risk aversion of Liu and Meyer (2013).

The paper also examines an alternative probability premium approach to comparative risk aversion that is based on the risk apportionment of Eeckhoudt and Schlesinger (2006) and Eeckhoudt et al. (2009), and shows that the alternative probability premium approach also leads to (*n*/*m*)th-degree Ross more risk aversion.

When all is said and done, the results of this paper combine with each other and those of Liu and Meyer (2013) to provide a characterization of comparative (*n*/*m*)th-degree risk aversion that is in the same spirit as Pratt's (1964) original characterization of comparative risk aversion. His seminal theorem consists of a set of equivalent conditions, some of which are mathematical in nature and some of which are behavioral. Here the mathematical conditions come from Liu and Meyer (2013), and in this paper they appear as equation (10) and Lemmas 2 and 3. To these our paper adds three new behavioral conditions: (i) a probability premium condition inspired by Pratt's and presented in Theorem 3; (ii) risk apportionment conditions inspired by Eeckhoudt and

Schlesinger (2006) and Eeckhoudt et al. (2009) and presented in Theorems 5 and 6; and (iii) a comparative statics condition inspired by Jindapon and Neilson (2007) and presented in Theorem 7. All of these behavioral conditions are equivalent, and therefore they help complete our understanding of the very general notion of (n/m)th-degree Ross more risk aversion.

This set of conditions can be compared to one based on the familiar risk premium approach. Importantly, the risk premium approach to comparative risk aversion can only lead to the notion of (n/1)th-degree Ross more risk aversion (Jindapon and Neilson 2007, Li 2009, and Denuit and Eeckhoudt 2010). Since (n/m)th-degree Ross more risk aversion includes the (n/1)th-degree Ross more risk aversion as a special case, the probability premium approach not only produces alternative measures of nth-degree risk aversion that are fundamentally equivalent to the risk premium measures, but also generate additional measures of nth-degree risk aversion. This may prove useful in future investigations of various factors that affect the intensity of higher-degree risk aversion.

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