Mutual Insurance Networks and Unequal Resource Sharing in Communities[*], *First Draft, February 2015*

Pascal Billand[a], Christophe Bravard[b], Sudipta Sarangi[c]

[a] *Université de Lyon, Lyon, F-69003, France ; Université Jean Monnet, Saint-Etienne, F-42000, France ; CNRS, GATE Lyon St Etienne, Saint-Etienne, F-42000, France.*

email: pascal.billand@univ-st-etienne.fr.

[b] *Université Grenoble 2 ; UMR 1215 GAEL, F-38000 Grenoble, France ; CNRS, GATE Lyon St Etienne, Saint-Etienne, F-42000, France. email: christophe.bravard@univ-st-etienne.fr.*

[c] *DIW Berlin and Louisiana State University. email: sarangi@lsu.edu*

Abstract

We study formation of mutual insurance networks in a model where agents who obtain more resources share a fixed amount of resources with all directly linked agents that obtain fewer resources. We identify the pairwise stable networks and efficient networks in a basic model where agents are identical. Then, we introduce in the model two types of heterogeneity: an exogenous one, where agents differs in their income or in their preferences over the transfer scheme, and an endogenous heterogeneity where the costs of linking to an agent depends on the number of links the latter has already formed in the network. We examine the impact of these heterogeneities on stability and efficiency.

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1 Introduction

A growing body of evidence has shown that while household income in developing countries varies greatly, consumption is remarkably smooth at a community level (e.g., Townsend, 1994, Paxson, 1992, Jacoby and Skouas, 1997). Given the lack of formal insurance especially in the rural areas, this suggests that informal institutions play a crucial role in helping households to counter the effects of income variation. In this paper we study the formation of these informal mutual insurance networks building on several stylized facts.

A first key feature is that informal insurance is not a village level phenomenon. Indeed, as has been well documented in the literature on social networks, in times of need individuals do not rely on the entire village, instead they seek help primarily through a network of mutual insurance relationships with friends and family (see Fafchamps and Lund, 2003, and Wellman and Currington, 1988). Another important feature is that mutual insurance networks are not complete within the observed set of individuals. That is, within any community, individuals do not enjoy the benefits of being insured by all others individuals of the village. Finally, the sharing of resources in times of need is not equal (Townsend, 1994). In fact to the best of our knowledge, this aspect of the formation of mutual insurance networks has not been addressed before.

Our goal is to provide a picture of the mutual insurance arrangements within the community from a networks perspective. We examine when symmetric and asymmetric network architectures can be stable among *ex ante* symmetric agents. We ask whether such arrangements may be locally complete, *i.e.*, involve every individual in a small group. We also study efficiency and the impact of agent heterogeneity in this problem.

In our model mutual insurance takes place between pairs of individuals in a village or a small community. A specific feature is the way agents "share" their resources (and hence the risk): individuals who draw high resources give a fixed amount of resources to individuals in their immediate neighborhood in the network who draw low resources. Thus, agents do not engage in equal sharing of resources. This type of sharing mechanism has two realistic features: *(i)* it ensures that individuals who draw high resources can always transfer resources to all their neighbors who draw low resources, and *(ii)* individuals who draw high resources always obtain higher benefits than individuals who draw low resources.

In our benchmark model, we assume that each agent obtains random resources which take on two values: high or low. If a person draws the high endowment state, then she gives an amount $\delta > 0$ of resources to each of her neighbors (agents with whom she has established a bilateral risk-sharing agreement) that has drawn the low endowment state. Conversely, if a person draws the low endowment state, then she obtains an amount δ of resources from each of her neighbors who has drawn the high endowment state.[1] Note that such a mutual insurance network exposes agents to the risk of their neighbors. Indeed, if two individuals decide to insure each other, then each of them increases her chances of obtaining a satisfying payoff when her own resources are low, but also increases her chances of reducing her payoffs when her own resources are high.

We also assume that informal agreements are not binding and hence to make them work agents need to invest time in their relationships. So, in our model, establishing such mutual insurance agreements is costly. More precisely, the cost of an agreement (a link) between two individuals depends on the number of agreements established by them. In particular, we assume that the marginal cost of individual i, when she forms a link with an individual j, is increasing with the number of links formed by i. This captures the idea that a mutual insurance agreement between two agents requires the agents to spend a minimal amount of time on it. Now the more links they have already formed, the less time they have to spend on any additional link, and so the higher is the cost of time. It follows that the cost of an additional link increases with the number of links.

Using this framework, we examine the formation of mutual insurance links and ask what structures will emerge when agents cannot coordinate link formation across the entire population. We use pairwise stability as the equilibrium concept (see Jackson and Wolinsky, 1996). In a pairwise stable network no pair of unlinked agents has an incentive to reach a mutual insurance agreement (add a link), and no individual has an incentive to break a mutual insurance agreement (remove one of her links). We contrast pairwise stable networks with the efficient networks for mutual insurance agreements. An efficient network is one which maximizes the

[1] In reality transfers can take a wide range of values depending on the incomes of the individuals and may be even the needs of the agents in a community. However, in our model individuals have either high or low incomes. Therefore it is reasonable to assume that in the high income state agents give a fixed amount to their neighbors who have drawn the low income state.

amount of total payoffs obtained by agents.

In the basic model, we have several findings.

- We establish that there exist pairwise stable networks, in which individuals are in asymmetric situations relative to the risk they support. Thus, agents have different risk-sharing outcomes, despite that they have identical preferences and their incomes are identically distributed. Moreover, in pairwise stable networks agents who obtain the smallest amount of insurance are always linked together.
- We show that in efficient networks agents always obtain similar amounts of insurance. In that case, an efficient network is a pairwise stable network (or a network very similar to a stable network), but the converse is not true. Thus there may exist a conflict between pairwise stable networks and efficient networks. We conclude that the mutual insurance mechanism described here does not always lead to efficient networks.

Then we extend the basic model by introducing two types of agents heterogeneity, an exogenous one and an endogenous one.

First, we consider situations where agents are exogenously heterogeneous: either they do not obtain the same income when they draw the high income state or they do not have the same preferences with respect to the generosity of the transfer scheme. In the first case, we assume that there exist two types of agents concerning the income they get when they draw the high income state. In that case, we show that there exist situations where only people who get the highest income when they draw the high income state, that is the rich people, get access to insurance, while people who get the lowest income when they draw the high income state, that is the poor people, will never be insured. It follows that the insurance mechanism increases the gap between the expected well-being of rich people and the expected well-being of poor people. In the second case, we assume that there exist two types of agents: the generous ones who are more giving more than the miserly ones. In that case, we establish that there exist conditions under which, in a pairwise stable network, generous agents are linked together, miserly agents are linked together, but there are no links between these two types of agents. This kind of pairwise stable networks are compatible with a result stressed by several empirical studies: the majority of transfers takes place only between sub-groups of agents (see Rosenzweig, 1988, and Udry, 1994).

Second, we consider situations where the cost of linking to an agent is increasing in the number of links that this agent has already formed. This captures the fact that insurance agreements are informal and are honored if agents involved in a relationship invest time. In such situation, it is more difficult to establish a relationship with an agent who already has numerous links since she has less time available. Note that in this situation agents heterogeneity arises endogenously in the model. As in the basic model, we show that agents who obtain the smallest amount of insurance are always linked together. However, by contrast with the basic model, we show that agents who have the highest amount of insurance are never linked together. Concerning efficiency, we show that unlike in the basic model, if a network is efficient, then it is not always pairwise stable (or a network very similar to a pairwise stable network). More precisely, an efficient network can be over connected with respect to stability. This result is due to the fact that when an agent decides whether to delete a link, she does not take into account the positive externality that would accrue to agents who will be still linked with this agent after the deletion of the link.

A recent theoretical literature about revenue sharing in developing economies examines the formation of risk-sharing networks. Bramoulle and Kranton (2006 and 2007) discuss the stability/efficiency dilemma of risk-sharing networks. A distinctive aspect of their work is that after the income realization occurs, linked pairs of agents meet (sequentially and randomly) and share their current money holding equally. The authors show that with many rounds of such meetings, an individuals money holding converges to the mean of realized income in her group,[2] that is in a group there is always equal revenue sharing among individuals. Belhaj and Deroian [1] also examine a situation where the bilateral partial risk-sharing rule is such that neighbors share equally a part of their revenue. However, they focus on the impact of informal risk-sharing on risk taking incentives when transfers are organized through a social network.
By contrast, in our paper we deal with situations where individuals do not engage in equal income sharing. In particular, after income sharing, an individual who has initially obtained high income always ends up better than an individual who has obtained low income. There is an interesting difference between our paper and the Bramoulle and Kranton papers concerning externality generated by links.

In our paper, when an agent i forms a link with an agent j, this link may have a negative

[2] In their paper, a group consists of agents who are both directly or indirectly linked.

impact on the utility of i's neighbors (there is a negative externality), since agent i will now have less time to spend on relations with her neighbors (*idem* for j's neighbors). It follows that when two agents have an incentive to form a link, this link may decrease social welfare. In the Bramoulle and Kranton model, when an agent i forms a link with an agent j, this link has a positive impact on the expected utility of i's neighbors. Indeed, in their model there is equal income sharing between all the agents of the components. Therefore due to the additional link between i and j, i's neighbors will share their income with an additional agent (agent j) and their expected utility will increase. It follows that it can be that two agents have no incentive to form a link, and this link increases the social welfare.

Some papers explain partial risk-sharing by self-enforcing mechanisms in networks (Bloch et al., 2008).[3] These models consider that individuals can use their links to punish individuals who deviate from the insurance scheme. For instance, if an agent deviates from the insurance scheme (*i.e.* fails to transfer money to directly connected agents that have negative income shocks), the victim will communicate such behavior to other connected agents who in turn will terminate the insurance scheme with the deviating household as a punishment. In this paper, we do not deal with the self-enforcement mechanism problem. Instead, we assume that establishing a relationship is costly and it commits the parties to future resource sharing, say, due to a social norm or a social punishment in case of non-sharing.[4] More precisely, we assume that the self-enforcement mechanism problem is solved when agents invest enough time and resources in the informal insurance agreements.[5]

The rest of the paper is organized as follows. In section 2, we present the definitions and the basic model setup. In section 3, we examine pairwise stable networks and efficient networks in the basic model context. In section 4, we extend the basic model by introducing different types of agents heterogeneity. In section 5, we conclude.

[3]This literature extends the literature about the robustness of mutual insurance (see for instance Genicot and Ray, 2003).

[4]This kind of relation can be illustrated with the marriage of daughters in India which are arranged to maximize gains from risk sharing, see Rosenzweig and Stark, 1989.

[5]The time an agent invests in the relationship and the social punishment in case of non-sharing are related since a bilateral relation in which agents have invested a lot of time can be more easily observed by the peers.

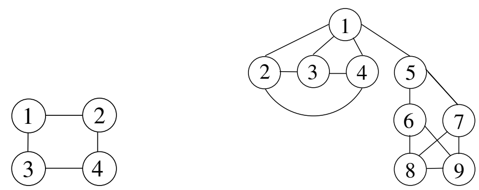
2 Basic model setup

Let $N = \{1, \ldots, n\}$ be a community of n, $n \geq 3$, *ex ante* identical agents. Agents receive an endowment and consume resources. Each agent's endowment is a random variable that takes two values. The low endowment state is called state 0 while the high endowment state is called state 1. Each agent i obtains an endowment 0 in state 0 while she obtains $\Theta > 0$ in state 1. State 1 occurs with probability $p > 0$ while the low endowment state occurs with probability $1 - p > 0$. The realizations of endowments are independent and identical across the agents.

Networks. To model bilateral mutual insurance agreements in a small population, we use tools from the theory of networks. Although the agreements themselves are bilateral, the amount of resources consumed by each agent depends on how many other agents she is connected with, and the endowments of these agents. Hence tools from network theory are useful for modeling such bilateral insurance networks. In the model, we assume that individuals i and j can have a mutual insurance agreement by forming a costly link between themselves. This assumption reflects the idea that there are always some costs (time at the least) to build a relationship.

We represent links and a network of links with the following notation: A network g is an $n \times n$ matrix, where $g_{ij} = 1$ when i and j are linked (i.e., have established a risk-sharing agreement) and $g_{ij} = 0$ otherwise. We assume that risk-sharing relations are mutual, so that $g_{ij} = g_{ji}$. By convention, $g_{ii} = 0$. Let $g + g_{ij}$ denote the network obtained by replacing $g_{ij} = 0$ in g by $g_{ij} = 1$. Similarly, let $g - g_{ij}$ denote the network obtained by replacing $g_{ij} = 1$ in g by $g_{ij} = 0$. We say that there is a *chain* between two agents i and j in the network g if there exists a sequence of agents $i_1, \ldots, i_k$ such that $g_{ii_1} = g_{i_1 i_2} = g_{i_2 i_3} = \ldots = g_{i_k j} = 1$. A subset of agents is *connected* if there is a chain between any two agents in the subset. A *component* of the network g is a maximal connected subset of agents. Moreover, network $g_{|N'}$ is a sub-network of g if it consists in the agents in $N' \subset N$, and $i \in N'$ and $j \in N'$ are linked in $g_{|N'}$ *if and only if* they are linked in g.

The *empty network* is the network where all agents have formed no links. The *complete network* is the network where each agent has formed links with all the agents. A *k-regular* network is a network where *all* agents have formed exactly k links. A *k_--regular network* is a network where all agents but one have formed k links; the agent who is the exception has formed $k-1$ links. An *k_+-regular network* is a network where all agents but one have formed k links; the agent who is the exception has formed $k+1$ links. An *almost-k-regular-networks* is either a k_--regular network, or k_+-regular network. We illustrate the notions of *k-regular* network, and *k_+-regular network* in Figure 1.



2-regular network 3_+-regular network

Figure 1: Networks architectures

In the following, the neighbors of agent i, that is agents with whom i has formed a link, will play a crucial role. Hence we define $N_i(g) = \{j \in N \mid g_{ij} = 1\}$ as the set of the neighbors of i. Let $n_i(g) = |N_i(g)|$ be the degree of agent i.

Payoffs. Having described the set of players and their strategies, we now ask: Given a network g, how are expected payoffs determined under different endowment realizations in the network? We consider a benchmark model where *ex ante* agents are identical: they get the same resources Θ when they draw state 1 and the same resources 0 when they draw state 0. Moreover, if an agent draws state 1, then she always gives $\delta \in (0,1)$ to each of her neighbors who has drawn state 0.[6] Conversely, if an draws state 0, then each of her neighbors who has drawn state 1 always gives her δ. Note that in our model, agents may receive different amounts of transfers depending on the network architecture. Moreover, we assume that $\Theta > (n-1)\delta$. In this paper we assume that the payoff obtained by an agent, say i, can be divided into two parts.

1. The benefits part which involves uncertainty captures the fact that each additional link

[6]Here, we assume that the transfer amount δ comes from some kind of social norm. Our goal in this paper is to study what is the architecture of the mutual insurance network within a community. Hence we do not explicitly model where δ comes from.

formed by i allows her to obtain additional insurance when she draws the bad state (0), and the fact that i has to insure more agents when she draws the good state (1).

2. The costs part which involves no uncertainty captures the fact that links are costly, with additional links being more costly.

We now present these two parts of an agent's payoff function.

Benefits. Working with a general payoff function in the context of a network formation problem poses tractability issues. Hence from now on we deal with the exponential utility function: $u_i(x) = 1 - \exp[-\rho x]$, where x is the income of agent i, and ρ is a positive parameter. Consequently, our model exhibits Constant Absolute Risk Aversion.[7] It follows that if agent i draws state 0 and k agents in her neighborhood draw state 1, then she obtains a benefit equal to $b^g(0,k) = u_i(k\delta) = 1 - \exp[-\rho k\delta]$. Conversely, if agent i draws state 1 and k agents in her neighborhood draw state 0, then she obtains a benefit equal to $b^g(1,k) = u_i(\Theta - (n_i(g) - k)\delta) = 1 - \exp[-\rho(\Theta - (n_i(g) - k)\delta)]$.
We now define the expected neighborhood benefits (ENB) function, $B_i(g)$, which captures the expected benefits obtained by an agent i given her neighborhood $n_i(g)$. We have:

$$B_i(g) = \phi(n_i(g)) = p \sum_{k=0}^{n_i(g)} \binom{n_i(g)}{k} p^k (1-p)^{n_i(g)-k} b^g(1,k) + (1-p) \sum_{k=0}^{n_i(g)} \binom{n_i(g)}{k} p^k (1-p)^{n_i(g)-k} b^g(0,k), \quad (1)$$

where $\binom{x}{y}$ is just the probability of y high resources out of x draws.
In the following, for each function f we set $\Delta f(x) = f(x) - f(x-1)$. Moreover, since we use non-continuous function, we use a slightly modified version of convexity and concavity. We say that f is concave if for all x, $\Delta f(x+1) - \Delta f(x) \leq 0$ and f is convex if for all x, $\Delta f(x+1) - \Delta f(x) \geq 0$.

Costs of links. In the first part of the paper, we assume that the costs of forming links of agent i depends on the number of links formed by i, but does not depend on the degree of the agents with whom firm i forms links. More precisely, we assume that the cost of links

[7] Here we use an exponential function, but we obtain the same qualitative results for some other CARA functions.

depends only on the number of links formed by i:

$$C_i(g) = f_1(n_i(g)),$$

where f_1 is a strictly increasing and convex function. In addition, we assume that $f_1(0) = 0$.

Expected payoffs function. The expected payoff function, $U_i(g)$, of each agent i, given the network g, is the difference between the ENB function and the cost function of forming links:

$$U_i(g) = B_i(g) - C_i(g) = \Phi(n_i(g)) = \phi(n_i(g)) - f_1(n_i(g)). \quad (2)$$

Pairwise stable networks and efficient networks. A network g is *pairwise stable* if no pair of unlinked agents would benefit by forming a link in g and if no agent would benefit from severing one of her existing links in g. Formally, following Jackson and Wolinsky (1996) we have (i) for all $g_{ij} = 1$, $U_i(g) \geq U_i(g - g_{ij})$ and $U_j(g) \geq U_j(g - g_{ij})$; and (ii) for all $g_{ij} = 0$, if $U_i(g) < U_i(g + g_{ij})$, then $U_j(g) > U_j(g + g_{ij})$.

An *efficient network* is one that maximizes the sum of the expected payoffs of the agents. Let $W(g) = \sum_{i \in N} U_i(g)$ be the total expected payoffs obtained in a network g. A network g^e is efficient if $W(g^e) \geq W(g)$ for all networks g.

3 Pairwise stable and efficient networks analysis

First, we analyze the ENB function. Second, we study pairwise stable networks. Third, we turn to efficient networks. Our first proposition provides some useful properties of the ENB function. We provide a sketch of proof of this proposition in the appendix.

Proposition 1 *Suppose that the ENB is given by equation 1. Then, the expected neighborhood benefits function of agent i is strictly increasing and strictly concave with the number of links she has formed.*

Proposition 1 states that the ENB obtained by agent i is increasing. In other words, each agent i prefers to be more insured than less insured when the cost of insurance (the cost of forming links) is sufficiently low. Moreover, Proposition 1 implies that Φ is concave: the marginal ENB that an agent i obtains from an additional link decreases with the number of links she has formed. Consequently, if the cost of forming links was constant, then the incentive of an agent to form an additional link would decrease with the number of links she has formed.

In the following proposition, we examine the incentive of agents to form an additional link when their income increases.

Proposition 2 *Suppose that the ENB is given by equation 1. Then, the marginal expected neighborhood benefits function of agent i increases with Θ.*

Proof By inspecting the proof of Proposition 1, we know that $\Delta B_i(g,ij)$ is equal to

$$\begin{aligned} \Delta B_i(g,ij) &= p(1-p)(1-\exp[-\rho\delta])(1+p(\exp[-\rho\delta]-1))^{n_i(g)} \\ &\quad -p(1-p)(\exp[\rho\delta]-1)\exp[-\rho\Theta](\exp[\rho\delta]+p(1-\exp[\rho\delta]))^{n_i(g)}. \end{aligned} \tag{3}$$

From this we have:

$$\frac{\partial \Delta B_i(g,ij)}{\partial \Theta} = \rho p(1-p)(\exp[\rho\delta]-1)\exp[-\rho\Theta](\exp[\rho\delta]+p(1-\exp[\rho\delta]))^{n_i(g)-1} > 0.$$

□

Let us explain the above result. Obviously, when agent i draws the low income state, then her ENB is not affected by her income. Suppose now that agent i draws the high income state. Due to the concavity of the utility function, when the income increases, the utility function of agent i is less affected by the loss of money she incurs when she has to help one of her neighbors. It follows that the marginal expected neighborhood benefits function of agent i increases with Θ.

We now deal with pairwise stable networks. We prove the existence and we characterize pairwise stable networks. To ensure the existence of pairwise stable networks, we use a theorem established by Erdös and Gallai. We need the following definition to present this theorem.

Definition 1 *A finite sequence $s=(d_1,d_2,\ldots,d_n)$ of nonnegative integers is graphical if there exists a network g whose nodes have degrees $d_1,d_2,\ldots,d_n$.*

Theorem 1 *(Erdös and Gallai, 1960) A sequence $s = (d_1, d_2, \ldots, d_n)$ of nonnegative integers, such that $d_1 \geq d_2 \geq \ldots \geq d_n$, and whose sum is even is graphical if and only if*

$$\sum_{i=1}^{r} d_i \leq r(r-1) + \sum_{i=r+1}^{n} \min\{d_i, r\}, \text{ for every } r,\ 1 \leq r < n.^{8} \tag{4}$$

In the following lemma we provide conditions that ensure the existence of three kinds of networks that turn out to be quite useful subsequently.

Lemma 1 *Let n and k be nonnegative integers with $n > k$.*

1. *Let n or k be even. Then, the sequence $s = (k, \ldots, k)$ is graphical.*
2. *Let n and k be odds. Then, the sequences $s = (k, k, \ldots, k, k+1)$, $s' = (k, k, \ldots, k, k-1)$ are graphical.*

Proof See Appendix. □

Since Φ is strictly concave, there exists $\hat{k}$ such that $\Phi(\hat{k}) > \Phi(k)$ for all $k \neq \hat{k}$. To present the next proposition, we need the additional definition: let $\mathcal{M}(g) = \{i \in N : n_i(g) \neq \hat{k}\}$ be the set of players who have formed $\hat{k}$ links in g.

Proposition 3 *Suppose that the benefits function is given by equation 1. Network g is pairwise stable if and only if for every agent $i \in \mathcal{M}(g)$, $n_i(g) < \hat{k}$, and $g_{|\mathcal{M}(g)}$ is complete. Moreover, (a) if n or $\hat{k}$ are even, then $\hat{k}$-regular networks are always pairwise stable; (b) If n and $\hat{k}$ are odd then, $\hat{k}_-$-regular networks are always pairwise stable.*

Proof Let g be a pairwise stable network. If an agent i has formed more than $\hat{k}$ links in g, then g is not a pairwise stable since $\hat{k}$ maximizes $\{\Phi(k) : k \in \{0, \ldots n-1\}\}$, and agent i has an incentive to remove a link. Similarly, if there exist two unlinked agents i and j who have formed less than $\hat{k}$ links in g, then g is not a pairwise stable network since i and j have an incentive to form a link together. It follows that for $i \in \mathcal{M}(g)$, $n_i(g) < \hat{k}$, and $g_{|\mathcal{M}(g)}$ is complete. Let us now assume that n or $\hat{k}$ are even. By Lemma 1, the sequence of degrees $s = (\hat{k}, \hat{k} \ldots, \hat{k})$ is graphical: we can build a $\hat{k}$-regular-network where no agent has an incentive to remove links and no couple of agents have an incentive to form an additional link. The

[8] The theorem can also be found in Harary, 1969, Chapter 6 pp. 59-62 and the statement here is based on his presentation.

result follows. Let us now assume that n and $\hat{k}$ are odd. By Lemma 1, the sequence of degrees $s = (\hat{k}, \hat{k} \ldots, \hat{k} - 1)$ is graphical: we can build a $\hat{k}_{-}$-regular-network where no agent has an incentive to remove links and where no couple of agents have an incentive to form an additional link. The result follows. $\square$

Suppose $\hat{k} = 3$. Network g drawn in Figure 2 satisfies properties given in Proposition 9. We have $\mathcal{M}(g) = \{1, 2, 3\}$. It is worth noting that since the players only take into account

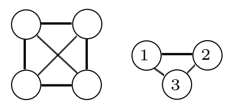


Figure 2: Network g

their degree when they form links, the identity of neighbors of players in $\mathcal{M}(g)$ cannot, in general, be known.

The following corollary establishes that when the costs of forming links are zero, the agents have to form some links in order to get some insurance from the others.

Corollary 1 *Suppose that the ENB is given by equation 1. If the costs of forming links are null, then the the pairwise equilibrium network is the complete network.*

Proof We know by Proposition 1 that the ENB is increasing in $n_i(g)$. By inspecting the proof of Proposition 1, we know that the ENB is positive for $n_i(g) = n - 2$. It follows that the complete network is the only pairwise equilibrium network when the costs of forming links are null. $\square$

Corollary 1 implies that if links had no costs, then all the agents would form insurance links with all other agents. By continuity, this result is true for small costs of forming links. We now focus on efficient networks.

Proposition 4 *Suppose that the ENB is given by equation 1. Suppose n or $\hat{k}$ are even, then $\hat{k}$-regular networks are the unique efficient networks. Suppose n and $\hat{k}$ are odd. If $\Phi(\hat{k} + 1) < \Phi(\hat{k} - 1)$, then $\hat{k}_{-}$-regular networks are the unique efficient networks. If $\Phi(\hat{k} + 1) > \Phi(\hat{k} - 1)$, then $\hat{k}_{+}$-regular networks are the unique efficient networks.*

Proof Suppose that n or $\hat{k}$ are even. By Lemma 1, $\hat{k}$-regular networks exist. In a $\hat{k}$-regular network g, each agent maximizes its expected payoffs. It follows that g is efficient. Suppose that n and $\hat{k}$ are odd. By Theorem 1, it is not possible to build a $\hat{k}$-regular network. By Lemma 1, $\hat{k}_{+}$-regular networks and $\hat{k}_{-}$-regular exist. In an almost-$\hat{k}$-regular network g, each agent except one, say i, maximizes its expected payoffs. Since i cannot form $\hat{k}$ links and Φ is concave, she maximizes her payoffs when she forms $k+1$ or $k-1$ links. If $\Phi(\hat{k}+1) < \Phi(\hat{k}-1)$, then the agent, who has not formed $\hat{k}$ links, forms $\hat{k}-1$ links in an efficient network; this agent forms $\hat{k}+1$ links in an efficient network if $\Phi(\hat{k}+1) > \Phi(\hat{k}-1)$. □

We note that if n or $\hat{k}$ are even then an efficient network is always pairwise stable while there exist pairwise stable networks that are not efficient. Suppose that n and $\hat{k}$ are odd. If $\Phi(\hat{k}+1) < \Phi(\hat{k}-1)$, then $\hat{k}_{-}$-regular network are efficient and pairwise stable networks. If $\Phi(\hat{k}+1) > \Phi(\hat{k}-1)$, then $\hat{k}_{+}$-regular networks are efficient. These networks are very similar to pairwise stable $\hat{k}_{-}$-regular networks.

4 Pairwise stable networks with heterogeneous agents

In this section, we assume that agents are heterogeneous. Although the following frameworks are harder to analyze than the previous one, we are able to obtain some insights by making some simplifying assumptions. We restrict attention to the linear cost function and assume that agent $i \in N$ incurs a cost $F > 0$ for each link she forms.[9]

First, we examine a situation where agents do not obtain the same income when they draw state 1. Agents in N^{Θ} obtain an income equal to Θ if they draw state 1 and agents in $N^{\Theta'}$ obtain an income equal to $\Theta' < \Theta$ if they draw state 1. In the following, we assume that $|N^{\Theta}|$ and $|N^{\Theta'}|$ are even. We denote by $\Phi^{\Theta}(n_i(g))$ the expected payoff of agent $i \in N^{\Theta}$ when she has formed $n_i(g)$ links and $\Phi^{\Theta'}(n_i(g))$ the expected payoff of agent $i \in N^{\Theta'}$ when she has formed $n_i(g)$ links. By Proposition 1, we know that the ENB is strictly concave. It follows that there exist $\hat{k}_{\Theta}$ which maximizes $\{\Phi^{\Theta}(k) : k \in \{0, \dots n-1\}\}$ and $\hat{k}_{\Theta'}$ which maximizes $\{\Phi^{\Theta'}(k) : k \in \{0, \dots n-1\}\}$. By Proposition 2, we know that $\partial \Delta B_i(g, ij)/\partial \Theta > 0$, so $\hat{k}_{\Theta} \geq \hat{k}_{\Theta'}$. For the following proposition we need some additional definitions. Let

[9] It is possible to do the analysis with the cost function assumed in section 2, but this would be an harder task that would not provide any additional insights.

$\mathcal{N}^{\Theta}(k_{\Theta}, g) = \{i \in N^{\Theta} : n_i(g) = k_{\Theta}\}$, and $\mathcal{N}^{\Theta'}(k_{\Theta'}, g) = \{i \in N^{\Theta'} : n_i(g) = k_{\Theta'}\}$. Finally, we set $\mathcal{M}'(g) = N \setminus (N^{\Theta}(g) \cup N^{\Theta'}(g))$.

Proposition 5 *Network g is pairwise stable if and only if for every agent $i \in \mathcal{M}'(g)$, $n_i(g) < \hat{k}_x$, $x \in \{\Theta, \Theta'\}$, and $g_{|\mathcal{M}(g)}$ is complete.*

Proof Suppose that g is pairwise stable. Then there is no agent who has an incentive to remove one of their links and no couple of unlinked agents (i, j) who have simultaneously an incentive to form a link together. It follows that no agent $i \in N^x$ forms more than $\hat{k}_x$ links, and agents $j \in \mathcal{M}'(g)$ are all linked together.
Suppose that for every agent $i \in \mathcal{M}'(g)$, $n_i(g) < \hat{k}_x$, $x \in \{\Theta, \Theta'\}$, and $g_{|\mathcal{M}'(g)}$ is complete. By construction, agents in $N \setminus \mathcal{M}'(g)$ have no incentive to modify their strategy. Similarly, agent $i \in \mathcal{M}'(g)$ does not have any incentive to remove a link; and i can form additional links only with agents in $N \setminus \mathcal{M}'(g)$ who will not accept any additional links. Consequently, g is pairwise stable. □

We now illustrate, through an example, that $\hat{k}_{\Theta} > \hat{k}_{\Theta'}$ does not imply that for $i \in N^{\Theta}$ and $i' \in N^{\Theta'}$, we have $n_i(g) \geq n_{i'}(g)$ in a pairwise stable network g. Suppose that $N^{\Theta} = \{1, 2\}$, $N^{\Theta'} = \{3, 4\}$, $\hat{k}_{\Theta} = 2$, and $\hat{k}_{\Theta'} = 1$. Then network g' drawn in Figure 3 is pairwise stable. We observe that $n_2(g') < n_3(g')$ while agent 2 wants to form more links than agent 3.

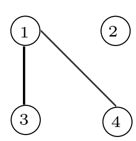


Figure 3: Networks g

Finally, there exist some costs of forming links such that no agent in $N^{\Theta'}$ has formed links and agents in N^{Θ} have formed links.

Corollary 2 *There exists F such that only agents in N^{Θ} have formed links.*

Proof By Proposition 2, we know that agents in N^{Θ} have more incentive to form links than agents in $N^{\Theta'}$. We set $\iota(x) = \frac{(\exp[\rho x]-\exp[\rho\delta])(\exp[\rho\delta]-1)}{\exp[\rho\delta]\exp[\rho x]}$. By Corollary 1 and the concavity of the ENB, if $F \in \{\iota(\Theta'), \iota(\Theta)\}$, then agents in $N^{\Theta'}$ have no incentive to form any links. Similarly, since $F < \iota(\Theta)$, agents in N^{Θ} have an incentive to form links. □

There exist situations where the incomes of the agents and the costs of forming links are such that all rich agents have formed links together while poor agents have formed no links, that is only rich people get access to insurance. In that case, the insurance mechanism increases the gap between the expected well-being of rich people and the expected well-being of poor people. We now propose an example where this happens.

Example 1 We assume $N = \{1, \ldots, 6\}$, $\Theta = 7$, $\Theta' = 6$, $N^{\Theta} = \{1, 2, 3\}$, $N^{\Theta'} = \{4, 5, 6\}$, $\rho = 0.2$, $\delta = 0.5$, and $F = 0.02$. Then network g, where $g_{N^{\Theta}}$ is complete and $g_{N^{\Theta'}}$ is empty, is the unique pairwise stable network.

Till now, we have assumed that all agents who draw state 1 give the same amount of resources, δ, to their neighbors who draw state 0. We now consider an extension of our benchmark model where this assumption is relaxed. In particular, one can imagine that agents do not have the same preferences with respect to the generosity of the transfer scheme. Some agents may be willing to give a large amount of resources to their unlucky neighbors, while some others may be willing to give them small amount of resources.

We will assume that the population is partitioned into two sets: the set of generous agents, N^G, and the set of miserly agents, N^M. If a generous agent draws state 1, then she gives δ^G to each of her neighbors who draws state 0. Similarly, if a miserly agent draws state 1, then she gives δ^M to each of her neighbors who draws state 0. Obviously, $\delta^G > \delta^M$.

We now assume that each agent has the same probability of drawing state 0 or state 1: it does not depend on her preference regarding the transfer scheme. Moreover, if there are $\alpha \geq 0$ agents who belong to N^G in the neighborhood of i and k agents in $N_i(g)$ who draw state 1, then there is a probability given by

$$\mathcal{Q}(\alpha, x, k) = \left[\binom{\alpha}{x}\binom{n_i(g)-\alpha}{k-x}\right] / \left[\sum_{\ell=0}^{k}\binom{\alpha}{\ell}\binom{n_i(g)-\alpha}{k-\ell}\right]$$

that x, $x \leq \alpha$, of these k agents belong to N^G.

We now define $\bar{b}^g(0, \alpha, k)$, which describes the expected benefits obtained by i in g when she draws state 0, α agents in her neighborhood are generous, and k agents in her neighborhood draw state 1, as follows:

$$\bar{b}^g(0,\alpha,k) = \sum_{y=0}^{\alpha} \mathcal{Q}(\alpha,y,k)(1-\exp\left[-\rho(y\delta^G+(k-y)\delta^M)\right]).$$

It is worth noting that if there exists only one type of agents, then both definitions of $b^g(0,\cdot)$ and $\bar{b}^g(0,\cdot,\cdot)$ are "equivalent". Indeed, given that the size of the neighborhood of agent i is $n_i(g)$, if we assume that all agents in the neighborhood of agent i are generous and give $\delta=\delta^G$ to their neighbors, then we have $\bar{b}^g(0,n_i(g),k) = b^g(0,k)$. Similarly, if we assume that all agents in the neighborhood of agent i are miserly and give $\delta=\delta^M$ to their neighbors, then $\bar{b}^g(0,0,k) = b^g(0,k)$.

We now define the expected neighborhood benefits function of agent $i \in N^x$, $x \in \{G,M\}$, given that her neighborhood contains $n_i(g)$ agents and α among them are generous agents.

$$\begin{aligned} \bar{B}_i(g) = \bar{\phi}(n_i(g),\alpha) \quad &= \quad p\sum_{k=0}^{n_i(g)} \binom{n_i(g)}{k} p^k(1-p)^{n_i(g)-k} b_x^g(1,k) \\ &\quad + (1-p)\sum_{k=0}^{n_i(g)} \binom{n_i(g)}{k} p^k(1-p)^{n_i(g)-k} \bar{b}^g(0,\alpha,k), \end{aligned} \tag{5}$$

where $b_x^g(1,k)$ is equal to $b^g(1,k)$ when agent i gives $\delta=\delta^x$, $x \in \{G,M\}$, to each of her neighbors who draw state 0. In the following, we are interested in the incentives of agent $i \in N^x$ to form a link with an agent $j \in N^y$, $y \in \{G,M\}$, given that the cost associated with each link is F.
We denote by $\mathcal{Z}_i^{x,y}(g) = \bar{B}_i(g+g_{i,j}) - \bar{B}_i(g)$ the marginal expected neighborhood benefits obtained by agent $i \in N^x$, $x \in \{G,M\}$, when she forms a link with an agent which belongs to $N^y, y \in \{G,M\}$, in g. For the next results, we always use the expected neighborhood benefits function given by equation 5. The next proposition shows that generous and miserly agents would prefer to be linked with generous agents.

Proposition 6 *We have $\mathcal{Z}_i^{G,G}(g) > \mathcal{Z}_i^{G,M}(g)$, and $\mathcal{Z}_i^{M,G}(g) > \mathcal{Z}_i^{M,M}(g)$.*

Proof We only prove that $\mathcal{Z}_i^{G,G}(g) > \mathcal{Z}_i^{G,M}(g)$, since $\mathcal{Z}_i^{M,G}(g) > \mathcal{Z}_i^{M,M}(g)$ is established using the same arguments. Let g^G be the network which is identical to g except that agent $i \in N^G$ has formed an additional link with agent $j \in N^G$, and let g^M be the network which is identical to g except that agent $i \in N^G$ has formed an additional link with agent $j' \in N^M$. Let $\alpha^G = |N_i(g^G) \cap N^G|$ and $\alpha^M = |N_i(g^M) \cap N^G|$. We have $\alpha^G = \alpha^M + 1$. Moreover, we have $\mathcal{Z}_i^{G,G}(g) - \mathcal{Z}_i^{G,M}(g) = (1-p)\sum_{k=0}^{n_i(g)} \binom{n_i(g)}{k} p^k (1-p)^{n_i(g)-k} (\bar{b}^{g^G}(0, \alpha^G, k) - \bar{b}^{g^M}(0, \alpha^M, k))$. Suppose that k agents respectively in $N_i(g^G)$ and in $N_i(g^M)$ draw state 1. Then, we define $H(x) = \mathcal{Q}(\alpha^G, x, k)/\mathcal{Q}(\alpha^M, x, k) = \left[\binom{\alpha^G}{x}\binom{n_i(g)+1-\alpha^G}{k-x}\right] / \left[\binom{\alpha^M}{x}\binom{n_i(g)+1-\alpha^M}{k-x}\right]$ which provides the ratio of the weights (when they are defined) associated with $1-\exp\left[-\rho(x\delta^G + (k-x)\delta^M)\right]$. $H(x)$ is an increasing function with the number of agents who draw state 1 in the neighborhood of agent i. Consequently, the weights associated with the most valuable income are higher in g^G than the weights associated with the most valuable income in g^M. It follows that $\mathcal{Z}_i^{G,G}(g) > \mathcal{Z}_i^{G,M}(g)$. □

In the following corollary, we establish that there exist parameters such that a network where agents are linked only with other agents of the same type is a pairwise stable network. We denote by $g^{G/M}$ the network such that $g_{i,j}^{G/M} = 1$ if and only if $i, j \in N^x$, $x \in \{G, M\}$. The network $g^{G/M}$ can be seen as two different networks: the complete network $g_1^{G/}$ which contains all agents in N^G and where $g_{i,j}^{G/} = g_{i,j}^{G/M}$ for all $i, j \in N^G$, and the complete network $g^{/M}$ which contains all agents in N^M and where $g_{i,j}^{/M} = g_{i,j}^{G/M}$. The ENB of each player is not the same in $g^{G/}$ and in $g^{/M}$ since agents do not give and receive the same amount of money in each of these networks. For this corollary, we first observe that if $\delta^M = 0$, then $0 = \mathcal{Z}_i^{M,M}(g) > \mathcal{Z}_i^{G,M}(g)$. The strict inequality holds when δ^M is sufficiently small.

Corollary 3 *Suppose that for $i \in N^M$, $i' \in N^G$, we have $\bar{B}_i(g^{G/M}) - \bar{B}_i(g^{G/M} - g_i^{G/M}) > \mathcal{Z}_{i'}^{G,M}(g^{G/M})$. Then, there exists $F > 0$ such that $g^{G/M}$ is a pairwise stable network.*

Proof We know that the ENB is concave and for all g, $\mathcal{Z}_i^{G,G}(g) > \mathcal{Z}_i^{G,M}(g)$. It follows that for $i \in N^G$, $\mathcal{A}_2 = \bar{B}_i(g^{G/M}) - \bar{B}_i(g^{G/M} - g_{i,j}^{G/M}) > \mathcal{Z}_i^{G,M}(g^{G/M}) = \mathcal{A}_1$. Moreover, by assumption, for $i, j \in N^M$, we have $\mathcal{A}_3 = \bar{B}_i(g^{G/M}) - \bar{B}_i(g^{G/M} - g_{i,j}^{G/M}) > \mathcal{A}_1$. We set $F < \min\{\mathcal{A}_2, \mathcal{A}_3\}$ and $F > \mathcal{A}_1$. First, since $F > \mathcal{A}_1$, no agent in N^G will accept to form a link with an agent in N^M. Second, since $F < \min\{\mathcal{A}_2, \mathcal{A}_3\}$ no agent $i \in N$ has an incentive to remove one of her links in $g^{G/M}$. The result follows. □

Corollary 3 shows that there exist situations where in a pairwise stable network agents are partitioned according to their type, that is generous agents are linked together, miserly agents are linked together, but there are no links between these two types of agents. It is worth noting that for sufficiently small costs of forming links, and close enough δ^G and δ^M, generous agents will consent to form links with miserly agents. Indeed, generous agents are risk averse and prefer to be linked with miserly agents to be less insured.

5 Costs of forming links depend on the neighborhood of agents

5.1 Cost function and ENB

Informal insurance arrangements are potentially limited by the presence of various incentive constraints. As a first cut, it appears that the most important constraint arises from the fact that these arrangements are informal, *i.e.*, not written on legal paper. It follows that they will be honored only if agents involved in such a relationship invest time. Since each agent has a limited amount of time, the costs for agent i of forming an additional link with some agent j should increase with the number of links formed by agent i. Moreover, these costs should also increase with the number of links formed by agent j. Indeed, it is more difficult to establish a relationship with an agent who already has numerous links since she has less time available.[10] We capture these ideas through the following cost function for link formation:

$$C_i(g) = f_1(n_i(g)) + \sum_{\ell \in N_i(g)} f_2(n_\ell(g)),$$

where f_2 is strictly increasing and convex. In addition, $f_1(0) = 0$. Given this cost function, an additional link formed with agent j induces a cost for agent i equal to

$$C_i(g + ij) - C_i(g) = \Delta f_1(n_i(g) + 1) + f_2(n_j(g) + 1).$$

Since f_1 is strictly increasing and $f_2 > 0$, $C_i(g + ij) - C_i(g) > 0$.

The expected payoff function, $U_i(g)$, of each agent i, given the network g, is the difference between the ENB function and the cost function of forming links. We assume the same ENB

[10] Another option that makes such informal arrangements feasible is the threat of punishment as in Bloch, Genicot and Ray, 2008.

as in section 2. We have:

$$U_i(g) = B_i(g) - C_i(g) = \phi(n_i(g)) - \left(f_1(n_i(g)) + \sum_{\ell \in N_i(g)} f_2(n_\ell(g)) \right). \qquad (6)$$

Proposition 1 allows us to characterize some properties of the marginal payoffs, $\Delta U_i(g, ij) = B_i(g + ij) - C_i(g + ij) - (B_i(g) - C_i(g))$, obtained by agent i in a network g when she forms an additional link with agent j. We have $\Delta U_i(g, ij) = \gamma(n_i(g) + 1, n_j(g) + 1)$ and

$$\gamma(n_i(g) + 1, n_j(g) + 1) = \Delta\phi(n_i(g) + 1) - \Delta f_1(n_i(g) + 1) - f_2(n_j(g) + 1),$$

Clearly, γ is strictly decreasing in its first argument since $\Delta\phi$ is strictly decreasing by Proposition 1 and Δf_1 is strictly increasing. Similarly, γ is strictly decreasing in its second argument since f_2 is strictly increasing. T. Morrill (2011) has defined the class of network formation game with degree-base utility function under negative externalities. In the following sections, we complete its results since he mainly focuses his attention on situations where transfers between agents are allowed.[11]

5.2 Pairwise stable networks

First, we present an existence result in this context. Let $k^\star$, $k^\star \in \{0, \ldots, n-1\}$, satisfy the two following conditions: (1) $\gamma(k^\star, k^\star) \geq 0$, and (2) there is no k, $k > k^\star$, such that $\gamma(k, k) \geq 0$. In other words, $k^\star$ is the number of links which allows an agent i who has formed $k^\star$ links to obtain a positive marginal expected payoff from a link with an agent j who has formed $k^\star$ links. Furthermore, it is a threshold since no k' higher than $k^\star$ satisfies this property. Recall that $\gamma(\cdot, \cdot)$ is strictly decreasing in its two arguments. Hence, we have $\gamma(k, k) > 0$ for all $k < k^\star$. For the next results in this section, we always use the payoff function given by equation 6.

Proposition 7 *A pairwise stable network always exists.*

Proof See Appendix. □

Proposition 7 shows that our setting is consistent with our steady state solution of pairwise stability. The next proposition imposes conditions that a pairwise stable network must satisfy

[11]In his paper, the author does not deal with the existence of pairwise stable networks. Moreover, he does not provide a full characterization of pairwise stable networks when transfers are not allowed.

and sheds light on when insurance arrangements will exhibit symmetric and asymmetric structures. In particular, this proposition does not exclude from the set of pairwise stable networks those networks where agents are in asymmetric positions relative to the amount of insurance they receive. To show this result, we need the following definition.

Let $S_\ell(g) = \{j \in N \mid n_j(g) = \ell\}$ be the set of agents who have formed ℓ links in g. It is worth noting that the set $S_\ell(g)$ is empty if no agents form ℓ links in g.[12]

Proposition 8 *If a network $g^\star$ is pairwise stable, then the two following conditions are satisfied:*

1. *(Q1) If $\ell, \ell' > k^\star$, then there is no link between an agent who belongs to S_ℓ and an agent who belongs to $S_{\ell'}$ in $g^\star$. If $\ell, \ell' < k^\star$, then there is a link between an agent who belongs to S_ℓ and an agent who belongs to $S_{\ell'}$ in $g^\star$.*

2. *(Q2) Suppose $\ell' \leq \ell < k^\star$ and $k^\star < k' \leq k$. If there is a link between an agent who belongs to S_ℓ and an agent who belongs to S_k in $g^\star$, then there is a link between an agent who belongs to $S_{\ell'}$ and an agent who belongs to $S_{k'}$ in $g^\star$.*

3. *(Q3) If $\gamma(k^\star, k^\star + 1) < 0$, then there is no agent who have formed more than $k^\star$ links.*

Proof Let $g^\star$ be a pairwise stable network. To introduce a contradiction, suppose that $g^\star$ does not satisfy property Q1. In particular, suppose that there is a link between an agent, say i, who belongs to S_ℓ and an agent, say j, who belongs to $S_{\ell'}$, with $\ell, \ell' > k^\star$, in $g^\star$. Then, the marginal payoff of agent j from the link with agent i is equal to $\gamma(n_j(g), n_i(g))$ with $(n_j(g), n_i(g)) \geq (k^\star + 1, k^\star + 1)$. Since γ is strictly decreasing in its two arguments, we have $\gamma(n_j(g), n_i(g)) \leq \gamma(k^\star + 1, k^\star + 1) < 0$. It follows that the link between agent i and j does not belong to $g^\star$, a contradiction. Similarly, suppose that there is no link between an agent, say i, who belongs to S_ℓ and a agent, say j, who belongs to $S_{\ell'}$, with $\ell, \ell' < k^\star$, in $g^\star$. Then, the marginal payoff associated with the link between agents i and j for agent j is equal to $\gamma(n_j(g), n_i(g))$ with $(n_j(g), n_i(g)) < (k^\star, k^\star)$. Since γ is strictly decreasing in its two arguments, we have $\gamma(n_j(g), n_i(g)) > \gamma(k^\star, k^\star) \geq 0$. Similarly, for agent i we have $\gamma(n_i(g), n_j(g)) > \gamma(k^\star, k^\star) \geq 0$. It follows that agents i and j have an incentive to be linked in $g^\star$, a contradiction.

[12] We use $S_\ell(g) = S_\ell$ when there is no doubt about the network being studied.

Next suppose that $g^\star$ does not satisfy property Q2. In particular, consider ℓ, ℓ', k, and k' such that $\ell' \leq \ell < k^\star$ and $k^\star < k' \leq k$ and suppose that there is a link between an agent, say i, who belongs to S_ℓ and an agent, say j, who belongs to S_k in $g^\star$ and there is no link between an agent, say i', who belongs to $S_{\ell'}$ and an agent, say j', who belongs to $S_{k'}$ in $g^\star$. Since there is a link between agents i and j we have $\gamma(n_i(g), n_j(g)) \geq 0$ and $\gamma(n_j(g), n_i(g)) \geq 0$. Moreover, since there is no link between agents i' and j' we have $\gamma(n_{i'}(g), n_{j'}(g)) < 0$ or $\gamma(n_{j'}(g), n_{i'}(g)) < 0$ with $n_i(g) \geq n_{i'}(g)$ and $n_j(g) \geq n_{j'}(g)$. These inequalities are not compatible with the fact that γ is strictly decreasing in its two arguments, a contradiction.
Finally, suppose that $\gamma(k^\star, k^\star+1) < 0$, then agents who have formed $k^\star$ links will never form a link with an agent who has $k' > k^\star$ neighbors in $g^\star$ since γ is decreasing in its second argument. Since agents who have formed more than $k^\star$ links do not formed links together, the neighbors of agent i who has formed $k' > k^\star$ links have formed $k < k^\star$ links. Agents, who have formed $k < k^\star$ links, form links together. It follows that agents who have formed $k < k^\star$ are less than $k^\star$. Consequently the number of neighbors of agent i is $k < k^\star < k'$, a contradiction.

□

We now graphically illustrate Proposition 8. In Figure 4, network g satisfies the properties given in $Q1$. Indeed, if we assume that $k^\star = 4$, we observe that agents 1 and 2 are involved in $k^\star + 1$ links, agents 3, 4 and 5 are involved in $k^\star$ links and agents 6 and 7 are involved in $k^\star - 1$ links, and they are linked.

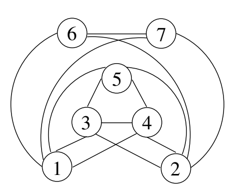


Figure 4: Network g satisfying $Q1$

Note that in Proposition 8 we do not examine agents who have formed exactly $k^\star$ links. Indeed, these agents can form links both with agents who have formed $x > k^\star$ links and with agents who have formed $y < k^\star$ links.
Proposition 8 highlights several properties of pairwise stable networks in a community of *ex ante* identical agents. First, agents who obtain insurance from more than $k^\star$ agents (they are

the most insured agents) are never linked together. This property illustrates the fact that some agents play a specific role in the provision of mutual insurance in the pairwise stable networks: these agents insure (and are insured by) a large part of the population. But an agent of this type does not interact with other agents of this type. In some sense there may exist "some insurance leaders" but these leaders themselves are not linked by mutual insurance agreements. Secondly, we find that there is a kind of solidarity among the less insured agents: agents who obtain the smallest amount of insurance are always linked together. This property illustrates the fact that agents, who do not have a sufficient amount of insurance, will always reach mutual insurance agreements. Finally, Q3 establishes that the asymmetric position of agents with regard to the insurance, can be reduced if the cost of forming links becomes sufficiently high when the neighbors has more than $k^\star$ neighbors.

From these two properties and the *pigeon hole principle*[13], we obtain the following corollary.

Corollary 4 *Agents who have formed a number of links higher than $k^\star$ (and obtain the highest amount of insurance) are always fewer than the rest of the population.*

Lastly, pairwise stable networks may divide the population into sets of agents who are in asymmetric positions relative to their risk exposure. In other words, in a pairwise stable network, some agents are better off since they obtain insurance from others agents, who are involved in few mutual insurance agreements themselves. Note that in our model, agents can have different numbers of bilateral insurance agreements, but this difference is bounded. Indeed, let $\overline{n}(g) = \max_{i \in N}\{n_i(g)\}$ be the number of links formed by the agents who have formed the highest number of links and let $\underline{n}(g) = \min_{i \in N}\{n_i(g)\}$ be the number of links formed by the agents who have formed the smallest number of links.

Corollary 5 *Let $g^\star$ be a pairwise stable network. Then (a) $\overline{n}(g^\star) \leq \sum_{\ell \leq k^\star} |S_\ell(g^\star)|$ and (b) $\underline{n}(g^\star) \geq \sum_{\ell < k^\star} |S_\ell(g^\star)|$.*

The above corollary allows us to bound the difference of links in which the highest insured agents and the lowest ones are involved. We have $\overline{n} - \underline{n} \leq |S_{k^\star}(g^\star)|$.

[13]This principle states the following. Let n and k be positive integers, and let $n > k$. Suppose we have to place n identical balls into k identical boxes, where $n > k$. Then there will be at least one box in which we place at least two balls.

Moreover, Proposition 8 establishes that networks where agents are in symmetric positions and networks where agents are in asymmetric positions are candidates to be pairwise stable. In the next corollary, we show that there always exist some "symmetric" networks in the set of pairwise stable networks. In other words, there always exist pairwise stable networks where either all agents obtain an identical, or a very similar amount of insurance.

Corollary 6 *If n or $k^\star$ are even, then there exists a k-regular network which is pairwise stable. If n and $k^\star$ are odd, then, there exists an almost-k-regular network which is pairwise stable.*

Proof This result follows the proof of Proposition 7. □

Corollary 6 establishes that the set of pairwise stable networks always contains either a $k^\star$-regular network, or an almost-$k^\star$-regular network. The fact that agents are in "symmetric" positions in a pairwise stable mutual insurance network does not mean that these agents obtain the same amount of benefits. Indeed, in our model agents who draw state 0 always obtain a lower amount of benefits than the benefits obtained by agents who draw state 1.

Finally, we highlight the fact that in our setting agents can be partitioned into distinct components in a pairwise stable network. Moreover some of these stable insurance networks may also be locally complete as the network between agents 1, 2 and 3 in the following example.

Example 2 Suppose $N = \{1, \ldots, 12\}$ and $k^\star = 2$. Then, network g shown in Figure 5 is a pairwise stable network.

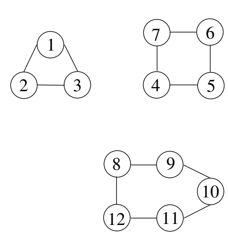


Figure 5: Network g

5.3 Efficient Networks

We now deal with the efficient networks. In this section, we always use the payoff function given by equation 6. Let $\eta(k) = \phi(k) - f_1(k) - kf_2(k)$. By assumption, ϕ is concave, and f_1 is convex. Moreover, $kf_2(k)$ is convex, since f_2 is increasing and convex. It follows that η is concave and admits a unique maximum. Moreover, note that the neighbors of agent i are connected to an agent with $n_i(g)$ neighbors. Consequently, we have $W(g) = \sum_{i \in N}[\phi(n_i(g)) - f_1(n_i(g)) - n_i(g)f_2(n_i(g))] = \sum_{i \in N} \eta(n_i(g))$. We know that there exists $k^e \in \{0, \ldots, n-1\}$ such that $\eta(k^e)$ is maximal. Therefore $\sum_{i \in N} \eta(k^e) \geq \sum_{i \in N} \eta(n_i(g))$ for all $n_i(g) \in \{0, \ldots, n-1\}$. Moreover, we have $\arg\max_{k \neq k^e} \eta(k) \subset \{k^e - 1, k^e + 1\}$, since η is concave and maximum for $k = k^e$. Therefore, we have $\sum_{i=1}^{n-1} \eta(k^e) + \max\{\eta(k^e - 1), \eta(k^e + 1)\} \geq \sum_{i=1}^{n-1} \eta(k^e) + \eta(n_n(g))$ for $n_n(g) \neq k^e$. These observations are summarized in the next proposition.

Proposition 9 *Let g^e be an efficient network. Then, g^e is either a k^e-regular network, or an almost-k^e-regular network.*

Proof We use Lemma 1 to obtain the existence of one of these three networks. □

It follows from Corollary 6 that if k or n are even, then only $k^\star$-regular networks can be both pairwise stable and efficient. However, in the following corollary we show that $k^\star$-regular pairwise stable networks and efficient networks do not always coincide. More precisely, we establish that in the $k^\star$-regular stable network, agents will form at least the same number of links as in the efficient network. Observe that in this case, a non-efficient pairwise stable network is always over-connected from the efficiency perspective. To simplify the presentation, we assume that n is even.

Corollary 7 *Suppose n is even. Then in the $k^\star$-regular pairwise stable network agents will form at least the same number of links as in the efficient network.*

Proof Suppose n is even. Then by Lemma 1, network $g^\star$ where all agents have formed $k^\star$ links, and network g^e where all agents have formed k^e links, exist. By Corollary 6, we know that $g^\star$ is pairwise stable, and by Proposition 9, we know that g^e is efficient. Moreover, we have $\gamma(k, k) - (\eta(k) - \eta(k-1)) = (k-1)(f_2(k) - f_2(k-1)) \geq 0$, for $0 < k < n$ since f_2 is strictly increasing. Now, by definition of k^e, we have $\eta(k^e) - \eta(k^e - 1) \geq 0$. It follows that we have $\gamma(k^e, k^e) \geq \eta(k^e) - \eta(k^e - 1) \geq 0$. We know that by definition of $k^\star$, we have $\gamma(k, k) < 0$, for all $k > k^\star$. It follows that $k^\star$ is at least equal to k^e. □

The intuition behind this result is as follows. Let $g^\star$ be the regular stable network where all agents have formed $k^\star$ links, and let i, j, k be three agents such that $g^\star_{ij} = g^\star_{ik} = 1$, and $g^\star_{kj} = 0$. If agents i deletes her link with agent j, then agent k will benefit from this deletion, since her payoffs will increase by $f_2(n_i(g^\star) - 1) - f_2(n_i(g^\star))$. However, when i decides whether to delete her link with j in $g^\star$, she does not take into account the positive externality that would accrue to k from the deletion of this link.

6 Conclusion

In this paper, we study situations where agents form some mutual informal insurance arrangements on their own. More precisely, we examine when agents will create agreements with their neighbors concerning the following transfer scheme: each agent helps her neighbors who draw the low income state when she herself draws the high income state and each agent is helped by her neighbors who draw the high income state when she herself draws the low income state.
We find that efficient networks are either k-regular networks, or almost-k-regular networks. In other words, only networks where agents obtained a very similar level of insurance are efficient networks. By contrast, this is certainly not true for pairwise stable networks. Under certain conditions they can be asymmetric, with those having the lowest levels of insurance always connecting to each other, while those with the highest levels of insurance never forming any links with each other.
Then, we introduce in the model two types of heterogeneity: an exogenous one, where agents differs in their income or in their preferences over the transfer scheme, and an endogenous heterogeneity where the costs of linking to an agent depends on the number of links the latter has already formed in the network. We examine the impact of these heterogeneities on stability and efficiency. We obtain several distinctive results. In particular, when agents do not obtain the same income when they draw the high income state, we show that the insurance mechanism may increase the gap between the expected well-being of rich people and the expected well-being of poor people. Indeed, in some situation only agents who obtain the highest income when they draw the high income form insurance links. Moreover, when there exist two

types of agents: the generous ones who are more giving more than the miserly ones, we show that in some situation agents are partitioned in the network according to their type: generous agents are linked together, miserly agents are linked together, but there are no links between these two types of agents.

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7 Appendix

To simplify notation, we extend $b^g(1,\cdot)$ to $b^g(1,-1) = 1 - \exp[-\rho(\Theta - (n_i(g) - k - 1)\delta)]$. To demonstrate Proposition 1, we need to compute the difference between $B_i(g + ij)$ and $B_i(g)$, called $\Delta B_i(g, ij)$. We set $P(n_i(g), k) = \binom{n_i(g)}{k} p^k (1-p)^{n_i(g)-k}$.

Proof of Proposition 1..

1. We calculate the marginal expected benefits associated with the addition of a link. We present successively the two situations which can arise when agent i forms an additional link with agent j in g.

Suppose j draws state 1. This occurs with probability p. Then the benefits obtained by agent i when she forms a link with agent j is

$$\left(p \sum_{k=0}^{n_i(g)} P(n_i(g), k) b^g(1, k)\right) + \left((1-p) \sum_{k=0}^{n_i(g)} P(n_i(g), k) b^g(0, k+1)\right)$$

Suppose j draws state 0. This occurs with probability $1 - p$. Then the benefits obtained by agent i when she forms a link with j is

$$\left(p \sum_{k=0}^{n_i(g)} P(n_i(g), k) b^g(1, k-1)\right) + \left((1-p) \sum_{k=0}^{n_i(g)} P(n_i(g), k) b^g(0, k)\right)$$

We set $\Delta b^g(0,k+1) = (\exp[-\rho\delta])^k(1-\exp[-\rho\delta]) > 0$, and $\Delta b^g(1,k) = (\exp[\rho\delta])^{n_i(g)-k}$ $\exp[-\rho\Theta](\exp[\rho\delta]-1)$

By using the binomial theorem and straightforward computations we obtain

$$\begin{aligned}\frac{\Delta B_i(g,ij)}{p(1-p)} &= \sum_{k=0}^{n_i(g)} P(n_i(g),k)\Delta b^g(0,k+1) - \sum_{k=0}^{n_i(g)} P(n_i(g),k)\Delta b^g(1,k)\\ &= (1-\exp[-\rho\delta])(1+p(\exp[-\rho\delta]-1))^{n_i(g)} \qquad (7)\\ &\quad -(\exp[\rho\delta]-1)\exp[-\rho\Theta](\exp[\rho\delta]+p(1-\exp[\rho\delta]))^{n_i(g)}.\end{aligned}$$

2. We show that the expected neighborhood benefits function of agent i is strictly concave. In order to prove this statement we need to assign a sign to the difference between the marginal benefits. To obtain this result, we assume that agent i adds the link $g_{ik}=1$ to the network $g+g_{ij}$. Following the same steps as in Point 1. we have:

$$\begin{aligned}\frac{\Delta B_i(g+ij,ik)}{p(1-p)} &= (1-\exp[-\rho\delta])(1+p(\exp[-\rho\delta]-1))^{n_i(g)+1}\\ &\quad -(\exp[\rho\delta]-1)\exp[-\rho\Theta](\exp[\rho\delta]+p(1-\exp[\rho\delta]))^{n_i(g)+1}.\end{aligned}$$

We now determine the sign of the difference between $\Delta B_i(g+ij,ik)$ and $\Delta B_i(g,ij)$. We observe that

$$\frac{(1+p(\exp[-\alpha\delta]-1))^{n_i(g)+1}}{(1+p(\exp[-\alpha\delta]-1))^{n_i(g)}} = 1+p(\exp[-\alpha\delta]-1<1,$$

and

$$\frac{(\exp[\alpha\delta]+p(1-\exp[\alpha\delta]))^{n_i(g)+1}}{(\exp[\alpha\delta]+p(1-\exp[\alpha\delta]))^{n_i(g)}} = (1-p)\exp[\alpha\delta]+p>1.$$

It follows that $\Delta B_i(g+ij,ik)-\Delta B_i(g,ij)<0$.

3. Finally, we show that the expected neighborhood benefits function of agent i is strictly increasing. From point 2. we know that the marginal expected neighborhood benefits is the lowest for $n_i(g)=n-2$. So we have to show that $\Delta B_i(g,ij)>0$, when g is such that $n_i(g)=n-2$. For $n_i(g)=n-2$, we have:

$$\begin{aligned}\frac{\Delta B_i(g,ij)}{p(1-p)} &= (1-\exp[-\rho\delta])(1+p(\exp[-\rho\delta]-1))^{n-2}\\ &\quad -(\exp[\rho\delta]-1)\exp[-\rho\Theta](\exp[\rho\delta]+p(1-\exp[\rho\delta]))^{n-2}.\end{aligned}$$

Since $\Theta > \delta n$, we have $\exp[\rho\Theta] > \exp[\rho\delta n]$. Consequently,

$$\frac{(1-\exp[-\rho\delta])(1+p(\exp[-\rho\delta]-1))^{n-2}}{(\exp[\rho\delta]-1)\exp[-\rho\Theta](\exp[\rho\delta]+p(1-\exp[\rho\delta]))^{n-2}} > \frac{(1-\exp[-\rho\delta])\exp[\rho\delta n](1+p(\exp[-\rho\delta]-1))^{n-2}}{(\exp[\rho\delta]-1)(\exp[\rho\delta]+p(1-\exp[\rho\delta]))^{n-2}}$$

$$= \exp[\rho\delta] > 1.$$

The result follows.

□

Proof of Lemma 1. We prove successively that the three sequences are graphical.

1. Suppose n or k is even. Let $n > k > 0$. Since either n, or k is even, the sum of the sequence $s = (k, k \ldots, k)$ is even. Equation 4 can be written as

$$rk \leq r(r-1) + \sum_{i=r+1}^{n} \min\{k, r\}, \text{ for every } r,\ 1 \leq r < n. \tag{8}$$

   There are two cases. Suppose $r \leq k$. Then equation 8 is satisfied if

$$rk \leq r(r-1) + (n-r)r \Rightarrow k \leq (r-1) + (n-r) \Rightarrow k \leq n-1.$$

   This equation is always satisfied. Suppose $r > k$. Then equation 8 is

$$rk \leq r(r-1) + (n-r)k \tag{9}$$

   If $k = n-1$, then $s = (n-1, \ldots, n-1)$ is a graphical sequence since the complete network supports this sequence. Similarly, if $k = 0$, then $s = (0, \ldots, 0)$ is a graphical sequence since the empty network supports this sequence. We now deal with k, $0 < k < n-1$. We have

$$r(r-1) + (n-r)k - rk = r^2 - r(1+2k) + nk = \left(r - \frac{2k+1}{2}\right)^2 + nk - \left(\frac{2k+1}{2}\right)^2.$$

   Since $nk \geq (k+2)k = (k+1)k + k$ and $\left(\frac{2k+1}{2}\right)^2 = (k+1)k + 1/4$, we have $nk - \left(\frac{2k+1}{2}\right)^2 \geq 0$, for $0 < k < n-1$.

2. Suppose n and k are odd, with $n-1 > k > 0$ ($k \neq n-1$ since k and n are odd). Since n is odd, $n-1$ is even and since k is odd, $k+1$ is even. Consequently, the sum of the sequence $s = (k+1, k \ldots, k)$ is even.

For $r = 1$, equation 4 is satisfied since $k + 1 \leq (n - 1)$ for $0 < k < n - 1$. For $r \geq 2$, equation 4 is equal to

$$k + 1 + (r - 1)k \leq r(r - 1) + \sum_{i=r+1}^{n} \min\{k, r\}, \text{ for every } r,\ 2 \leq r < n. \tag{10}$$

There are two cases. (1) Suppose $r \leq k$, with $k \leq n - 2$, and $r \geq 2$. Then equation 10 is

$$k + 1 + (r - 1)k \leq r(r - 1) + (n - r)r \Rightarrow k \leq (r - 1) + (n - r) - \frac{1}{r} \Rightarrow k \leq (n - 1) - \frac{1}{r}.$$

This equation is always satisfied since $k < n - 1$ and $1/r < 1$ for $r > 2$. (2) Suppose $r > k$. Then equation 10 is

$$rk + 1 \leq r(r - 1) + (n - r)k. \tag{11}$$

We first deal with the case where $k = n - 2$. In that case $r = n - 1$. Therefore, we have:

$$(n - 1)(n - 2) + 1 \leq (n - 1)(n - 2) + n - 2,$$

and since $n \geq 3$, this equation is always satisfied. We now deal with $k < n - 2$, we have

$$r(r-1)+(n-r)k-rk-1 = r^2-r(1+2k)+nk-1 = \left(r - \frac{2k+1}{2}\right)^2 + nk - \left(\frac{2k+1}{2}\right)^2 - 1.$$

Since $nk \geq (k + 3)k = (k + 1)k + 2k$ and $\left(\frac{2k+1}{2}\right)^2 + 1 = (k + 1)k + 5/4$, we have $nk - \left(\frac{2k+1}{2}\right)^2 - 1 > 0$, for $0 < k < n - 2$.

3. Suppose n and k are odd. Since n is odd, $n - 1$ is even and since k is odd, $k - 1$ is even. Consequently, the sum of the sequence $s = (k, \ldots, k, k - 1)$ is even. equation 4 is equal to

$$rk \leq r(r - 1) + \sum_{i=r+1}^{n-1} \min\{k, r\} + \min\{k - 1, r\}, \text{ for every } r,\ 1 \leq r < n - 1. \tag{12}$$

There are two cases. (1) Suppose $r \leq k - 1$, with $k < n - 1$ ($k \neq n - 1$ since n and k are odd). Then equation 12 becomes

$$rk \leq r(r - 1) + (n - r)r, \text{ for every } r,\ 1 \leq r < n.$$

We have already shown in point 1., equation 9, that this equation is always satisfied. (2) Suppose $r > k - 1$. Then equation 12 becomes

$$rk \leq r(r - 1) + (n - r)k - 1 \Rightarrow rk + 1 \leq r(r - 1) + (n - r)k. \tag{13}$$

We first deal with the case where $k = n - 2$. In that case either $r = n - 1$, or $r = n - 2$. We have shown in point 2., equation 11, that the previous equation is satisfied when $r = n - 1$ and $k = n - 2$. If $r = n - 2$ and $k = n - 2$, we have

$$(n-2)(n-2)+1 \leq (n-2)(n-3)+2(n-2) \Rightarrow (n-2)(n-2)+1 \leq (n-2)(n-2)+(n-2).$$

This equation is always satisfied since $n \geq 3$. Finally, we have shown in point 2., equation 11, that equation 13 is satisfied when $0 < k < n - 2$.

□

Proof of Proposition 7. Let n or $k^\star$ be even. Then, we build the network g^r where all agents form $k^\star$ links. We know by Lemma 1 that g^r exists. We now show that g^r is pairwise stable. First, no agent has a strict incentive to remove a link since $\gamma(k^\star, k^\star) \geq 0$ in g^r. Second, no agent has an incentive to add a link since $\gamma(k^\star + 1, k^\star + 1) < 0$ in g^r. Therefore g^r is a pairwise stable network.

Suppose now that k and n are odd. There are two cases: either (a) $\gamma(k^\star + 1, k^\star) \geq 0$ and $\gamma(k^\star, k^\star + 1) \geq 0$, or (b) $\gamma(k^\star + 1, k^\star) < 0$ or $\gamma(k^\star, k^\star + 1) < 0$. We first deal with case (a): $\gamma(k^\star + 1, k^\star) \geq 0$ and $\gamma(k^\star, k^\star + 1) \geq 0$, with $k^\star \neq 0$. We build the network g where one agent, say i, forms $k^\star + 1$ links and all other agents form $k^\star$ links. By Lemma 1, the network g exists. Using the same argument as above we can see that no agent has an incentive to add or remove one link in g. Next, we deal with case (b): $\gamma(k^\star + 1, k^\star) < 0$ or $\gamma(k^\star, k^\star + 1) < 0$, with $k^\star \neq 0$. We build network g' where one agent, say i, forms $k^\star - 1$ links and all other agents form $k^\star$ links. By Lemma 1, the network g' exists. If agent i forms a link with agent j, then i obtains a marginal payoff associated with this link equal to $\gamma(k^\star, k^\star + 1)$, while j obtains a marginal payoff associated with this link equal to $\gamma(k^\star + 1, k^\star)$. By assumption, $\min\{\gamma(k^\star, k^\star + 1), \gamma(k^\star + 1, k^\star)\} < 0$. Therefore agent i or agent j has no incentive to form this link. Moreover, no agent has an incentive to remove one of her links and no agent $j \in N \setminus \{i\}$ has an incentive to add a link with an agent $j' \in N \setminus \{i\}$ in g', since $\gamma(k^\star + 1, k^\star + 1) < 0$. This completes the proof. □