Dempster-Shafer Equilibrium and Unambiguous Types

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\section*{Abstract}

This paper explores the impact of the assumption of unambiguous types on the Dempster-Shafer equilibrium introduced by Eichberger and Kelsey (2004). It is shown that if the types of the Sender are perceived unambiguous, then the Receiver’s conditional Choquet preference derived by the Dempster-Shafer updating rule is expected utility, regardless of whether the observed massage is ambiguous or not. This property of the updating rule narrows down the equilibrium behavior of the pooling but not separating Dempster-Shafer equilibrium to the same behavior of perfect Bayesian equilibrium. Moreover, if one relaxes the assumption of unambiguous types but maintains the belief persistence axiom of Ryan (2002a), then any separating Dempster-Shafer equilibrium is behaviorally equivalent to perfect Bayesian equilibrium.

\textit{Keywords:} Ambiguity, Choquet expected utility, unambiguous events, updating, signaling games, Dempster-Shafer equilibrium, refinements

\textit{JEL:} C72, D82, D83

\section{1. Introduction}

This paper revisits the Dempster-Shafer equilibrium (DSE) notion, a novel solution concept for signaling games under ambiguity introduced by Eichberger and Kelsey (2004). Ambiguity is accommodated into games via Choquet expected utility (CEU) theory à la Schmeidler (1989). Players’ subjective beliefs are represented by capacities (i.e., non-additive probabilities). A primary motivation for introducing non-additive beliefs into games is to

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explain strategic behavior that cannot be captured as an equilibrium within the standard Bayesian framework. In line with such motivation, this paper aims to shed new light on the scope and limitation of the DSE concept for modeling strategic communication with ambiguous signals.

The key feature of Eichberger and Kelsey’s equilibrium notion is the updating rule for non-additive beliefs. The Receiver observes a message sent by the Sender, and revises his beliefs by applying the updating rule introduced by Dempster (1968) and Shafer (1976). Gilboa and Schmeidler (1993) provides an axiomatic foundation for the Dempster-Shafer updating rule. The revision rule underlies the idea of pessimistic belief change; that is, the conditioning message is always regarded as “not-a-good-news.” The attractiveness of the Dempster-Shafer rule relies on the fact that the conditional capacity is well-defined on the events of measure zero, provided that the event is ambiguous. In signaling games, this property enables updating on the information sets that are off-the-equilibrium-path.

Our first goal is to present another property of the Dempster-Shafer updating rule. Typically, ambiguity designates a situation where some events are unambiguous while others might be ambiguous. For instance, in games under incomplete information, it is often taken for granted that the probability distribution on the types of the players is common knowledge. When beliefs are subjective, it is reasonable to assume that the Receiver incorporates the known probabilities into his ex-ante beliefs by revealing the Sender’s types to be unambiguous. To accommodate the assumption of unambiguous types, we apply Nehring’s (1999) behavioral notion of non-ambiguity. When some events are unambiguous (e.g. types) while other events are ambiguous (e.g. messages), it is important to know whether the ex-ante unambiguity remains preserved by conditional preferences. Our first result demonstrates that it is the case under the Dempster-Shafer conditioning rule. More precisely, if the conditional capacity is derived according to the Dempster-Shafer rule, then any ex-ante unambiguous event is also revealed to be unambiguous by the conditional Choquet preference.

This result has an important implication for Choquet preferences taken with respect to a capacity defined on a Cartesian product of two finite sets, e.g., a set of types and a set of messages. The implication is that if the types are perceived ex-ante unambiguous and the conditioning event is an ambiguous message, then the conditional Choquet preference derived by the Dempster-Shafer rule coincides with the expected utility form. Both results hold true irrespective of whether the ex-ante capacity is convex or non-convex.
(i.e., the Receiver exhibits ambiguity-aversion or not (see Schmeidler, 1989)).

Consequently, when types are assumed to be unambiguous, the only ambiguity that the Receiver may perceive is about the Sender’s messages. After a message is observed, the ambiguity is resolved. Our game theoretic analysis begins by exemplifying a signaling game with a separating equilibrium behavior that cannot be depicted as an equilibrium under the Bayesian paradigm. Although the Receiver exhibits conditional expected utility preferences, the pessimism inherent in the Dempster-Shafer updating rule hinders him from learning (i.e., ascribing probability value one to) the true type revealed by the Sender. This sort of behavior is inconsistent with the standard perfect Bayesian equilibrium (PBE), where learning is a consequence of the Sender’s separating behavior.

Recently, Ryan (2002a) observed that the Dempster-Shafer updating rule may violate the so-called belief persistence axiom. Roughly speaking, the axiom requires new information to be embedded with the smallest possible change to ex-ante beliefs. For instance, an updated capacity that attaches a positive value to the states that were regarded ex-ante as impossible to occur (i.e., states outside of the support of the ex-ante capacity) violate the belief persistence axiom. To avoid such inconsistencies, Ryan suggested a refinement of DSE obtained by eliminating conditional beliefs that violate the belief persistence postulate. However, the refinement process has a severe implication. When the Sender’s types are perceived unambiguous then the Receiver’s refined beliefs also preclude ambiguity about the massages. More specifically, at any separating DSE the Receiver has to exhibit ex-ante expected utility preferences.

The question rises whether deviations from PBE are compatible at all with the belief persistence axiom. For that to be possible, the assumption of unambiguous types needs to be abandoned. However, when types and messages are ambiguous, it has to be clarified first under what condition the belief persistence is respected. As it turns out, if the the support of an ex-ante capacity is perceived unambiguous, then the conditional capacity derived by the Dempster-Shafer updating rule maintains the belief persistence axiom. This result suggests that the violation of the principle of belief persistence is not caused by the equilibrium notion itself, rather by the fact that the states outside of the ex-ante support are not necessarily Savage-null; and thus, such states may contribute to the formation of conditional preferences. However, if the support of the Receiver’s ex-ante capacity is an unambiguous event, then any separating DSE is “behaviorally equivalent” to the standard PBE.
notion, even though the massages and types may be ambiguous.

Finally, we scrutinize the effect of the assumption of unambiguous types on pooling DSE. For pooling behavior, the assumption has a more serious consequence. When the Sender’s types are perceived unambiguous, then any pooling DSE behavior can be supported by a pooling PBE. In other words, the ax-ante ambiguity about the Sender’s massages does not suffice to explain a pooling behavior that is incompatible with the standard PBE notion.

The structure of the paper is organized as follows. Section 2 recalls the capacity model together with the preference-based notion of unambiguous events suggested by Nehring (1999). In Section 3, the Dempster-Shafer updating rule is defined and the decision-theoretical results are presented. Section 4 introduces the Dempster-Shafer equilibrium notion. In Section 5, the tension between the belief persistence axiom and the Dempster-Shafer updating rule is scrutinized. Section 6 studies the behavior at pooling Dempster-Shafer equilibrium. Finally, we conclude in Section 7.

2. Choquet Preferences and Unambiguous Events

Let $S$ be a finite set of states. An event $E$ is a subset of $S$ and $\Sigma = 2^S$ the algebra of all events. For each $E \in \Sigma$, the complementary event $S \setminus E$ is denoted by $E^c$. A capacity $\nu : \Sigma \to \mathbb{R}$ is a monotone and normalized set function: (i) $\nu(\emptyset) = 0$, $\nu(S) = 1$ and (ii) $\nu(E) \leq \nu(F)$ for all $E \subseteq F \subseteq S$. A capacity $\nu$ on $\Sigma$ is called convex, if it satisfies (iii) $\nu(E \cup F) + \nu(E \cap F) \geq \nu(E) + \nu(F)$ for all $E, F \subseteq S$.$^1$ However, the results of this paper are not restricted to convex capacities.

Let $X$ be a set of consequences. An act $f : S \to X$ is a mapping from states to consequences. The set of all acts is denoted by $A = \{f \mid f : S \to X\}$. For $f, g \in A$, the act $fEg$ assigns the outcome $f(s)$ to all $s \in E$, and $g(s)$ to all $s \in E^c$. A preference relation $\succeq$ on $A$ is said to admit Choquet expected utility representation if there exists a capacity $\nu$ on $\Sigma$ and a utility function $u : X \to \mathbb{R}$ with respect to which each $f \in A$ is evaluated via Choquet integration (see Choquet, 1954).

$^1$Convex capacities express aversion towards ambiguity (see Schmeidler, 1989).
Definition 2.1. The Choquet integral of $f \in A$ with respect to $\nu$ and $u$ is

$$\int_S u(f(s)) \, d\nu(\{s\}) = \sum_{j=1}^{n-1} \left[ u(f(s_j)) - u(f(s_{j+1})) \right] \nu(s_1, ..., s_j) + u(f(s_n)), $$

where $u(f(s_1)) \geq \cdots \geq u(f(s_n))$.

The maximization of Choquet integrals, as a decision criterion, has been behaviorally justified in different setups by Schmeidler (1989), Gilboa (1987), Wakker (1989), Sarin and Wakker (1992), Nakamura (1990), Chew and Karni (1994). Throughout the paper, $\succeq$ is Choquet expected utility preference. If the Choquet integral is taken with respect to an additive capacity, then $\succeq$ is subjective expected utility preference (SEU).

In many choice problems under ambiguity, decision makers are informed about probabilities for some events whereas for other events such information is missing. For instance, in the classical 3-color experiment of Ellsberg (1961) subjects are informed that an urn is filled with 90 balls, 30 of them are red and 60 are either black or yellow without further information on the composition. In order to incorporate probabilistic information into preferences, one needs to ensure that the events for which likelihoods are given are also subjectively perceived as unambiguous events. For that purpose, we adopt the ambiguity notion of Nehring (1999). For Nehring, an event $U \in \Sigma$ is unambiguous if the measure attached to that event in the Choquet integration (Definition 2.1) is independent upon the act being evaluated; otherwise $U$ is ambiguous. Nehring’s notion can be expressed equivalently in terms of a capacity $\nu$.

Definition 2.2. For a given capacity $\nu$, an event $U \in \Sigma$ is revealed to be unambiguous if for any $A \in \Sigma$ it is true that

$$\nu(A) = \nu(A \cap U) + \nu(A \cap U^c),$$

(1)

otherwise the event is ambiguous.

Nehring’s concept of unambiguous events admits an axiomatic underpinning due to Sarin and Wakker (1992) and Dominiak and Lefort (2011). An event $U \in \Sigma$ is revealed to be unambiguous by $\succeq$, if $\succeq$ satisfies Savage’s Sure-Thing Principle constrained to $U$ and $U^c$ (see Dominiak and Lefort, 2011, Proposition 3.1). That is, if for any $f, g, h, h' \in A$ it holds true that

$$fUh \succeq gUh \iff fUh' \succeq gUh',$$

(2)
and Condition (2) is also satisfied when $U$ is everywhere replaced by $U^c$. Choquet preferences with respect to a capacity satisfying Condition (1) are said to be additively-separable across the unambiguous events. That is, if $U \in \Sigma$ is unambiguous, then the Choquet integral of $f \in \mathcal{A}$ can be linearly decomposed:

$$\int_S u(f(s)) \, d\nu(\{s\}) = \int_U u(f(s)) \, d\nu(\{s\}) + \int_{U^c} u(f(s)) \, d\nu(\{s\}).$$ (3)

Eichberger and Kelsey (1999) axiomatized an interesting class of capacities, called $E$-capacities, which make the inclusion of probabilistic information tractable. Let $p$ be a “known” probability distribution on $S$ and let $\mathcal{P} = \{E_1, \ldots, E_n\}$ be a fixed partition of $S$. Assume that $\mathcal{P}$ is coarser than the finest partition of $S$. Denote by $Q$ the set of all probability measures that agree with $p$ on $\mathcal{P}$; that is, for all $q \in Q$ and $E \in \mathcal{P}$, $q(E) = p(E)$. Each measure in $Q$ is referred to as the $p$-consistent assessment.

**Definition 2.3.** For any $A \in \Sigma$, the $E$-capacity $\nu_{q,\rho}(\cdot)$ based on a $p$-consistent assessment $q \in Q$ and $\rho \in [0, 1]$ is defined to be

$$\nu_{q,\rho}(A) := \sum_{i=1}^{n} \left[ \rho \cdot q(A \cap E_i) + (1 - \rho) \cdot p(E_i) \cdot \beta_i(A) \right],$$

where $\beta_i(A) = \begin{cases} 1 & \text{if } E_i \subset A, \\ 0 & \text{otherwise.} \end{cases}$

A decision maker with an $E$-capacity incorporates probabilistic information encoded by $p$, but only about the events in $\mathcal{P}$. That is to say, the decision maker is confident that the distribution $p$ truthfully describes the likelihoods of the events in $\mathcal{P}$, but otherwise distorts probabilities by a degree of confidence $\rho$. Let $\Sigma(\mathcal{P})$ be the algebra generated by the events in $\mathcal{P}$. The following lemma demonstrates that any $E$-capacity reveals the events in $\Sigma(\mathcal{P})$ to be unambiguous. Further, if the elements of partition $\mathcal{P}$ have the capacity value strictly larger than zero, then $\Sigma(\mathcal{P})$ is the only algebra of unambiguous events.

**Lemma 2.4.** Fix a partition $\mathcal{P}$ of $S$ and a probability distribution $p$ on $S$. Let $\nu_{q,\rho}(\cdot)$ be an $E$-capacity based on a $p$-consistent assessment $q \in Q$ with a degree of confidence $\rho \in [0, 1]$. Then, each $E \in \mathcal{P}$ is unambiguous. Moreover, if $p(E) > 0$ for all $E \in \mathcal{P}$, then each $F \in \Sigma \setminus \Sigma(\mathcal{P})$ is ambiguous.
It is worth mentioning that for the case of convex capacities, the Nehring’s notion of unambiguous events coincides with the standard additivity definition. That is, if \( \nu \) is convex and \( \nu(U) + \nu(U^c) = 1 \) for some \( U \in \Sigma \), then \( U \) is unambiguous in the sense of Definition 2.2 (see Nehring, 1999, Proposition 2 and Theorem 3). However, when capacity is non-convex, the standard additivity notion is weaker than the additive-separability condition. This point is exemplified below.

**Example 2.1.** Let \( \mathcal{P} = \{\{s_1\}, \{s_2, s_3\}\} \) be a partition of \( S = \{s_1, s_2, s_3\} \). Consider the following capacity. For any \( A \subset S \), the capacity \( \nu \) is defined as

\[
\begin{align*}
\nu(\{s_1\}) &= \frac{1}{3}, \\
\nu(\{s_2, s_3\}) &= \frac{2}{3}, \\
\nu(\{s_2\}) &= \frac{1}{6}, \\
\nu(\{s_1, s_2\}) &= 1, \\
\nu(\{s_3\}) &= \frac{1}{6}, \\
\nu(\{s_1, s_3\}) &= 1.
\end{align*}
\]

Apparently, \( \nu \) is not convex. It is additive on \( \mathcal{P} \). Yet the events in \( \mathcal{P} \) are ambiguous.

\[1 = \nu(\{s_1, s_2\}) \neq \nu(\{s_1\}) + \nu(\{s_2\}) = \frac{1}{2}.
\]

### 3. Dempster-Shafer Updating and Ambiguous Events

In dynamic choice situations, the decision maker observes new information in form of an event \( E \in \Sigma \). The new piece of information is incorporated into the decision making process by updating the unconditional preference \( \succeq \). The conditional preference, denoted by \( \succeq_E \), is represented by Choquet integrals with respect to the same unconditional utility function \( u \) and an updated capacity \( \nu_E \).

Throughout the paper, ex ante beliefs are revised by applying the Dempster-Shafer updating rule introduced by Dempster (1968) and Shafer (1976).

**Definition 3.1.** Let \( \nu \) be an unconditional capacity and \( E \in \Sigma \) a conditioning event. For any \( A \in \Sigma \), the Dempster-Shafer updating rule for \( \nu \) is defined as

\[
\nu_E(A) = \frac{\nu((A \cap E) \cup E^c) - \nu(E^c)}{1 - \nu(E^c)},
\]

whenever \( \nu(E^c) < 1 \).

The Dempster-Shafer rule has been behaviorally justified by Gilboa and Schmeidler (1993) and interpreted as pessimistic updating rule. Conditioning on an event \( E \), the decision maker believes that the best consequence
belongs to the complement of the event $E$ and so it is impossible to occur. Thus the conditioning events are always regarded as “not-a-good-news.”

We also remark that the Dempster-Shafer updating rule respects consequentialism, a fundamental property of conditional preferences introduced by Hammond (1988), but not dynamic consistency. It is well-known that one of these two properties must be relaxed when modeling ambiguity-sensitive behavior (see Ghirardato, 2002).\(^2\) In a recent experimental study, Dominiak, Dürrsch, and Lefort (2012) provide an evidence confirming the view that subjects behave more in line with consequentialism than dynamic consistency.

The Dempster-Shafer updating rule have been extensively studied in the economics literature.\(^3\) Proposition 2.2 derives an interesting property of the pessimistic belief change in face of an ambiguous information. It is shown that the Dempster-Shafer revision rule preserves the additive-separability condition across the unambiguous events (see Equation (1)). That is to say, all the events that are perceived ex-ante unambiguous remain unambiguous with respect to the conditional Choquet preferences $≽_E$, even though the event $E$ is ambiguous.\(^4\)

**Proposition 3.2.** Let $≽$ be an unconditional Choquet preference relation with respect to a capacity $\nu$, and $\mathcal{P} = \{U_1, ..., U_m\}$ be a partition of $S$ consisting of unambiguous events. If the conditional capacity $\nu_E$ is derived by the Dempster-Shafer updating rule for any $E \in \Sigma$, then the partition $\mathcal{P}$ remains unambiguous after updating. That is, for any $A \in \Sigma$:

$$\nu_E(A) = \sum_{k=1}^{m} \nu_E(A \cap U_k).$$  \hspace{1cm} (4)

This result becomes even more meaningful when capacities are defined over a product space of two finite sets $\mathcal{T}$ and $\mathcal{M}$. Consider a capacity $\nu$ defined on

\(^2\)For Choquet preferences, these two properties can be maintained simultaneously if and only if the conditioning event is unambiguous (see Dominiak and Lefort, 2011).

\(^3\)For instance, Eichberger, Grant, and Kelsey (2010) studies the Dempster-Shafer rule for updating the so-called neo additive capacities axiomatized by Chateauneuf, Eichberger, and Grant (2007). In Dominiak, Eichberger, and Lefort (2012), the Dempster-Shafer rule is used as an explanation of the existence of speculative trade.

\(^4\)It merits emphasis that another updating rule for non-additive beliefs, the Full-Bayesian updating rule of Walley (1991) and Jaffray (1992), does not maintain this property.
Suppose that the partition $P_T = \{ \{ t \} \times M \mid t \in T \}$ is unambiguous while $P_M = \{ T \times \{ m \} \mid m \in M \}$ is ambiguous with respect to $\nu$. Let $\Sigma(P_M)$ be the algebra generated by the events in partition $P_M$. It turns out that if the conditioning event $E$ is an element of $\Sigma(P_M)$, then the conditional preference $\succ_E$ admits the subjective expected utility (SEU) representation. Lemma 3.3 formally states this observation.

Lemma 3.3. Let $\succ$ be an unconditional Choquet preference relation with respect to a capacity $\nu$ on $2^{T \times M}$. Suppose that the states in $T$ are unambiguous and that for each $E \in \Sigma(P_M)$, the conditional preference $\succ_E$ is obtained by applying the Dempster-Shafer updating rule. Then, the conditional preference $\succ_E$ admits SEU representation.

The result seems intuitive. In the product space where one coordinate is assumed to be perceived as being unambiguous while the other is not, a state consists of two components: ambiguous and unambiguous state. After the uncertainty governing the ambiguous state is resolved, the true state now only depends upon the component that was ex-ante perceived unambiguous. The Dempster-Shafer rule fulfills the intuition by preserving the ex-ante unambiguity and letting the decision maker to behave as an expected utility maximizer with respect to an updated probability distribution on $T$.

In the following sections, we apply the results to signaling games and explore how restrictive the conditional SEU preference is for DSE behavior.

4. Dempster-Shafer Equilibrium in Signaling Games

The class of signaling games studied here is described as follows. There are two players called the Sender ($S$) and the Receiver ($R$). In the ex ante stage, Nature draws a type for the Sender from the set of types $T = \{ t_j \}_{j=1}^J$ according to a probability distribution $p$. The Sender learns his type and then chooses a massage from the set of messages $M = \{ m_k \}_{k=1}^K$. In the interim stage, the Receiver observes the massage, but not the type, and selects a response from the set of responses $R = \{ r_l \}_{l=1}^L$, and the game ends. Final payoffs are given by $u^i : T \times M \times R \to \mathbb{R}$ for $i \in \{S,R\}$. We assume $J,K,N \geq 2$ and denote this class of signaling games by $\Gamma$.

The players hold beliefs about the opponent’s behavior. For the Sender, a capacity $\nu^S_m$ defined on $\Sigma^S$, the set of all subsets of $R$, reflects his beliefs about the Receiver’s action(s) that will be chosen in response to each massage $m \in M$. Similarly, the Receiver’s capacity $\nu^R$ is defined on $\Sigma^R$, the set of
all subsets of $\mathcal{T} \times \mathcal{M}$, and it represents the Receiver’s ex-ante joint belief about the Sender’s strategic choice of messages and her possible types. For notational convenience, let $T_j := \{t_j\} \times \mathcal{M}$ and $M_k := \mathcal{T} \times \{m_k\}$ denote the marginal events of the product space $\mathcal{T} \times \mathcal{M}$. We also follow the notations from the previous chapters: $\mathcal{P}_\mathcal{T} := \{T_j\}_{j=1}^J$ and $\mathcal{P}_\mathcal{M} := \{M_k\}_{k=1}^K$ denote the partition of types and messages, and $\Sigma(\mathcal{P}_\mathcal{T})$ and $\Sigma(\mathcal{P}_\mathcal{M})$ denote the algebras generated by the respective partitions.

In the interim stage, the Receiver observes a massage sent by the Sender and revises his beliefs in accordance with the Dempster-Shafer updating rule. The Receiver’s conditional beliefs, represented by the updated capacities $\{\nu^R_m\}_{m \in \mathcal{M}}$, reflect his inference about the Sender’s type from her messages. In games under incomplete information, it is taken for granted that the probability distribution $p$ over types is common knowledge between the players. To incorporate the probabilistic information into the Receiver’s ex-ante beliefs, we follow the approach of Eichberger and Kelsey (2004), and assume that the Receiver’s capacity agrees with the probability distribution $p$ on $\mathcal{T}$.

**Assumption 4.1.** The Receiver’s capacity $\nu^R$ agrees with $p$ on $\mathcal{T}$. That is,

$$\nu^R(T_j) = p(T_j) \quad \text{for } j = 1, \ldots, J. \quad (5)$$

However, a (non-convex) capacity which agrees with $p$ on $\mathcal{T}$ does not necessarily reveal the events in partition $\mathcal{P}_\mathcal{T}$ to be unambiguous. For that to be true, the additive-separability condition of Definition 2.2 is further required.

**Assumption 4.2.** The Receiver’s capacity $\nu^R$ reveals the types of the Sender to be unambiguous. That is, for any $A \subseteq \mathcal{T} \times \mathcal{M}$:

$$\nu^R(A) = \sum_{j=1}^J \nu^R(A \cap T_j). \quad (6)$$

Note that the two Assumptions together assure that the perception of ambiguity solely stems from the messages going to be sent by the Sender.

The equilibrium definition relies on a support notion. The support of a capacity reflects the actions of a player that are perceived possible by his opponent. To ensure non-emptiness, the definition of Dow and Werlang (1994) is adopted.\(^5\) The Dow-Werlang (DW) support is the smallest event
whose complement has zero capacity value.

**Definition 4.3.** A DW-support of a capacity \( \nu \) is an event \( D \subset S \) such that \( \nu(D^c) = 0 \) and \( \nu(F^c) > 0 \) for any \( F \subset D \).

The Dempster-Shafer equilibrium (DSE), introduced by Eichberger and Kelsey (2004), constitutes an equilibrium in beliefs.\(^6\) The definition consists of three components. The first two conditions require the consistency of real behavior with the players’ beliefs about the opponents’ actions. That is, any action which belongs to the support of a player must be a best response of his opponent. The last condition requires the Receiver to follow the Dempster-Shafer updating rule whenever possible.

**Definition 4.4.** A Dempster-Shafer equilibrium (DSE) is a family of beliefs \([\nu^S, \nu^R, \{\nu^R_m\}_{m \in \mathcal{M}}]\) for which there exist the associated supports that satisfy:

\[
\begin{align*}
(i)\quad (t^*, m^*) &\in \text{supp}(\nu^R) \Rightarrow m^* \in \text{argmax}_{m \in \mathcal{M}} \int_{\mathcal{R}} u^S(m, r(m), t) d\nu^S, \\
(ii)\quad r^*(m) &\in \text{supp}(\nu^R_m) \Rightarrow r^*(m) \in \text{argmax}_{r \in \mathcal{T}} \int_{\mathcal{T}} u^R(m, r, t) d\nu^R_m \quad \forall m \in \mathcal{M}, \\
(iii)\quad \nu^R_m &\text{ is derived by the Dempster-Shafer updating rule.}
\end{align*}
\]

If all the ex-ante capacities are additive then the definition of DSE coincides with the definition of perfect Bayesian equilibrium (PBE) in behavioral strategies.\(^7\) For any game in \( \Gamma \), the existence of a DSE is guaranteed (see Proposition 2, Eichberger and Kelsey, 2004). The next proposition, implied by Lemma 3.3, provides the first implication of the assumption of unambiguous types on the DSE notion.

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\(^6\) The DSE is an extension of the notion of “equilibrium in beliefs” in static games introduced by Dow and Werlang (1994); Eichberger and Kelsey (2000); and further studied by Marinacci (2000) and Haller (2000).

\(^7\) Note that the Dempster-Shafer rule coincides with Bayes rule if a capacity is additive.
Proposition 4.5. Consider a signaling game in $\Gamma$, and assume that the Receiver’s capacity $\nu^R$ reveals the types to be unambiguous. In any DSE, the Receiver’s conditional preference exhibits SEU.

Under the assumptions of unambiguous types, the Receiver can only exhibit ambiguity-sensitive preferences in the ex ante stage. After a message is observed, however, the Receiver does not perceive ambiguity anymore and uses a conditional probability to make an inference about the type of the Sender.

In signaling games, two types of equilibria are of special interest: the separating and pooling equilibria in pure strategies. Following the convention, the separating equilibrium refers to a DSE in which each type of the Sender sends different messages, and the pooling equilibrium is a DSE in which all types of the Sender sends the same message. Denote by $\psi_t(\nu^R) = \{m \in \mathcal{M} | (m, t) \in \text{supp}(\nu^R)\}$, the set of messages of the Sender which are in the support of the Receiver’s equilibrium capacity $\nu^R$.

Definition 4.6. A DSE, $[\nu^S, \nu^R, \{\nu^R_m\}_{m \in \mathcal{M}}]$, is called

(i) separating if $\psi_t(\nu^R) \cap \psi_{t'}(\nu^R) = \emptyset$ for any $t, t' \in \mathcal{T}$, and

(ii) pooling if $\psi_t(\nu^R) = \psi_{t'}(\nu^R)$ for any $t, t' \in \mathcal{T}$.

5. Unambiguous Types and Belief Persistence Axiom

The assumption of unambiguous types forces the Receiver’s conditional preference to be an expected utility preference. If it is further assumed that the Sender is a SEU maximizer, the only possible source of ambiguity is the massages which the Receiver may be perceived ambiguous before game is played. Given this “least” departure from the Bayesian framework, it is interesting to ask whether the DSE notion is flexible enough to accommodate behavior that differs from the PBE behavior. In this section, this issue is scrutinized firstly in the context of a separating DSE behavior.

As exemplified below, the ambiguity perceived by the Receiver at the ex-ante stage is sufficient for the existence of a separating behavior that is incompatible with the standard PBE notion.

Example 5.1. Consider the game in Figure 1 which is a variant of Beer and Quiche game of Cho and Kreps (1987). It can be verified that this game has neither pooling nor separating (pure-strategy) PBE. However, there exists the
unique PBE in which the weak type mixes two messages while the strong type only sends the message Q. Notice that the separating equilibrium does not exist because the Sender has an incentive to deviate once he anticipates that the Receiver will learn the Sender’s true type from a message. For instance, the separation where the strong type sends Q, while the weak types sends B cannot constitute a PBE. Once the Receiver learns that message Q has been sent by the strong type, he responds by playing D. Knowing that D is played after Q is sent, the Sender of the weak type should deviate to achieve higher payoff and send message Q instead of B.

However, when the Receiver’s beliefs are allowed to be non-additive, the separating behavior where the strong type sends Q and weak B can be supported by a DSE. The family of beliefs that constitutes the separating DSE is defined in Table 1, 2, and 3. For given $\alpha_1$ and $\alpha_2$, the Receiver’s conditional Choquet expected utility is the following:

$$
\int_{T} u^R(m,r,t) \, dv^R_m = \begin{cases} 
\frac{0.1}{1-\alpha_2} & \text{if } (m,r) = (Q,D), \\
\frac{0.9-\alpha_2}{1-\alpha_2} & \text{if } (m,r) = (Q,F), \\
\frac{0.1-\alpha_1}{1-\alpha_1} & \text{if } (m,r) = (B,D), \\
\frac{0.9}{1-\alpha_1} & \text{if } (m,r) = (B,F).
\end{cases}
$$
Hence, it is optimal for the Receiver to respond with $F$ regardless of the message he observes. Due to the pessimistic updating the Receiver does not infer the true type and this induces the Sender to separate. That is, by anticipating that the Receiver is incapable to learn the true type, the Sender will secure the certain payoff $1$ rather than exposing herself to uncertainty.

Table 1: Receiver’s ex-ante capacity.

<table>
<thead>
<tr>
<th>$E$</th>
<th>$\nu^R(E)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${(ts, Q)}$</td>
<td>$\alpha_1$</td>
</tr>
<tr>
<td>${(ts, B)}$</td>
<td>0</td>
</tr>
<tr>
<td>${(tw, Q)}$</td>
<td>0</td>
</tr>
<tr>
<td>${(tw, B)}$</td>
<td>$\alpha_2$</td>
</tr>
<tr>
<td>${(ts, B), (ts, Q)}$</td>
<td>0.1</td>
</tr>
<tr>
<td>${(tw, B), (tw, Q)}$</td>
<td>0.9</td>
</tr>
<tr>
<td>${(ts, Q), (tw, Q)}$</td>
<td>$\alpha_1$</td>
</tr>
<tr>
<td>${(ts, B), (tw, B)}$</td>
<td>$\alpha_2$</td>
</tr>
<tr>
<td>${(ts, B), (ts, Q)}$</td>
<td>0</td>
</tr>
<tr>
<td>${(ts, Q), (tw, B)}$</td>
<td>$\alpha_1 + \alpha_2$</td>
</tr>
<tr>
<td>${(ts, B), (ts, Q), (tw, B)}$</td>
<td>$0.1 + \alpha_2$</td>
</tr>
<tr>
<td>${(tw, B), (tw, Q), (ts, Q)}$</td>
<td>0.1</td>
</tr>
<tr>
<td>${(tw, B), (tw, Q), (ts, B)}$</td>
<td>$0.9 + \alpha_1$</td>
</tr>
</tbody>
</table>

$0 < \alpha_1 \leq 0.1$ and $0 < \alpha_2 < 0.8$

$\text{supp}(\nu^R) = \{(ts, Q), (tw, B)\}$

Table 2: Receiver’s conditional capacity.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$E$</th>
<th>$\nu^R_m(E)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>${(ts, Q), (tw, Q)}$</td>
<td>$T_s \frac{0.1}{1-\alpha_2}$</td>
</tr>
<tr>
<td></td>
<td>${(ts, B), (tw, B)}$</td>
<td>$T_w \frac{0.9-\alpha_2}{1-\alpha_2}$</td>
</tr>
<tr>
<td></td>
<td>${(ts, B), (ts, Q)}$</td>
<td>$T_s \frac{0.1-\alpha_1}{1-\alpha_1}$</td>
</tr>
<tr>
<td></td>
<td>${(tw, B), (tw, Q)}$</td>
<td>$T_w \frac{0.9}{1-\alpha_1}$</td>
</tr>
</tbody>
</table>

$\text{supp}(\nu^R_B) = \{T_s, T_w\}$

$\text{supp}(\nu^R_Q) = \{T_s, T_w\}$

Table 3: Sender’s capacity.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$E$</th>
<th>$\nu^S_m(E)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>${D}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>${F}$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>${D}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>${F}$</td>
<td>1</td>
</tr>
</tbody>
</table>

The driving force behind such DSE is the pessimistic attitude of the Dempster-Shafer updating rule. Notice that prior to observing a message, the support of the Receiver’s capacity contains the event that the strong type sends Quiche but does not include the event that the weak type sends Quiche. However, after Quiche is observed, the Receiver’s conditional capacity, which is a probability, attaches positive values to both types. That is, the Receiver distrusts the message by taking into account that Quiche could also have been
sent by the weak type. As a consequence of the pessimistic belief change, the Receiver is hindered from inferring the true type, although the Sender reveals her private information by sending different massages. This behavior is impossible under the Bayesian paradigm where at any separating PBE, the Receiver learns the Sender’s private information by ascribing probability one to the type who sent the massage observed.

However, Ryan (2002a) criticized such DSE. He pointed out that the beliefs on-the-equilibrium-path violate the so-called belief persistence axiom. Broadly speaking, the belief persistence postulate is a qualitative requirement for the updated beliefs to reflect the prior beliefs as accurate as possible (see Battigalli and Bonanno, 1997).

**Definition 5.1.** For a given capacity $\nu$ on $\Sigma$, a conditional capacity $\nu_E$ is said to respect the belief persistence if for all $E \in \Sigma$, it satisfies

$$\text{supp}(\nu) \cap E \neq \emptyset \implies \text{supp}(\nu_E) = \text{supp}(\nu) \cap E.$$  

(7)

In Example 5.1, the Receiver’s conditional capacity assigns a strictly positive value to the weak type after observing Quiche although the event that the week type sends Quiche was not in the ex-ante support. Put differently, the support of the Receiver’s conditional capacity expands, relative to the ex-ante support, by adding the states that were deemed impossible at the ex-ante stage. To eliminate such irregularities, Ryan advocated to consider the DSEs that comply with the the principle of belief persistence. However, if the belief persistence axiom is exogenously imposed on the beliefs on-the-equilibrium path of a separating DSE, then a remarkable result follows.

**Proposition 5.2.** Consider a signaling game in $\Gamma$ with $|\mathcal{M}| = |\mathcal{T}| \geq 2$. Assume that the Receiver’s capacity $\nu^R$ reveals the types to be unambiguous. At any separating DSE, provided it exists, if the Receiver’s capacity respects the belief persistence axiom on any $E \in \Sigma(\mathcal{P}_\mathcal{M})$, then the Receiver must exhibit (ex-ante) SEU preferences.

Under the joint assumption of unambiguous types and belief persistence, the separating equilibrium precludes any ambiguity perception of the Receiver. Ryan provided an example for intrinsically identical limitation. In his particular game, the Receiver’s capacity is assumed to be an $E$-capacity.\(^8\) Since the

\(^8\)For the sake of completeness, Ryan’s game is recalled in Appendix B.
$E$-capacity reveals the types of the Sender to be unambiguous (see Lemma 2.4), Ryan’s result (2002a, Proposition 4.1) can be stated as a corollary of Proposition 5.2.

**Corollary 5.3.** In any separating DSE of the Ryan’s game which respects the belief persistence, the Receiver displays an (ex-ante) SEU preference.

To model a separating equilibrium behavior that is due to ambiguity perception of the Receiver, one needs to relax either the belief persistence axiom or the assumption of unambiguous types (or both).\(^9\) Let us assume for a moment that the Receiver’s capacity does not reveal the types to be unambiguous. When both the types and massages are ambiguous, then the Receiver’s conditional beliefs may again violate the belief persistence axiom. Two questions rise. First, is there a natural way to respect the belief persistence under the pessimistic updating? Second, if the belief persistence is maintained, what behavior can be captured by the DSE notion under ambiguous types and massages?

Let us firstly address the former question. The main reason for the violation of the belief persistence lies on the fact that the Dempster-Shafer updating rule assigns positive values to the events which were ex-ante valued zero. However, the events with the capacity value zero do not need to be unambiguous; i.e., such events are not necessarily Savage-null. An event $E$ is said to be Savage-null if for any $f, g \in F$, $f \sim g_E f$. Clearly, the Savage-null events are the unambiguous events with the zero capacity value.

**Lemma 5.4.** Let $\nu$ be a capacity on $\Sigma$. An event $E \in \Sigma$ is Savage-null if and only if $E$ is unambiguous and $\nu(E) = 0$.

Indeed, the Receiver’s equilibrium capacity $\nu^R$ (see Table 1) reveals its ex-ante support to be an ambiguous event. That is, the states outside of the support (e.g., the event that the weak type sends Quiche) are not Savage-null. Since such events still contribute to the formation of unconditional preferences, they may also affect the formation of conditional preferences. However, the events that are ex-ante revealed to be Savage-null, remain Savage-null.

\(^9\)Of course, one could also assume that massages are unambiguous but types are not. However, if the conditioning events is unambiguous then the Dempster-Shafer updating rule coincides with the Bayes rule (Proposition 5.2, Dominiak and Lefort, 2011). In this case, the pessimism inherent in the Dempster-Shafer updating rule vanishes.
after the pessimistic belief change, making the events irrelevant for the conditional preferences.

**Lemma 5.5.** Let $\nu$ be a capacity on $\Sigma$ and $\text{supp}(\nu)$ its support. Suppose that $\text{supp}(\nu)$ is revealed to be unambiguous. Then, for any $E \in \Sigma$ such that $\text{supp}(\nu) \cap E \neq \emptyset$, the following is true

\[ \nu_E(A) = 0, \quad \forall A \subset [\text{supp}(\nu)]^c. \tag{8} \]

Lemma 5.5 suggests a natural way of respecting the belief persistence by assuming the support of the ex-ante capacity to be perceived unambiguous.\(^{10}\) When the support of an ex-ante capacity is assumed to be unambiguous, two properties follow: (i) the DW-support is unique, and (ii) the support of the conditional capacity cannot expand.

**Proposition 5.6.** Let $\nu$ be a capacity on $\Sigma$ and $\text{supp}(\nu)$ its DW-support. If the support is an unambiguous event, then (i) it is unique, and (ii) for any $E \in \Sigma$, the following is true

\[ \text{supp}(\nu) \cap E \neq \emptyset \implies \text{supp}(\nu_E) \subseteq \text{supp}(\nu) \cap E. \tag{9} \]

Having said that, the second question can be addressed. If the support of the Receiver’s ex-ante capacity is an unambiguous event, then the Receiver’s equilibrium beliefs will satisfy the belief persistence axiom (Definition 5.1) at any separating DSE. However, any such separating DSE corresponds to a behaviorally equivalent PBE, albeit the fact that the Receiver may perceive the types and massages as being ambiguous.

**Proposition 5.7.** Consider a signaling game in $\Gamma$, and assume that the support $\text{supp}(\nu^R)$ of the Receiver’s capacity $\nu^R$ is an unambiguous event. Then, for any separating DSE, there exists a separating PBE which captures the identical behavior of the DSE.

This result detects a serious limitation of the belief persistence postulate. The belief persistence forces the Receiver to learn the true type at any separating DSE. The ambiguity that the Receiver perceives about massages and types, as well as the pessimistic belief change, is immaterial. The strategic behavior

\[ \text{supp}(\nu) \neq \emptyset \implies \text{supp}(\nu_E) \subseteq \text{supp}(\nu) \cap E. \]

---

\(^{10}\)The requirement for the support of a capacity to be an unambiguous event is equivalent to saying that the states outside of the support are Savage-null.
will not differ from that of PBE. Consequently, the deviation from PBE behavior presented in Example 5.1 cannot be modeled via DSE unless the belief persistence axiom is abandoned.

6. Unambiguous Types on Pooling Equilibrium

In this section, we examine the effect of the assumptions of unambiguous types on strategic behavior in pooling DSE. At pooling DSE, the support of the Receiver’s capacity (i.e., the massage on which the Sender pools) has to be ambiguous. Otherwise, the conditional capacity off-the-equilibrium-path is not well-defined, and the Dempster-Shafer updating rule loses its main purpose of application.

We start by remarking that in pooling DSE, the Receiver’s beliefs never violate the belief persistence axiom. Recall that the belief persistence axiom requires the support of a conditional capacity to be equal to the intersection of the support of an unconditional capacity and a conditioning event; provided the intersection is non-empty. Consider a pooling DSE and suppose the Sender’s types are unambiguous. If the message on-the-equilibrium-path is observed, the Receiver’s conditional capacity coincides with the prior distribution on the Sender’s types; and if a message off-the-equilibrium-path is received, then this message cannot belong to the support of the Receiver’s ex-ante capacity. This makes the belief persistence axiom being vacuously satisfied.

**Proposition 6.1.** Consider a signaling game in $\Gamma$, and assume that the Receiver’s capacity $\nu^R$ reveals the Sender’s types to be unambiguous. Then, in any pooling DSE, provided it exists, the following is true. For any $E \in \Sigma(P_M)$,

$$\text{supp}(\nu^R) \cap E \neq \emptyset \implies \text{supp}(\nu^R_E) = \text{supp}(\nu^R) \cap E.$$

Note that the coexistence of the assumption of unambiguous types and the belief persistence at pooling DSE does not restrict the Receiver’s preferences.

However, the assumption of unambiguous types substantially constrains the pooling behavior supported by DSE. In short, any pooling DSE can be explained by a PBE that captures the identical pooling behavior. Fix a pooling DSE. When the Sender’s types are perceived unambiguous, the Receiver’s conditional beliefs are additive regardless of whether the massage observed is on or off-the-equilibrium path. Further, the conditional beliefs on-the-equilibrium-path coincide with the prior distribution over the types,
and the beliefs off-the-equilibrium-path are well-defined although the conditioning event is measured zero. Thus, one can replace the Receiver’s ex-ante capacity with the prior probability and define the conditional beliefs off-the-equilibrium-path, which are arbitrary under ex-ante additivity, as dictated by the Dempster-Shafer updating rule. Given that these beliefs support an optimal behavior at the pooling DSE, these beliefs must also support the same pooling behavior in PBE.

**Proposition 6.2.** Consider a pooling DSE, and assume that the Receiver’s capacity $\nu^R$ reveals the Sender’s types to be unambiguous. Then, for any pooling DSE, there exists a behaviorally identical pooling PBE.

Although the message on which the Senders pools is perceived ambiguous by the Receiver, a pooling behavior inconsistent with PBE is neither compatible with the DSE notion unless the assumption of unambiguous types is relaxed.

However, when types are perceived ambiguous, then the Receiver’s conditional beliefs do not need to be additive, giving rise to a pooling behavior that is impossible under the PBE notion. We close up the discussion with an example illustrating a pooling behavior that cannot be explained under the regime of additive beliefs.

**Example 6.1.** In the signaling game described by the Figure 2, for any $p \in (0, 1)$, only pooling PBE exists:

1. **Pooling on $R$;** $\{(R, R), (u, u), \mu(t_1 | L) = q \in [0, 1], \mu(t_1 | R) = p_1 > p_2\}$
2. **Pooling on $R$;** $\{(R, R), (u, d), \mu(t_1 | L) = q \in [0, 1], \mu(t_1 | R) = p_1 < p_2\}$

For the Sender, if she pools on message $L$, then she always attains the payoff 1 regardless of the Receiver’s response. However, it is better for her to send message $R$ as long she believes that the Receiver will never play $C$ after observing $R$, and the payoff 100 will be guaranteed.

For the Receiver, playing $U$ after $L$ is dominant action for any type of the Sender. Also, as long as the Receiver holds additive beliefs, playing $C$ is never optimal after observing $R$. Thus, any equilibrium concept under the regime of additive beliefs of the Receiver will fail to explain the response $C$ although it assures the payoff $\alpha \in (0, 1)$ to the Receiver.

However, if the Sender believes $C$ to be responded to her message $R$, then she might want to consider to send $L$ instead of $R$ which brings the safe payoff 1. The following tables presents capacities that support this behavior as pooling
DSE at which the types are no longer assumed to be unambiguous, although the Receiver’s capacity is additive on types.

The Receiver exhibits an extremely pessimistic attitude after observing message $R$ as he only takes into account the lowest outcome possible:

$$\int_T u^R(R, r, t) \, d\nu^R_R = \begin{cases} 
0 & \text{if } r \in \{U, D\}, \\
\alpha & \text{if } r = C.
\end{cases}$$

The Receiver’s pessimism leads him to play $C$ and the Sender who has anticipated this will pool on $L$. 

Figure 2: Pooling behavior under ambiguous types and massages.
Table 4: Receiver’s ex-ante beliefs.

<table>
<thead>
<tr>
<th>(E)</th>
<th>(\nu^R(E))</th>
</tr>
</thead>
<tbody>
<tr>
<td>({(t_1, L)})</td>
<td>(\rho p_1)</td>
</tr>
<tr>
<td>({(t_2, L)})</td>
<td>(\rho p_2)</td>
</tr>
<tr>
<td>({(t_1, R)})</td>
<td>(0)</td>
</tr>
<tr>
<td>({(t_2, R)})</td>
<td>(0)</td>
</tr>
<tr>
<td>({(t_1, L), (t_1, R)})</td>
<td>(p_1)</td>
</tr>
<tr>
<td>({(t_2, L), (t_2, R)})</td>
<td>(p_2)</td>
</tr>
<tr>
<td>({(t_1, L), (t_2, L)})</td>
<td>(\rho)</td>
</tr>
<tr>
<td>({(t_1, R), (t_2, R)})</td>
<td>(0)</td>
</tr>
<tr>
<td>({(t_1, L), (t_2, L), (t_1, R)})</td>
<td>(\rho)</td>
</tr>
<tr>
<td>({(t_1, L), (t_2, L), (t_2, R)})</td>
<td>(\rho p_2)</td>
</tr>
<tr>
<td>({(t_1, L), (t_1, R), (t_2, R)})</td>
<td>(\rho p_1)</td>
</tr>
</tbody>
</table>

Table 5: Conditional beliefs.

<table>
<thead>
<tr>
<th>(m)</th>
<th>(E)</th>
<th>(\nu^R_m(E))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L)</td>
<td>(T_1)</td>
<td>(\rho p_1)</td>
</tr>
<tr>
<td></td>
<td>(T_2)</td>
<td>(\rho p_2)</td>
</tr>
<tr>
<td>(R)</td>
<td>(T_1)</td>
<td>(0)</td>
</tr>
<tr>
<td></td>
<td>(T_2)</td>
<td>(0)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(m)</th>
<th>(E)</th>
<th>(\nu^S_m(E; m))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L)</td>
<td>({U})</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>({D})</td>
<td>(0)</td>
</tr>
<tr>
<td>(R)</td>
<td>({C})</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>({U, D})</td>
<td>(0)</td>
</tr>
</tbody>
</table>

7. Concluding Remarks

In signaling games, the Receiver may update his ex-ante beliefs in pessimistic manner when incorporating the arrival of an ambiguous massage. The DSE notion is an equilibrium concept that captures such behavior and thus generalizes the standard perfect Bayesian equilibrium.

This paper has unfolded the tension between the additional explanatory power of the DSE notion and two assumptions imposed on equilibrium beliefs: the unambiguous types and the belief persistence axiom. It has been argued that one of these assumptions has to be abandoned in order to accommodate behavior that is incompatible with an equilibrium under the Bayesian paradigm.

If separating behavior is the subject matter of investigation, then the belief persistence axiom has to be relaxed; otherwise, the Receiver always learns the true type and any separating DSE behavior can be explained by the PBE notion. For pooling DSE behavior to be different from PBE, the assumption of unambiguous types has to be dropped.
Appendix A. Proofs:

In this appendix, we provide the proofs for the results in the paper.

**Lemma 2.4** Fix a partition $\mathcal{P}$ of $S$ and a probability distribution $p$ on $S$. Let $\nu_{q,\rho}(\cdot)$ be an E-capacity based on a $p$-consistent assessment $q \in Q$ with a degree of confidence $\rho \in [0,1)$. Then, each $E \in \mathcal{P}$ is unambiguous. Furthermore, if $p(E) > 0$ for all $E \in \mathcal{P}$, then each $F \in \Sigma \setminus \Sigma(\mathcal{P})$ is ambiguous.

*Proof.* Fix a partition $\mathcal{P} = \{E_1, \ldots, E_k, \ldots, E_n\}$, $p$ on $S$ and $\rho \in [0,1)$. Let $\nu_{q,\rho}(\cdot)$ be an E-capacity based on a $p$-consistent assessment $q \in Q$. The E-capacity is convex and additive on $\mathcal{P} = \{E_1, \ldots, E_n\}$. By Proposition 2 and Theorem 3 of Nehring (1999), for any $A \in \Sigma$ the following is true:

$$\nu_{q,\rho}(A) = \sum_{k=1}^{n} \nu(A \cap E_k).$$

Thus, the events in $\mathcal{P}$ are perceived as being unambiguous.

Suppose that $p(E) > 0$ for all $E \in \mathcal{P}$. Take an event $F \in \Sigma \setminus \Sigma(\mathcal{P})$ and assume that $F$ is unambiguous. Again, by convexity of $\nu_{q,\rho}(\cdot)$, Proposition 2, and Theorem 3 of Nehring (1999), we have

$$1 = \nu_{q,\rho}(F) + \nu_{q,\rho}(F^c).$$

Hence,

$$1 = \sum_{k=1}^{n} \rho \left[ q(F \cap E_k) + q(F^c \cap E_k) \right] + \sum_{k=1}^{n} (1 - \rho) p(E_k) \left[ \beta_k(F) + \beta_k(F^c) \right]$$

$$= \rho + (1 - \rho) \sum_{k=1}^{n} p(E_k) \left[ \beta_k(F) + \beta_k(F^c) \right]$$

$$= \sum_{k=1}^{n} p(E_k) \left[ \beta_k(F) + \beta_k(F^c) \right] \quad (A.1)$$

Since $F \in \Sigma \setminus \Sigma(\mathcal{P})$, there is at least one $E_k \in \mathcal{P}$ so that

$$F \cap E_k \neq \emptyset \quad \text{and} \quad F^c \cap E_k \neq \emptyset. \quad (A.2)$$

If Condition (A.2) is true for all $E_k \in \mathcal{P}$, then the right-hand side of Equation (A.1)
A.1) is equal to zero and a contradiction follows. Now, suppose that Condition (A.2) is not true for all $E_k \in P$. Let $Q$ be a collection of events from $P$ that satisfy Condition (A.2). Then, Equation (A.1) implies that $p(E) = 0$ for any $E \in Q$, contradicting what we assumed. \hfill $\Box$

**Proposition 3.2** Let $\succeq$ be an unconditional Choquet preference relation with respect to a capacity $\nu$, and $P = \{U_1, \ldots, U_m\}$ be a partition of $S$ consisting of unambiguous events. If the conditional capacity $\nu_E$ is derived by the Dempster-Shafer updating rule for any $E \in \Sigma$, then the partition $P$ remains unambiguous after updating. That is, for any $A \in \Sigma$:

$$\nu_E(A) = \sum_{k=1}^{m} \nu_E(A \cap U_k).$$

**Proof.** Fix a conditioning event $E \in \Sigma$ and an unambiguous partition $P = \{U_1, \ldots, U_m\}$. For any event $A \in \Sigma$, consider first the Dempster-Shafer update for $A \cap U_k$ for some $k \in \{1, \ldots, m\}$:

$$(1 - \nu(E^c))\nu_E(A \cap U_k) = \nu((A \cap U_k) \cap E) \cup E^c) - \nu(E^c)$$

Since $P$ is unambiguous, $\xi_k$ we can be expressed in the following way:

$$\xi_k = \nu((A \cap E) \cap U_k) \cup E^c) - \nu(E^c)$$

$$\xi_k = \sum_{j=1}^{m} \nu((A \cap E) \cap U_k) \cup E^c) \cap U_j) - \sum_{j=1}^{m} \nu(E^c \cap U_j)$$

$$\xi_k = \nu((A \cap E) \cap U_k) + \nu(E^c \cap U_k)$$

The last equality follows because $U_k \cap U_j = \emptyset$ for all $k \neq j$. Thus, summing over $k$ delivers

$$\sum_{k=1}^{m} \xi_k = \sum_{k=1}^{m} \left[ \nu((A \cap E) \cap U_k) - \nu(E^c \cap U_k) \right]$$

$$\sum_{k=1}^{m} \xi_k = \nu((A \cap E) \cup E^c) - \nu(E^c)$$

$$\sum_{k=1}^{m} \xi_k = (1 - \nu(E^c))\nu_E(A)$$

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Therefore,

\[ \nu_E(A) = \sum_{k=1}^{m} \nu_E(A \cap U_k) \]

\[ \square \]

**Lemma 3.3** Let \( \succcurlyeq \) be an unconditional Choquet preference relation with respect to a capacity \( \nu \) on \( 2^{T \times M} \). Suppose that the states in \( T \) are unambiguous and that for each \( E \in \Sigma(P_M) \), the conditional preference \( \succcurlyeq_E \) is obtained by applying the Dempster-Shafer updating rule. Then, the conditional preference \( \succcurlyeq_E \) admits SEU representation.

**Proof.** Let \( \mathcal{P}_T = \{\{t_j\} \times M \mid k = 1, \ldots, J\} \) be a partition consisting of unambiguous events, and denote each event in the partition by \( T_j \) for each \( j \). We show that \( \nu_E \) is additive for any \( E \in \Sigma(P_M) \); i.e., \( \sum_{j=1}^{J} \nu_E(T_j) = 1 \).

Fix \( E \in \Sigma(P_M) \), and observe that the following is true.

\[
\nu((T_j \cap E) \cup E^c) - \nu(E^c) \\
= \sum_{k=1}^{J} \nu((T_j \cap E) \cup E^c \cap T_k) - \sum_{k=1}^{J} \nu(E^c \cap T_k) \\
= \sum_{k=1}^{J} \nu((T_j \cap E \cap T_k) \cup (E^c \cap T_k)) - \sum_{k=1}^{J} \nu(E^c \cap T_k) \\
= \nu(T_j) - \nu(T_j \cap E^c) \\
= \nu(T_j) - \nu(T_j \cap E^c)
\]
Now summing up $j$ gives the following:

\[
(1 - \nu(E^c)) \sum_{j=1}^{J} \nu_E(T_j) = \sum_{j=1}^{J} \nu((T_j \cap E) \cup E^c) - \nu(E^c)
\]

\[
= \sum_{j=1}^{J} [\nu(T_j) - \nu(T_j \cap E^c)]
\]

\[
= 1 - \nu(E^c)
\]

Therefore, we conclude that $\nu_E$ is additive and thus $\succsim_E$ admits SEU preferences. \hfill \Box

**Proposition 5.2** Consider a signaling game in $\Gamma$ with $|M| = |T| \geq 2$. Assume that the Receiver’s capacity $\nu^R$ reveals the types to be unambiguous. At any separating DSE, provided it exists, if the Receiver’s capacity respects the belief persistence axiom on any $E \in \Sigma(P_M)$, then the Receiver must exhibit (ex-ante) SEU preferences.

*Proof.* Without loss of generality, assume the Sender sends $m_k$ when his type is $t_k$ for $k = 1, 2, ..., J$, at any separating equilibrium. Due to the definitions of separating equilibrium (Definition 4.6) and of DW support (Definition 4.3), the ex-ante equilibrium capacity $\nu^R$ must assign positive values only on-the-diagonal states $(t_k, m_k)$, $k = 1, ..., J$ and zero values off-the-diagonal states; i.e., $\nu^R$ is such that

\[
\nu^R((t_j, m_k)) = \begin{cases} 
0 & \text{for } j \neq k, \\
\alpha_j & \text{for } j = k.
\end{cases}
\]

Note that any conditional capacity of the Receiver is additive by Proposition 4.5. Thus the belief persistence axiom requires the support of the conditional capacity to satisfy $\nu_M(T_j) = 0$ for any $M \subseteq M_j^c$. Take $M = M_j^c$, then we have:

\[
\nu^R_{M_j^c}(T_j) = \frac{\nu^R((T_j \cap M_j^c) \cup M_j) - \nu^R(M_j)}{1 - \nu^R(M_j)}
\]

Note that at any separating equilibrium, $\nu^R(M_j) < 1$ holds for any $j$ and
\( \nu_{M^c_j}(T_j) = 0 \) if and only if

\[
\nu^R((T_j \cap M_j^c) \cup M_j) = \nu^R(M_j).
\]

However, the unambiguous types implies the following: for any \( j \),

\[
\nu^R((T_j \cap M_j^c) \cup M_j) = \nu^R(M_j)
\]

\[
\nu^R(T_j) + \sum_{k \neq j} \nu^R(T_k \cap M_j) = \sum_{i=1}^{J} \nu^R(T_i \cap M_j)
\]

\[
\nu^R(T_j) = \nu^R(T_j \cap M_j)
\]

Since \( \nu^R(T_j \cap M_k) = 0 \) for any \( j \neq k \), \( \nu^R \) is additive. (to be continued) \( \Box \)

**Lemma 5.4** Let \( \nu \) be a capacity on \( \Sigma \). An event \( E \in \Sigma \) is Savage-null if and only if \( E \) is unambiguous and \( \nu(E) = 0 \).

**Proof.** We show that an event \( E \) is dummy if and only if \( E \) is unambiguous and \( \nu(E) = 0 \).

For sufficiency, assume \( E \) is dummy. Then \( \nu(E) = 0 \); otherwise \( \nu(\emptyset \cup E) > 0 \) which contradicts the assumption that \( E \) is dummy. Also note that for any \( D \subset E \) is also dummy: \( \nu(F \cup D) = \nu((F \cup D) \cup E) = \nu(F \cup E) = \nu(F) \) for any \( F \in \Sigma \). Thus we have the following: for any \( F \in \Sigma \),

\[
\nu(F) = \nu\left( \left( \bigcap_{E \subset F} (F \cap E) \right) \cup (F \cap E^c) \right)
\]

\[
= \nu(F \cap E^c)
\]

\[
= \nu(F \cap E) + \nu(F \cap E^c)
\]

For necessity, assume \( E \) is unambiguous and \( \nu(E) = 0 \). Then the following is straight forward: for any \( F \in \Sigma \setminus \emptyset \),

\[
\nu(F \cup E) = \nu\left( \left( (F \cup E) \cap E \right) \cup (F \cup E) \cap E^c \right)
\]

\[
= \nu(F \cap E) + \nu(F \cap E^c)
\]

\[
= \nu(F)
\]

For \( F = \emptyset \), \( E \) is dummy because \( \nu(E) = 0 \).
Therefore, we conclude that $E$ is Savage-null if and only if $E$ is unambiguous and $\nu(E) = 0$.

\[ \square \]

**Lemma 5.5** Assume $\nu^R$ reveals $\text{supp}(\nu^R)$ to be unambiguous. For any $E$ such that $\text{supp}(\nu^R) \cap E \neq \emptyset$, the following is true:

$$\nu^R_E(A) = 0, \quad \forall \ A \subset [\text{supp}(\nu^R)]^c$$

**Proof.** Fix $E$ and $A$, and denote $\text{supp}(\nu^R) = D$. Note that $\nu^R(A) = 0$ as $A \subseteq D^c$. The following is straightforward:

$$\nu^R_E(A) = \frac{\nu^R((A \cap E) \cup E^c) - \nu^R(E^c)}{1 - \nu^R(E^c)}$$

$$= \frac{\nu^R(([A \cap E] \cup E^c) \cap D) - \nu^R(E^c \cap D)}{1 - \nu^R(E^c \cap D)}$$

$$= \frac{\nu^R((A \cap E \cap D) \cup (E^c \cap D)) - \nu^R(E^c \cap D)}{1 - \nu^R(E^c \cap D)}$$

$$= 0$$

\[ \square \]

**Proposition 5.6** Assume $\nu^R$ reveals $\text{supp}(\nu^R)$ to be unambiguous. Then for any $E \in \Sigma^R$, the following is true:

$$\text{supp}(\nu^R) \cap E \neq \emptyset \implies \text{supp}(\nu^R_E) \subseteq \text{supp}(\nu^R) \cap E$$

**Proof.** Denote $\text{supp}(\nu^R)$ by $D$ and $\text{supp}(\nu^R_E)$ by $D_E$. Assume $D \cap E \neq \emptyset$ but $D_E \notin D \cap E$. Then there exists $\{\omega\} \in \Sigma^R$ such that $\omega \in D_E$ and $\omega \notin D \cap E$.

We exhibit a contradiction below. Note that $\nu^R_E(\{\omega\}) > 0$; otherwise $\nu^R_E((D_E \setminus \{\omega\})^c) = 0$ contradicting the definition of DW-support. Since $\omega \notin D \cap E$, it is either $\omega \notin D$ or $\omega \notin E$ (or both). However, $\nu^R_E(\{\omega\}) = 0$ trivially for the case of $\omega \notin E$, and by the Lemma 5.5 for the case of $\omega \notin D$. This contradicts $\nu^R_E(\{\omega\}) > 0$ and thus we conclude that $\text{supp}(\nu^R_E) \subseteq \text{supp}(\nu^R) \cap E$.

\[ \square \]

**Proposition 5.7** Consider a signaling game in $\Gamma$, and assume that the support $\text{supp}(\nu^R)$ of the Receiver’s capacity $\nu^R$ is an unambiguous event. Then,
for any separating DSE, there exists a separating PBE which captures the identical behavior of the DSE.

Proof. Fix a separating DSE \([\nu^S, \nu^R, \{\nu^R_m\}_{m \in M}]\). Then \(\nu^S\) is additive and without loss of generality, \(\nu^R\) can be characterized as the following:

\[
\nu^R((t_i, m_j)) = \begin{cases} 
\alpha_i & \text{if } i = j, \\
0 & \text{if } i \neq j, \ \alpha_i \leq p_i.
\end{cases}
\]

By Lemma 5.5, the conditional capacity satisfies the following:

\[
\nu^R_{m_j}(T_i) = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{cases}
\]

Now replacing \(\alpha_i\) with \(p_i\), we show that \([\nu^S, \pi^R, \{\nu^R_m\}_{m \in M}]\) constitutes the PBE with the same behavior.

\[
\pi^R(t_i, m_j) = \begin{cases} 
\alpha_i & \text{if } i = j, \\
0 & \text{if } i \neq j, \ \alpha_i \leq p_i.
\end{cases}
\]

\(\square\)

Proposition 6.2 For each pooling DSE, there exists pooling PBE whose equilibrium behaviors are identical.

Proof. (sketch) Take a pooling DSE \([\nu^S, \nu^R, \{\nu^R_m\}_{m \in M}]\). Notice that the family \(\{\nu^R_m\}_{m \in M}\) consists of additive capacities. Thus replacing \(\nu^R\) with its additive part should also satisfy the definition of DSE which in turn is PBE. This proof is not complete yet. \(\square\)
Appendix B. The game of Ryan (2002)

This section briefly introduces the note of Ryan (2002a).

**Figure B.3: Ryan (2002)**

**Example Appendix B.1.** Consider the signaling game in Figure B.3. For the Sender, sending $L$ when his type is $t_1$ and $R$ when $t_2$ is the strictly dominant strategy. Therefore, Ryan argues that any reasonable equilibrium should capture the separating equilibrium behavior.

Let $p(t_1) = p(t_2) = \frac{1}{2}$ be the probability distribution on types that is commonly known and so perceived as being unambiguous. Let $\pi$ be a probability distribution on $T \times M$ such that $\pi(\{t_1, L\}) = \pi(\{t_2, R\}) = \frac{1}{2}$. The tuple $[\nu^S, \nu^R, \nu^R_L, \nu^R_R]$ constitutes a DSE with the associated supports following:

$$\text{supp}(\nu^S) = \{D\}, \text{supp}(\nu^R) = \{(t_1, L), (t_2, R)\}$$
and

\[ \nu^S(E; t) = \begin{cases} 
1 & \text{if } E = D, \\
0 & \text{if } E = U,
\end{cases} \]

for all \( t \in T \), \( \beta_1 + \beta_2 \leq 1 \)

\[ \nu^R(E) = \frac{1}{2} \pi(E) + \frac{1}{2} \sum_{i=1}^{2} w_{t_i}(E)p_{t_i}, \]

\[ w_{t_i}(E) = \begin{cases} 
1 & \text{if } \{t_i\} \times M \subseteq E, \\
0 & \text{otherwise},
\end{cases} \]

\[ \nu^L_{R}(E) = \begin{cases} 
\frac{2}{3} & \text{if } E = T_1, \\
\frac{1}{3} & \text{if } E = T_2;
\end{cases} \]

\[ \nu^R_{R}(E) = \begin{cases} 
\frac{1}{3} & \text{if } E = T_1, \\
\frac{2}{3} & \text{if } E = T_2.
\end{cases} \]

The table below better illustrates \( \nu^R \) at the DSE above. Due to the definition of separating equilibrium and DW-support (Definitions 4.6 and 4.3), \( \nu^R \) at a separating DSE should have the following form: Note that \( \nu^R \) agrees with \( p \)

\[
\begin{array}{c|cc|c}
L & R & & \\
\hline
\text{t}_1 & \frac{1}{3} & 0 & p(\text{t}_1) = \frac{1}{2} \\
\text{t}_2 & 0 & \frac{1}{3} & p(\text{t}_2) = \frac{1}{2}
\end{array}
\]

Table B.6: \( \nu^R \) at separating equilibrium

on \( T \) and the belief persistence axiom implies that the support of the conditional capacities \( \{\nu^R_m\}_{m \in M} \) derived from \( \nu^R \) should not include any event in \( \{(\text{t}_1, R), (\text{t}_2, L)\} \). However, the support of the conditional capacities includes both as they were valued \( \frac{1}{3} \). Thus the Receiver’s conditional belief violates the belief persistence axiom. Filtering out the ex ante capacity distribution which violates the belief persistence provides a striking result. The remaining DSE is additive; i.e., the Receiver exhibits ex ante SEU preference (Proposition 4.1 Ryan, 2002a).


