

What You Don't Know Can Help You in School Assignment

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Abstract

It is well known that it is impossible for a strategyproof mechanism to Pareto dominate the celebrated Deferred Acceptance algorithm (hereafter DA). However, it is unknown whether or not a mechanism can Pareto dominate DA in equilibrium when students use weakly undominated strategies. We demonstrate a surprising result. A mechanism designer can do better by learning less about students preferences when making a school assignment. Specifically, we demonstrate that running DA but limiting students to only two applications always has an equilibrium in weakly undominated pure strategies that Pareto dominates DA. We also show that no mechanism that tries to Pareto improve DA directly has this property: no mechanism that Pareto improves DA with respect to submitted preferences actually Pareto improves DA in equilibrium. Therefore, these mechanisms do not improve DA in practice. Finally, we introduce a new algorithm, Application with Automatic Appeals, and demonstrate that it also dominates DA in equilibrium. Unlike the 2-school DA, the Application with Automatic Appeals is nonwasteful.

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1 Introduction

Due in large part to the contribution of economists, many U.S. school districts now allow students to choose which school they wish to attend. However, there is no perfect mechanism for making the student assignment since an assignment must balance numerous and sometimes conflicting objectives.

When designing a process for a *school choice system*, the market designer faces three constraints. First, only a limited number of students may attend any particular school, and it is typically impossible to assign all students to their favorite or even second-favorite school. In addition to this physical constraint, a school district typically prioritizes students at each particular school. These priorities can be based on performance based characteristics, such as an entrance exam score. Alternatively, the priorities can be based on school-specific characteristics such as whether or not a student has a sibling attending the school or if she lives within walking distance to the school. This constitutes the second design constraint. As these priorities represent a student's "claim" to a school, a board typically desires *fairness* in the sense that low priority students are not admitted to a school at the expense of higher priority students. Finally, the market designer faces a strategic constraint. She does not know the preferences of the students; therefore, she can only achieve her design objectives to the extent that the students have been incentivized to reveal their true preferences.

Unfortunately, these constraints are fundamentally incongruent. It is well known that there does not exist a fair and Pareto efficient assignment algorithm (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Moreover, it is impossible for a strategyproof mechanism to Pareto improve the student-optimal fair assignment.¹ Therefore, to resolve

¹Under a strategyproof mechanism, reporting true preference over schools weakly Pareto dominates any other strategies. When students act truthfully, Kesten (2010) demonstrates that no strategyproof and efficient mechanism Pareto dominates DA. (Abdulkadiroğlu et al., 2009) demonstrate that no strategyproof mechanism Pareto improves DA with single tie breaking. We do not consider indifferences in this paper, so the DA with single tie breaking and what we refer to as DA are equivalent in our environment.

the inefficiency associated with making a fair assignment, we must look to manipulatable mechanisms.²

This paper asks the following question: does there exist an assignment process that Pareto dominates the DA assignment in equilibrium with weakly undominated strategies? Specifically, we view an assignment process as a mechanism. The most common processes, DA, Top Trading Cycles (TTC), and the Boston mechanism (BM), are direct mechanisms. Students are asked their preferences and an assignment is determined. However, in general a process need not take as inputs the students preferences over all schools. Many school districts limit the number of schools a student may apply to (see Haeringer and Klijn, 2009, and Calsamiglia, Haeringer, and Klijn, 2010). Alternatively, an algorithm such as Kesten’s Efficiency Adjusted Deferred Acceptance mechanism (hereafter EADAM) asks for both preferences and a message indicating whether or not a student is willing to allow her priorities to be violated. But all such mechanisms ultimately result in a school assignment. We focus on the set of assignments that are supported by an equilibrium of the associated mechanism. Our objective is to find a mechanism that always has an equilibrium assignment induced by weakly undominated strategies that weakly Pareto dominates the DA assignment.³

At first glance, one might think the correct approach is to ask the students for their preferences, calculate the DA assignment, and then Pareto improve this assignment. Running DA and then running TTC on this assignment takes this approach. Kesten’s EADAM also takes this approach but in a more subtle manner.⁴ It is well known that such a mechanism

²We say a mechanism is manipulable if it is not strategy-proof.

³Specifically, we consider only equilibria in which agents play weakly undominated strategies. It is not possible to always Pareto dominate the DA assignment since for some problems the DA assignment is Pareto efficient. However, we seek a mechanism that always has as an equilibrium assignment (in undominated strategies) that is either the DA assignment or an assignment that Pareto dominates the DA assignment. We further require that for some problems it has an equilibrium assignment that Pareto dominates the DA assignment.

⁴Dur et al. (2015) characterize the class of mechanisms Pareto dominating DA mechanism.

cannot be incentive compatible;⁵ however, the equilibrium properties of any such are unknown. Our Theorem 1 demonstrates that no such mechanism actually Pareto dominates DA once we solve for the equilibria.

Specifically, our Theorem 1 introduces a general negative result: no algorithm that Pareto improves DA whenever DA is inefficient can Pareto dominate DA in equilibrium.⁶ That is to say, there always exist problems where in every equilibrium of the associated mechanism, some student is made worse off relative to her DA assignment. This demonstrates that these algorithms only dominate DA in a simplistic way. Once students respond to their incentives to reveal information, the algorithms no longer improve upon DA and in fact some students can be made worse off.

Next, we find a positive result. We introduce a mechanism that always has an equilibrium that weakly Pareto dominates the DA assignment. Here, for tractability we limit each school to have a capacity of one. A surprising mechanism has superior equilibrium properties to DA: running DA but limiting each student to submit only two schools. We call this the **2-school DA**. The intuition is as follows. Whenever we learn that a student i prefers school a to school b , this imposes a constraint on the designer. If we assign i to b , then we can only assign students to a that have higher priority at a than i . In order to improve upon the DA assignment, the designer must learn both what a student's DA assignment is and what is an alternative that she prefers. However, the designer does not want to learn all of the schools a student prefers to her DA assignment. Each school she reveals imposes a new constraint on the designer. When all schools are revealed, the only assignment that can satisfy all of the constraints is the DA assignment. Therefore the designer does better by limiting the information a student is able to reveal. By allowing the student to rank only two schools, in equilibrium she lists her DA assignment second and a school she prefers

⁵Otherwise it would be an efficient, strategyproof mechanism that always selects the fair and efficient assignment when it exists. This would violate the impossibility theorem in Kesten (2010).

⁶In the rest of the paper, whenever we say equilibrium we mean equilibrium with weakly undominated strategies.

first, and the outcome weakly Pareto dominates the DA assignment.⁷

This mechanism dominates DA in equilibrium, but due to miscoordination, out of equilibrium students may be left unassigned. In particular, it is wasteful. Next, we introduce a more complicated mechanism that is non-wasteful and also always has an equilibrium assignment that Pareto dominates the DA assignment. We call this process the Application with Automatic Appeals (AAA) mechanism. This mechanism is intended to be a more reasonable way of implementing the superior assignment.

For our equilibrium results, we have assumed complete information. This is for tractability and is the same approach as taken by several papers in the literature (for example, Pathak and Sönmez 2008, and Ergin and Sönmez, 2006). However, our results only depend on a student's DA assignment being predictable.⁸ The preferences of the other students and indeed the assignments of the other students are only relevant to student i to the extent that they impact i 's assignment. Therefore, in a sufficiently large market, our results will continue with imperfect information to the extent that students are able to predict what their fair outcome will be. Abdulkadiroğlu et al. (2006) provides anecdotal evidence that parents in Boston had grown quite adept at this. Moreover, the typical "cut-off" for a school is information that a school board could easily provide.

2 Relationship to the Literature

Haeringer and Klijn (2009) is the first paper to consider DA when students may only submit a limited number of applications. At first glance, our paper may seem at odds with theirs. They note that once students are limited in the number of schools, DA is no

⁷More precisely, in one equilibrium students submit preferences in this manner. There are other equilibria in which students list a different fair assignment second, and in these equilibria, a student may be assigned to a worse school than her DA assignment.

⁸Formally, if each student can predict her DA assignment, then she can play a strategy where she ranks her DA school second, and a school she prefers first. If every student plays such a strategy, then the outcome of the two-school DA will weakly Pareto dominate the DA outcome.

longer strategyproof, and they study Nash equilibria of the associated preference revelation game. They find that any equilibrium assignment when students are limited to listing k schools remains an equilibrium when students may list $l > k$ schools. This would imply that the mechanism where students can list only two schools does no better than the mechanism where students may list all schools in equilibrium. The difference between our analysis and that of Haeringer and Klijn (2009) is that they consider Nash equilibria where students may play weakly dominated strategies. Specifically, for any equilibrium in which students list only k schools, it remains an equilibrium for every student to continue to list the same k schools even when they are allowed to list more schools.⁹ This is a Nash equilibrium, but it is a weakly dominated strategy to only submit k schools when you find more than k schools acceptable. It is not true that all of the equilibria in weakly undominated strategies when students are constrained to list only k schools continue to be equilibria in weakly undominated strategies when students rank $l > k$ schools. Haeringer and Klijn (2009) also provide an example that demonstrates that the equilibria assignments cannot all be Pareto ranked relative to the DA assignment. Here again, our results do not conflict. Our paper finds that there always exists an equilibrium that Pareto dominates the DA assignment. Their paper notes that not all equilibria Pareto dominate DA. Note that our Theorem 3 demonstrates that all equilibria (that survive iterated elimination of weakly dominated strategies) can be Pareto ranked relative to the school proposing deferred acceptance algorithm.

Calsamiglia et al. (2010) is also closely related to the current paper. They run an experiment to test the impact of limiting the number of schools a student is able to rank.¹⁰ Specifically, they find a decrease in efficiency and an increase in justified envy when students are constrained to only listing three schools. We consider this a strong argument for using

⁹Loosely speaking, their proof proceeds as follows. Any student who can profitably deviate and receive school s may profitably deviate by ranking s first, and therefore would have a profitable deviation whether they can rank one, two, or any number of schools.

¹⁰Their experimental design limits students to ranking three schools, so it does not directly test our mechanism which limits students to two schools. However, we believe their findings remain informative for our mechanism.

the AAA algorithm instead of limiting the number of applications. We expect significant miscoordination when there are relatively few schools and when there is little information about the average ranking required to be admitted to a school. The AAA algorithm is specifically designed to eliminate the wastefulness associated with this miscoordination.

Our paper is a contribution to the recent literature on school choice introduced by Balinski and Sönmez (1999) and pioneered by Abdulkadiroğlu and Sönmez (2003). Pathak (2011) is an excellent survey on the literature. Our paper is similar in spirit to Erdil and Ergin (2008) and Abdulkadiroğlu et al. (2009). Erdil and Ergin (2008) demonstrate that the procedure used to break ties in student priorities can lead to significant inefficiencies in the assignment made by DA. Abdulkadiroğlu et al. (2009) study a similar problem in the context of the NYC school assignment. Their paper both quantifies the extent of the efficiency loss associated with DA and proves that no strategyproof mechanism can Pareto dominate DA with single tie-breaking.

Most similar to our paper, and indeed the motivation for our paper, is Kesten (2010). Kesten has three results which were the motivation for the current paper. Similar to Abdulkadiroğlu et al. (2009), he demonstrates that no strategyproof and efficient mechanism can Pareto dominate DA. Second, he identifies the precise source of DA's inefficiencies. Finally, he introduces an algorithm, the Efficiency Adjusted Deferred Acceptance algorithm, that Pareto improves DA when students submit their true preferences.¹¹ To the best of our knowledge, the equilibrium properties of EADA are unknown. However, our Theorem 1 demonstrates that EADAM does not dominate DA in equilibrium.

Our paper addresses the problem of making a fair assignment more efficient. There are several papers that have focused on making an efficient assignment fairer. Kesten (2004) introduces the Equitable Top Trading Cycles algorithm in order to reduce the number of priority violations induced by Top Trading Cycles. Morrill (2015) addresses the same problem by introducing the algorithm Clinch and Trade. These papers are complementary

¹¹Tang and Yu (2014) is useful for understanding EADAM. They provide an algorithm that is outcome equivalent to Kesten's EADAM but is simpler and more intuitive.

to our paper as each seeks find more of balance between efficiency and fairness than the known algorithms provide.

3 Model

In a school choice problem (Abdulkadiroğlu and Sönmez, 2003) there is a finite set of students I and a finite set of schools A . Each school $a \in A$ has a finite number of available seats. Let q_a be the capacity of school a and $q = (q_a)_{a \in A}$ be the capacity vector. Let \emptyset denote being unassigned option. Each student has a strict preferences (complete, transitive and antisymmetric relations) over all schools and being unassigned option. We denote the preferences of student $i \in I$ by P_i and preferences profile of all students by $P = (P_i)_{i \in I}$. For each $i \in I$, let R_i denote the at-least-as-good-as relation associated with P_i . We assume that each student $i \in I$ considers all schools acceptable, i.e., $aP_i\emptyset$ for all $a \in A$. Each school has a strict priority order (complete, transitive and antisymmetric relations) over all students. We denote the priority order of school $a \in A$ by \succ_a and priority profile of all schools by $\succ = (\succ_a)_{a \in A}$. Throughout the paper we fix the set of students I , the set of schools A , the quota vector q , and the priority profile \succ . We represent a problem with the preference profile P .

A **matching** $\mu : I \rightarrow A \cup \{\emptyset\}$ is a function such that each student is assigned to at most one school and the number of students assigned to a school is less than or equal to its capacity. Let \mathcal{M} be the set of all matchings. We denote the assignment of student i and the set of students assigned to school a by μ_i and μ_a , respectively.

A matching $\mu \in \mathcal{M}$ **Pareto dominates** another matching $\nu \in \mathcal{M}$ if $\mu_i R_i \nu_i$ for each student $i \in I$ and $\mu_j P_j \nu_j$ for at least one student $j \in I$. A matching μ is **Pareto efficient** if there does not exist another matching $\nu \in \mathcal{M}$ which Pareto dominates μ .

A matching μ is **non-wasteful** if there does not exist a student school pair (i, a) such that $|\mu_a| < q_a$ and $aP_i\mu_i$. A matching μ is **individually rational** if $\mu_i R_i \emptyset$ for all $i \in I$. A

matching μ is **fair** if there does not exist a student school pair (i, a) where $a P_i \mu_i$ and $i \succ_a j$ for some $j \in \mu_a$. A matching is **stable** if it is non-wasteful, individually rational, and fair. A stable matching μ is student-optimal stable matching if it Pareto dominates any other stable matching.

A **mechanism** Φ is a procedure which selects a matching for each message profile $\theta = (\theta_i)_{i \in I}$. Here θ_i is the message submitted by student i . θ_i is not limited to be a preference order over schools and being unassigned option.¹² We view the 2-school DA and AAA as indirect mechanisms. A student's type is the preferences she has over all objects. In the 2-school DA, the message space consists of a ranking of two schools; therefore, a student is not able to express her type.¹³ In AAA, the message space consists of the ranking of all schools plus an auxiliary message. In this case, we let $\theta_i = (\alpha_i, P_i)$ where α_i is the auxiliary message of student i .¹⁴ The matching selected by mechanism Φ for a given message profile θ is denoted by $\Phi(\theta)$ and the assignment of each student $i \in I$ and the set of schools assigned to school $a \in A$ are denoted by $\Phi_i(\theta)$ and $\Phi_a(\theta)$, respectively.

A mechanism Φ is **Pareto efficient (stable)** if for any message profile $\theta = (\alpha, P)$ its outcome $\Phi(\theta)$ is Pareto efficient (stable) under preference order P .

A mechanism Φ is **strategy-proof** if a student i cannot get a better school than the one he gets when he reports his true preferences order P_i for any message profile of the other students and his auxiliary message.

In the Appendix A, we provide formal definitions of the classic school choice algorithms: Top Trading Cycles, the Boston Mechanism, EADAM, and Deferred Acceptance plus Top Trading Cycles. The student proposing version of the Deferred Acceptance (DA) algorithm

¹²For instance, students do not only submit preference order over the schools under the school choice mechanisms introduced in Kesten (2010) and Abdulkadiroğlu et al. (2011).

¹³Note that we can also define the 2-school DA as a direct mechanism where an agent submits her entire preference profile, and the mechanism ignores all but her top two schools.

¹⁴For the well known school choice mechanisms like DA, top trading cycles and Boston mechanism, $\theta_i = P_i$.

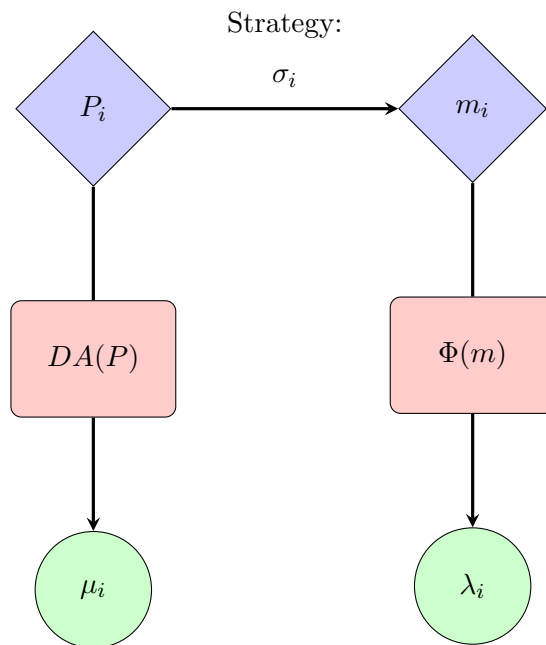
is defined as follows. In the first round, each student proposes to her most preferred school. Each school tentatively accepts students up to its capacity and rejects the lowest priority students beyond its capacity. In every subsequent round, each student rejected in the previous round proposes to her most preferred school that has not already rejected her. Each school tentatively accepts the highest priority students up to its capacity and rejects all others. The algorithm terminates when there are no new proposals and tentative assignments are made final. DA was first introduced in Gale and Shapley (1962). Roth and Sotomayor (1990) is an excellent resource for the properties of DA.

4 Results

We seek a school assignment mechanism where the associated outcome weakly Pareto dominates the DA assignment. We require that in this equilibrium that each student plays a weakly undominated strategy. Specifically, consider Figure 1. DA is an incentive compatible direct mechanism. In the associated preference revelation game, there is a unique equilibrium assignment in weakly undominated strategies. In Figure 1, this is labeled by μ . It is well known that this assignment eliminates justified envy and Pareto dominates any other assignment eliminating justified envy. However, it is not Pareto efficient. A strategyproof mechanism cannot Pareto improve DA when students play their weakly undominated strategies; therefore, we must look to manipulatable mechanisms in order to make the DA assignment more efficient.

We consider both direct mechanisms, where students submit their entire preferences over schools, and indirect mechanisms where students submit a message different from their preferences. Any indirect mechanism induces a game. Agents simultaneously choose which message to send according to strategies σ . Consistent with the literature, we assume players have complete information, and we consider simple Nash equilibria of this game. We say that λ is an **equilibrium assignment** for mechanism Φ if there exist strategies σ such that each σ_i is a best response to σ_{-i} and $\Phi(\sigma(P)) = \lambda$.

Figure 1: Comparing the outcome of indirect mechanism Φ when students play weakly undominated strategies σ to the DA assignment.



We wish to compare the equilibrium assignments of a mechanism Φ to what the assignment would have been under DA.

Definition 1. *A mechanism Φ Pareto dominates DA in equilibrium if:*

- *For every assignment problem, there exists an equilibrium assignment, λ , of Φ such that for every student i , $\lambda_i R_i \mu_i$ where μ is the DA assignment under true preferences.*
- *For some assignment problem, there exists an equilibrium assignment, λ , of Φ such that for every student i , $\lambda_i R_i \mu_i$, and for some student i , $\lambda_i P_i \mu_i$ where μ is the DA assignment under true preferences.*

More generally, we can compare any two mechanisms this way. For a mechanism Φ and assignment problem Γ , let $\Lambda_\Phi(\Gamma)$ denote the set of all assignments supported by an equilibrium (where no student plays a dominated strategy). We say a mechanism Φ **Pareto dominates in equilibrium** a mechanism Ψ if for every assignment problem Γ and for every $\mu \in \Lambda_\Psi(\Gamma)$, either $\mu \in \Lambda_\Phi(\Gamma)$ or there exists a $\nu \in \Lambda_\Phi(\Gamma)$ such that ν Pareto dominates μ . Moreover, we require that for some problem Γ , if $\mu \in \Lambda_\Psi(\Gamma)$, then there exists a $\nu \in \Lambda_\Phi(\Gamma)$ such that ν Pareto dominates μ .

If Φ Pareto dominates DA in equilibrium, this does not necessarily make Φ a superior mechanism to DA. Φ may contain very bad equilibrium outcomes in addition to the good outcomes. However, it can be interpreted that in the best case scenario, Φ is a superior mechanism.

There are several algorithms that Pareto improve DA relative to the submitted preferences. Kesten's EADAM and running DA + TTC are two examples of this. We define a mechanism Φ to **improve DA directly** if for any set of preferences P and any student i , $\Phi_i(P) R_i DA_i(P)$. While it is known that no efficient mechanism that improves DA directly can be strategyproof, little is known about the equilibrium properties of any such mechanism. Our first result is to demonstrate that no efficient mechanism that improves DA directly can dominate DA in equilibrium. Directly making DA more efficient creates

perverse incentives. Once students respond, such an algorithm cannot be Pareto ranked relative to DA. In particular, neither EADAM nor running DA+TTC Pareto dominates DA in equilibrium.

Theorem 1. *Consider any efficient mechanism Φ that improves DA directly. Φ does not Pareto dominate DA in equilibrium with weakly undominated strategies.*¹⁵

Proof. Let Φ be any mechanism that Pareto improves DA under submitted preferences. Consider a problem with three students, $I = \{i, j, k\}$, three schools each with unit capacity, $A = \{a, b, c\}$, and the following preferences and priorities:

P_i	P_j	P_k	\succ_a	\succ_b	\succ_c
a	a	b	k	j	j
c	b	c	i	k	k
	c	a	j	i	i

When true preferences are submitted, DA assigns i to a , j to b , and k to c . As this is Pareto efficient, Φ makes the same assignment. However, if k instead submits preferences $P'_k : b, a, c$, then DA assigns i to c , j to b , and k to a . This is Pareto inefficient and the unique matching Pareto dominating it assigns i to c , j to a , and k to b . Therefore, Φ must assign i to c , j to a , and k to b . This is the unique equilibrium assignment of Φ in weakly undominated strategies. To see this, first consider student j . j has the highest priority at b , so DA never assigns her to an object worse than b . Therefore, Φ never assigns her to an object worse than b . However, she can only be assigned to a if she ranks it ahead of b . Therefore, for student j submitting her true preferences weakly dominates any other strategy under Φ . Similarly, if k submits P'_k , then Φ must assign k to a or b regardless of the preferences i submits. Since assigning j to b and k to a would be Pareto inefficient, when i submits P'_k , Φ assigns her to a regardless of what i submits. Therefore,

¹⁵In particular, this is true for any mechanism that improves DA whenever possible and selects DA outcome whenever DA outcome is Pareto efficient.

the unique equilibrium assignment induced by weakly undominated strategies under Φ for this problem is to assign i to c , j to a , and k to b . Therefore, in every weakly undominated equilibrium of Φ , i receives a worse assignment under Φ than under DA. \square

Kesten’s EADAM is a subtle and complicated algorithm. One might think that students will not be sophisticated enough or that they will not have enough information to successfully manipulate the algorithm. However, the proof of Theorem 1 demonstrates that manipulating EADAM is fairly straightforward and requires little information. The only information a student needs to know is which schools are overdemanded. Consider the following case:

- student i ’s three favorite schools are a , b , and c , respectively.
- i has very low priority at a .
- i has one of the highest priorities at school c .
- a and c are highly demanded schools whereas b is underdemanded.

Since i has very low priority at a and a is highly demanded, DA will never assign her to a . If i submits her true preferences to EADAM, then she will be assigned to b as this is fair and efficient. However, if she instead ranks a first and c second, then she has created an inefficiency with DA. DA assigns her to c , but so long as there is a symmetric student at a (a student with very high priority at a , low priority at c , but one who prefers c to a), then the DA assignment is inefficient. EADAM “corrects” this inefficiency by assigning i to a . In summary, any student with very high priority at a highly demanded school has a simple strategy for manipulating EADAM. She should rank her favorite school first and the safety school second. In particular, she should not rank any underdemanded school ahead of a unless it is her favorite school.¹⁶

¹⁶Further note that parents can discover this manipulation without every understanding the algorithm. A parent group just needs to observe what preferences were submitted and what the outcome was. Such

The two most prominent ways of Pareto improving DA are Kesten’s EADAM and running DA followed by running TTC. An immediate corollary of Theorem 1 is that neither of these mechanisms dominate DA in equilibrium.

Corollary 1. *In equilibrium with weakly undominated strategies, running Deferred Acceptance followed by running TTC does not Pareto dominate running DA.*

Corollary 2. *In equilibrium with weakly undominated strategies, EADAM with all students consenting does not Pareto dominate running DA.¹⁷*

Kesten (2010) proved that there cannot be an efficient and strategyproof mechanism that Pareto dominates DA.¹⁸ Note that since a strategyproof mechanism has a unique equilibrium outcome induced by weakly undominated strategies, this is also a corollary of Theorem 1.

Corollary 3. *[Kesten, 2010] No strategyproof and efficient mechanism Pareto dominates DA.*

Next, we consider running DA but allowing students to rank only two school. This is part of a broader class of mechanisms first studied by Haeringer and Klijn (2009). We demonstrate that this mechanism always has an equilibrium with weakly undominated strategies that Pareto dominates DA when schools can only be assigned to one student. Kesten (2010) identifies the source of DA’s inefficiency. A student i can temporarily hold a

group will notice that a parent who submits a, b, c is assigned to b while a parent who submits a, c is assigned to a .

¹⁷Theorem 1 actually demonstrates something stronger. So long as at least one student consents, EADAM does not Pareto dominate DA in equilibrium. The proof of Theorem 1 only relies on the one student i consenting. It is interesting to contrast this result with Kesten’s Proposition 3 which says that no student is harmed by consenting. This is true so long as all other students submit the same preferences. But this demonstrates that a student can be harmed by consenting in equilibrium.

¹⁸Kesten’s impossibility result is stronger than this. He proves that no strategyproof and efficient mechanism always selects the fair and efficient assignment when it exists. A direct corollary of this result is that no strategyproof and efficient mechanism can Pareto dominate DA.

seat at a school a , cause a student j to be rejected from a , but then later be rejected from a in favor of a higher ranked student. Kesten calls such a student an *interrupter*. Since i does not benefit and j is potentially harmed, this can lead to an inefficient assignment. A Kesten interrupter envies another student's assignment but she herself cannot benefit from a change to the assignment. We will refer to an objection by a Kesten interrupter as a *petty objection*.¹⁹

We Pareto improve DA by limiting petty objections. We must learn the DA assignment, and we must also learn at least one school that a student prefers to her DA assignment. However, in order to eliminate Kesten interrupters, we must make it costly for a student to apply to a school. Under DA it is costless to apply to a school. One way to do this is by limiting the number of schools a student is allowed to apply to. This gives the student the ability to express the school she “deserves” (DA) and a school she prefers, but it forces her to be judicious about which school she applies to. We define the **2-school DA** as follows. Each student is allowed to rank at most two schools and then DA is run on the submitted two-school lists.

Under 2-school DA and under DA, a student does not benefit from being an interrupter. However, it is costly for a student to apply to such a school under 2-school DA since she is limited in the number of applications she may submit. Since there is a cost but no benefit, in equilibrium a student does not apply to a school where she is an interrupter.²⁰

Theorem 2. *When each object has a capacity of one, there is always a Nash equilibrium of 2-school DA where no student plays a weakly dominated strategy and the resulting assignment weakly Pareto dominates the student optimal fair assignment.*

Proof. For convenience, we assume that each student finds every school acceptable, and

¹⁹See Morrill (2016) for a full study of petty objections. Petty objections are closely related to λ -equity, introduced in Alcalde and Romero-Medina (2015), and reasonable stability, introduced by Cantala and Pápai (2015), and partial fairness, introduced by Dur et al. (2015).

²⁰The exception to this is if her DA assignment is her second favorite school and she is a Kesten interrupter at her favorite school. In equilibrium, she applies to her two favorite schools.

that the total number of students equals the total capacity of all schools. These assumptions are only for technical convenience. Let $\mu = DA(P)$, the DA assignment when each student i submits her true preferences P_i . Each student i submits a preference list composed of only two schools and being unassigned option. Let $r_{P_i}(k)$ be the k^{th} school under P_i . Let P_i^0 be the initial strategy of student i such that $r_{P_i}(1)P_i^0\mu_iP_i^0\emptyset$ if $\mu_i \neq r_{P_i}(1)$ and $\mu_iP_i^0r_{P_i}(2)P_i^0\emptyset$ otherwise. A key point is that if all other students j submits P_j^0 , then when i submits P_i^0 she is ensure μ_i or a school she prefers. We show starting from P^0 we can construct an equilibrium and that it supports a (weak) Pareto improvement of μ .

This strategy induces a graph as follows. Draw a directed edge from μ_i to i . Draw a directed edge from i and a if i is the highest ranked student that ranks a at the top of her submitted list. Call this graph G^1 . Note that each school points to exactly one student. Further, note that each school has at most one student pointing to it. Therefore, the graph partitions the students and schools into either cycles or paths. It is straightforward to verify that students in a cycle will be assigned to the school they are pointing to (a Pareto improvement of μ if the cycle is composed of more than two students), and students in a path will be assigned to the school pointing to it (their assignment under μ).

Consider a student i and consider a school a where there is a path from a to i . Define a school a to be achievable for i if i has higher priority than any student ranking it first under P^0 . Note that μ_i is always achievable for i (if j points at μ_i , it must be that i has higher priority at μ_i than j or else j would have justified envy of i under μ). Further note that the last school on the path to i is always achievable for i since no other student is ranking it first. Intermediate schools on the path to i may or may not be achievable depending on whether or not i has higher priority at the school than the student currently pointing at it. For any other school, if i applies to it she will either be rejected.

A given set of strategies constitute a Nash equilibrium if no student prefers an achievable school to her assignment. Suppose some student has a profitable deviation under G^1 (she prefers an achievable school to her DA assignment). Label this student i_1 , and change i_1 's strategy so that she ranks her favorite achievable school at the top of her submitted

preference list. i_1 was part of a path $(j_1, \mu_1, j_2, \mu_2, \dots, j_n, \mu_n)$ where $i_1 = j_n$, μ_k is j_{k+1} 's DA assignment, and μ_{k-1} is $k-1$'s favorite school. i_1 's deviation changes the graph in three potential ways. First, there is now a cycle including i_1 . Call this cycle C_1 . Second, i_1 is no longer ranking her favorite school a at the top of her submitted list. If she was not previously the highest ranked student ranking a at the top, then this has no effect on the graph. But suppose i was the highest ranked student ranking a at the top, then this changes the path to the students “downstream” from i in potentially two ways.²¹ In G^1 , suppose there is a path from a student k to i , and a path from i to a student j . After i 's deviation, there is now no longer a path from k to j . But second, if other students were also ranking a at the top, then now there is a new highest priority student ranking a at the top and now there is a new path upstream from a . Finally, if i_1 ranks an intermediate school at the top, then her deviation breaks off the beginning of the path.

If the preference profile obtained after this deviation is an equilibrium, stop. If not, then there is a second student $i_2 \neq i_1$ with a profitable deviation. Have i_2 deviate by changing her top rank school to her favorite achievable school. i_2 is not part of C_1 since these students (other than i_1) are already getting their favorite school. Label the cycle i_2 creates C_2 .

Now consider the graph with C_1 and C_2 . i_1 is currently part of a cycle $(i_1, \mu_{i_1}, 2, \mu_2, \dots, n, \mu_n)$. If i_1 deviates and does not rank μ_n at the top, then i_1 's deviation means there could be a new agent pointing to μ_n . However, this could only increase the length of the path pointing to i_1 . Importantly, the path still includes all the members of C_1 . Therefore, we let i_1 change the school she's ranking at the top to if she wants. If she now has a better deviation, it must be to a school farther down on the path (if she wanted to shorten the cycle, she would have shortened it initially). Relabel this cycle, if there is one, C_1 . C_1 either has not changed or else is a strict superset of the original C_1 . Moreover, every student in C_1 other than i_1 receives her favorite assignment. Further, note that even if C_1 did change, it does not intersect with C_2 . Since C_2 is a cycle, it is not part of any path to i_1 . Moreover, i_1

²¹ j is downstream from i if there is a path from i to j . j is upstream from i if there is a path from j to i .

changing C_1 cannot change the set of schools available to i_2 .

If this is an equilibrium, stop. If not, select a third student with a profitable deviation, i_3 . Since i_1 and i_2 do not have any further deviations, $i_3 \notin \{i_1, i_2\}$. Since the agents in $C_1 \cup C_2 \setminus \{i_1, i_2\}$ all get their first choice, $i_3 \notin C_1 \cup C_2$. Label the cycle i_3 creates C_3 . The key points are that this cycle cannot intersect with C_1 or C_2 (each school points to one agent and an agent points to at most one school). i_1 and/or i_2 may not want to change their top ranked school, but this can only increase the cycle they are in. Since there is no path between i_1 and i_2 , expanding either cycle would not effect the other. Finally, after i_1 and i_2 expand their cycle (if they want), it remains that i_1 , i_2 , and i_3 have no profitable deviations and all other agents in C_1 , C_2 , and C_3 get their top choice.

This process must eventually terminate, and when it does, no agent has a profitable deviation. Moreover, when this process terminates, we reach a preference profile in which (1) if μ_i is i 's top choice, then i ranks her top two choices (2) otherwise, i ranks μ_i as second choice and a better school as a top choice. One can easily verify this is a weakly undominated strategy. \square

Due to the similarities between 2-school DA and Shanghai mechanism (Chen and Kesten (2015)), one can wonder whether the set of equilibrium outcomes induced by weakly undominated strategies. In Appendix, we provide an example showing that this is not true.

In general, the 2-school DA does not have a unique equilibrium and not all equilibria Pareto dominate the student-optimal stable assignment. However, our next result is to show that in all equilibria that survive iterated elimination of weakly dominated strategies that no student receives worse than a fair assignment. Specifically, each student receives either their school-proposing DA assignment or a school they strictly prefer.

Theorem 3. *Consider any school capacities, and any equilibrium of 2-school DA that survives iteratively eliminating weakly dominated strategies. No student is assigned to worse than a fair assignment.*

Proof. Consider the first round of the school-proposing DA algorithm, and consider any student i that gets multiple proposals. Let a be her favorite school that she is proposed to in the first round, and let b be any of her other proposals (in particular, $a P_i b$). Note that she is one of the q_a and q_b highest ranked students at a and b , respectively. If she ever were to list either a or b , she will be assigned to that school or the other school in her list. A key point is that i does not play the following strategies: ranking a third school c first and b second and ranking b first. This strategies are weakly dominated by ranking c first and a second and ranking a first, respectively. A second point is that in any equilibrium outcome of 2-school DA, she will not be assigned to a school worse than a .

For the inductive step, consider the k^{th} round of the school-proposing DA algorithm. Consider a student j who is holding onto a proposal from school a and has already rejected school b . Our inductive hypothesis is that j does not rank a third school c first and b second and b first.²² In the k^{th} round, consider any student i who has received more than one proposal. Let a be her favorite of all such proposals and let b be any other proposal. By the inductive hypothesis, in any equilibrium of 2-school DA that survives iteratively eliminating dominated strategies, no student who has previously rejected either a or b ever lists a or b as her first choice or ranks as her second choice and does not rank the school she is not tentatively holding in step $k - 1$ as first choice (they have a school that they strictly prefer that they are sure to get into). Therefore, if i lists a (b) as first choice, she will be assigned to a (b). Similarly, if i lists another school c first and lists a (b) as second choice, she will be assigned to either c or a (b). Hence, ranking b as first choice and a third school c first and b second is weakly dominated by applying to a first or c first and a second, respectively. Moreover, i is never assigned to a school worse than she is holding onto in the k^{th} round of the school-proposing DA algorithm.

Since this is true for any k , in any equilibrium of 2-school DA that survives iterated elimination of weakly dominated strategies, no student is assigned to a worse school than she receives under the school-proposing DA algorithm. \square

²²Note that it cannot be an equilibrium for her to list a and b but rank b ahead of a .

5 Application with Automatic Appeals Algorithm

Theorem 2 demonstrates that under 2-school DA it is possible to Pareto improve DA in equilibrium with weakly undominated strategies. However, the equilibrium requires coordination among the students. With imperfect information or miscoordination, it may be that a student is rejected from both of her application schools while a seat at a desired school is available. Calsamiglia, Haeringer, and Klijn (2010) provide evidence of the welfare loss due to miscoordination. In this section, we introduce a new mechanism which Pareto dominates DA in equilibrium with weakly undominated strategies but is designed to mitigate the harm from any miscoordination among students. In particular, the mechanism is non-wasteful.

We call this mechanism Application with Automatic Appeals (AAA). In this mechanism, a student i submits an application to one school and a preference order over schools in the order in which she wishes to appeal. We denote the school that student i applies by α_i and the order of schools that student i wishes to appeal by Θ_i . Before giving the definition of the algorithm finding the outcome of the AAA mechanism, we introduce a priority structure which is used in the algorithm.

In our analysis, we assume that each student $i \in I$ considers all schools acceptable, i.e., $aP_i\emptyset$ for all $a \in S$.

The Application with Automatic Appeals Algorithm (AAA):

A student i submits an application to one school, a_i , and a list of schools, Θ_i in the order in which she wishes to appeal.

A school has two sets of priorities: an application priority and an appeals priority. The application priority equals \succ_a which is the normal student priorities. The appeals priority for a school a , $\tilde{\succ}_a$, is defined as follows.

Definition 2. *Given application priorities \succ_a and any two students i and j , $i \tilde{\succ}_a j$ iff*

1. i ranks a higher on her appeals list than j does; or
2. i and j rank a the same on their appeals but $i \succ_a j$.

The algorithm proceeds in rounds.

Application Round: The top q_a highest ranked student (under \succ_a) to apply to a is temporarily assigned to a . All other students are rejected.

Appeals Round: After a student is rejected, she can appeal being unassigned. In each appeals round, the students that were rejected in an earlier round appeal to the highest school on their appeals list that has not already rejected their appeal.

Denote the set of all applying and appealing students to school a in this round with A_a^1 and A_a^2 , respectively. Then consider the following cases:

- $|A_a^1 \cup A_a^2| \leq q_a$: Tentatively accept all students in $A_a^1 \cup A_a^2$.
- $|A_a^1 \cup A_a^2| > q_a$: We iteratively reject students until q_a students remain. If $A_a^1 = \emptyset$ (all students appealed to a), then reject the lowest ranked student according to $\tilde{\succ}_a$. If $A_a^2 = \emptyset$ (all students applied to a), then reject the lowest ranked student according to \succ_a . If $A_a^1 \neq \emptyset$ and $A_a^2 \neq \emptyset$, then let i be the lowest ranked (according to \succ_a) student in A_a^1 and j be the lowest ranked (according to $\tilde{\succ}_a$) student in A_a^2 . Reject i if $j \succ_a i$ and $j \tilde{\succ}_a i$. Otherwise, reject j . Proceed until only q_a students remain. Tentatively accept these students.

The algorithm terminates when there are no new appeals.

We illustrate the dynamics of the algorithm in the following example.

Example 1. There are 5 schools, $S = \{a, b, c, d, e\}$, and 6 students, $I = \{1, 2, 3, 4, 5, 6\}$. School a has two available seats and any other school has one available seat. The priorities are given as:

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>
5	3	4	6	4
2	4	3	4	
4	5	6	5	
1	6			
6				

The submitted message profile is:

	1	2	3	4	5	6
<i>α</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>b</i>	<i>b</i>	<i>c</i>
<i>θ</i>	<i>a</i>	<i>e</i>	<i>b</i>	<i>b</i>	<i>d</i>	<i>d</i>
		<i>a</i>		<i>a</i>	<i>a</i>	<i>a</i>
				<i>d</i>		<i>b</i>
				<i>c</i>		<i>c</i>
				<i>e</i>		

AAA algorithm selects its outcome according to the steps below:

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>
Application Step	<u>1, 2</u>	<u>3, 4, 5</u>	<u>6</u>		
Appeal Step 1	<u>1, 2</u>	<u>3, 4</u>	<u>6</u>	<u>5</u>	
Appeal Step 2	<u>1, 2, 4</u>	<u>3</u>	<u>6</u>	<u>5</u>	
Appeal Step 3	<u>1, 2</u>	<u>3</u>	<u>6</u>	<u>5, 4</u>	
Appeal Step 4	<u>1, 2</u>	<u>3</u>	<u>6, 4</u>	<u>5</u>	
Appeal Step 5	<u>1, 2</u>	<u>3</u>	<u>4</u>	<u>5, 6</u>	
Appeal Step 6	<u>1, 2, 5</u>	<u>3</u>	<u>4</u>	<u>6</u>	
Appeal Step 7	<u>1, 5</u>	<u>3</u>	<u>4</u>	<u>6</u>	<u>2</u>

Here, the students in red are the ones appealing to that school for a given step. Underbarred students are the tentatively assigned students in that step.

Example 2 gives the intuition behind the equilibrium of AAA that Pareto dominates DA.

Example 2. Suppose there are five schools, $\{a, b, c, d, e\}$, each with a capacity of one; five students, $\{1, 2, 3, 4, 5\}$; and preferences and priorities as follows.

1	2	3	4	5	a	b	c	d	e
a	b	c	b	d	2	1	4	1	
b	a	a	c	e	3	2		2	
c	c	d			1	4		3	

DA proceeds as follows:

	a	b	c	d	e
DA Round 1:	1	2,4	3	5	
DA Round 2:	1	2	3,4	5	
DA Round 3:	1,3	2	4	5	
DA Round 4:	3	1,2	4	5	
DA Round 5:	2,3	1	4	5	
DA Round 6:	2	1	4	3,5	
DA Final:	2	1	4	3	5

Note that 3 is a Kesten-interrupter. When she applies to a , she initiates a rejection chain that ultimately results in her being rejected from a . The DA assignment is Pareto inefficient as reassigning 1 to a and 2 to b is a Pareto improvement. This assignment is not fair since 3 makes a petty objection. However, consider the following strategies for AAA:²³

	1	2	3	4	5
α	a	b	c	b	d
θ	b	a	d	c	d

²³For brevity, we have only listed the first school a student appeals to as this is all that is relevant.

AAA algorithm selects its outcome according to the steps below:

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>
Application Step	1	2, 4	3	5	
Appeal Step 1	1	2	3 , 4	5	
Appeal Step 2	1	2	4	3, 5	
Appeal Step 3	1	2	4	3	5

Note that this is the assignment that Pareto dominates DA. Is this an equilibrium? Note that only 3 has the ability to file a successful appeal. If she instead appealed to *a*, she would initially win her appeal. However, after 1 is rejected, she successfully appeals to *b*. After 2 is rejected from *b*, she successfully appeals to *a* causing 3 to be rejected. Therefore, 3 does not benefit from appealing to *a* instead of appealing to *c*. In fact, 3 is harmed. Note that now when she appeals to *d*, her appeal is rejected. Although she has a higher priority than 5 at *d*, since she did not appeal to *d* first, she has a lower appeals priority than 5. Therefore, her appeal is rejected.

One can think of DA as honoring all objections and BM as honoring essentially none. To improve on DA, we need to make it costly for a student to apply to a school. If there is a cost associated with an application, then we will avoid some of the petty objections as interrupters won't apply to a school that they cannot be assigned to. However, applications must be "cheap" enough that students are willing to apply to multiple schools. This is necessary to improve DA as we must know both what a student's fair assignment is and assignments she prefers. Intuitively, we strike a balance by giving each student one "free" application and then making all later applications costly. We do this by allowing students to apply to only one school. In the application round, assignments are made based purely on priorities. Specifically, the highest ranked student to apply to a school is assigned to that school. After the application round, we allow students to appeal their assignment. In order to make appeals costly, we make a student's priority on appeal to be conditional on how she ranks the object. Specifically, the appeals priority is based first on how highly

the student ranks the object and second based on the students priority at the object. Therefore, in equilibrium, an agent only appeals to a school that she will be accepted to. Importantly, a petty objection makes any future appeal that the student makes less likely to be successful.

Theorem 4. *When each object has a capacity of one, there is always a Nash equilibrium of AAA with weakly undominated strategies that weakly Pareto dominates the student optimal fair assignment.*²⁴

Our appeals process is similar to the Boston Mechanism. Ergin and Sönmez (2006) demonstrate that the DA assignment is a Nash equilibrium of the Boston Mechanism.²⁵ Intuitively, this equilibrium will also be an equilibrium of the appeals process in AAA. However, AAA gives students one application that is free in the sense that it does not harm their appeals priority at any school. Therefore, a student can use the appeals process to ensure that she is not assigned to a school worse than her assignment under DA mechanism and use the application process to potentially get a better assignment.

Proof. For convenience, we assume that each student finds every school acceptable, and that the total number of students equals the total capacity of all schools. These assumptions are only for technical convenience. Each student's strategy consists of two things: the school she applies to, α_i , and her list of schools in the order she wishes to appeal to, Q_i . In equilibrium, each student appeals to only one school (unless no student envies her assignment). Let $\mu = DA(P)$, the DA assignment when each student i submit her true preferences P_i . Initially, for each i set α_i to be i 's favorite school, and set Q_i to be any ordering that ranks μ_i first.²⁶ A key point is that if all other students j rank μ_j first on appeals, then when i ranks μ_i first on appeals she is ensure μ_i or a school she prefers. We

²⁴By weakly Pareto dominates, we mean that every student weakly prefers the AAA outcome to the student-optimal fair assignment.

²⁵In particular, they show that any stable matching is a Nash equilibrium of the Boston Mechanism.

²⁶It may be that ranking μ_i first is a weakly dominated strategy. For example, this occurs if i is assigned to her least favorite school. In this case, set Q_i to any undominated strategy.

show that this is an equilibrium and that it supports a (weak) Pareto improvement of μ .

This strategy induces a graph as follows. Draw a directed edge from μ_i to i . Draw a directed edge from i and a if i is the highest ranked student that applies to a . Call this graph G^1 . Note that each school points to exactly one student. Further, note that each school has at most one student pointing to it. Therefore, the graph partitions the students and schools into either cycles or paths. It is straightforward to verify that students in a cycle will be assigned to the school they are pointing to (a (weak) Pareto improvement of μ), and students in a path will be assigned to the school pointing to it (their assignment under μ).

Consider a student i and consider a school a where there is a path from a to i . Define a school a to be achievable for i if i has higher priority than any student applying to a . Note that μ_i is always achievable for i (if j points at μ_i , it must be that i has higher priority at μ_i than j or else j would have justified envy of i under μ). Further note that the last school on the path to i is always achievable for i since no other student has applied to it. Intermediate schools on the path to i may or may not be achievable depending on whether or not i has higher priority at the school than the student currently pointing at it. For any other school, if i applies to it she will either be rejected in the application stage or the appeals stage.

A given set of strategies constitute a Nash equilibrium if no student prefers an achievable school to her assignment under those strategies. Suppose some student has a profitable deviation under G^1 (she prefers an achievable school to her DA assignment). Label this student i_1 , and change i_1 's strategy so that she applies to her favorite achievable school and ranks her DA assignment at the top of her appeals list. i_1 was part of a path $(j_1, \mu_1, j_2, \mu_2, \dots, j_n, \mu_n)$ where $j_n = i_1$, μ_k is j_k 's DA assignment, and μ_{k-1} is $k-1$'s favorite school. i_1 's deviation changes the graph in three potential ways. First, there is now a cycle including i_1 . Call this cycle C_1 . Second, i_1 is no longer applying to her favorite school a . If she was not previously the highest ranked student applying to a , then this has no effect on the graph. But suppose i was the highest ranked student applying to a , then

this changes the line to the students “downstream” from i in potentially two ways.²⁷ In G^1 , suppose there is a path from a student k to i , and a path from i to a student j . After i ’s deviation, there is now no longer a path from k to j . But second, if other students were also applying to a , then now there is a new highest priority student applying to a and now there is a new path upstream from a . Finally, if i_1 applies to an intermediate school, then her deviation breaks off the beginning of the path.

If this is an equilibrium, stop. If not, then there is a second student i_2 with a profitable deviation. Have i_2 deviate by changing her application to her favorite achievable school. i_2 is not part of C_1 since these students (other than i_1) are already getting their favorite school. Label the cycle i_2 creates C_2 . i_1 is currently part of a cycle $(i_1, \mu_1, 2, \mu_2, \dots, n, \mu_n)$. If i_1 does not apply to μ_n , then i_2 ’s deviation means there could be a new agent pointing to μ_n . However, this could only increase the length of the path pointing to i_1 . Importantly, the path still includes all the members of C_1 . Therefore, we let i_1 change the school she’s applying to if she wants. If she now has a better deviation, it must be to a school farther down on the path (if she wanted to shorten the cycle, she would have shortened it initially). Relabel this cycle, if there is one, C_1 . C_1 either has not changed or else is a strict superset of the original C_1 . Moreover, every student in C_1 other than i_1 receives her favorite assignment. Further, not that even if C_1 did change, it does not intersect with C_2 . Since C_2 is a cycle, it is not part of any path to i_1 . Moreover, i_1 changing C_1 cannot change the set of schools available to i_2 .

If this is an equilibrium, stop. If not, select a third student with a profitable deviation, i_3 . Since i_1 and i_2 do not have any further deviations, $i_3 \notin \{i_1, i_2\}$. Since the agents in $C_1 \cup C_2 \setminus \{i_1, i_2\}$ all get their first choice, $i_3 \notin C_1 \cup C_2$. Label the cycle i_3 creates C_3 . The key points are that this cycle cannot intersect with C_1 or C_2 (each school points to one agent and an agent points to at most one school). i_1 and/or i_2 may not want to change their application school, but this can only increase the cycle they are in. Since there is no path between i_1 and i_2 , expanding either cycle would not effect the other. Finally, after i_1

²⁷ j is downstream from i if there is a path from i to j . j is upstream from i if there is a path from j to i .

and i_2 expand their cycle (if they want), it remains that i_1 , i_2 , and i_3 have no profitable deviations and all other agents in C_1 , C_2 , and C_3 get their top choice.

This process must eventually terminate, and when it does, no agent has a profitable deviation.

When this process terminates in the final strategy profile each student is applying to a school better her DA assignment if she is not assigned to her top choice under DA and her DA assignment if she is assigned to her top choice under DA. Moreover, she ranks her DA assignment at the top of her appeals list if she is not assigned to her last choice under DA and her top choice otherwise. One can easily verify that there does not exist any strategy profile weakly dominating the final strategy profile. Hence, when this process terminates, we reach a weakly undominated strategy profile. \square

In general, the AAA algorithm does not have a unique equilibrium and not all equilibria Pareto dominate DA. However, our next result is to show that in all equilibria that survive iterated elimination of weakly dominated strategies that no student receives worse than a fair assignment. Specifically, each student receives either their school-proposing DA assignment or a school they strictly prefer.

Theorem 5. *Consider any school capacities, and any equilibrium of AAA that survives iteratively eliminating weakly dominated strategies. No student is assigned to worse than a fair assignment.*

Proof. Consider the first round of the school-proposing DA algorithm, and consider any student i that gets multiple proposals. Let a be her favorite school that she is proposed to in the first round, and let b be any of her other proposals (in particular, $a P_i b$). Note that she is one of the q_a and q_b highest ranked students at a and b , respectively. Therefore, if she ranks either a (or b) first on appeals and she is rejected by her application school, then she will be assigned to a (resp., b). Since she prefers a to b , ranking b first on appeals is weakly dominated by ranking a first on appeals. Note that in any equilibrium of AAA,

she will not be assigned to a school worse than a .

For the inductive step, consider the k^{th} round of the school-proposing DA algorithm and assume that no student who rejects a school a in an earlier round ranks that school first on appeals. If a student i has received more than one proposal, let a be her favorite of all such proposals and let b be any other proposal. By the inductive hypothesis, in any equilibrium of AAA that survives iteratively eliminating dominated strategies, no student who has previously rejected either a or b ranks it first on appeals. Therefore, if i ranks a (b) first on appeals, she will be assigned to a (b). Since i prefers a to b , ranking b first on appeals is weakly dominated by ranking a first. Therefore, in equilibrium under AAA, i does not rank first on appeals a school she rejects in round k , and i is not assigned to a school worse than the school she is holding onto in round k of the school-proposing DA algorithm.

Since this is true for any k , in any equilibrium of AAA that survives iterated elimination of weakly dominated strategies, no student is assigned to a worse school than she receives under the school-proposing DA algorithm. \square

6 Conclusion

Fairness and strategyproofness are highly desirable properties in an assignment mechanism. However, the most fundamental normative criterion in economics is Pareto efficiency. If it is possible to make students better off without harming anyone, then we should. It is well known that the Deferred Acceptance algorithm (DA) makes Pareto inefficient assignments. However, with DA, we only discover that it is possible to make a Pareto improving assignment *ex-post*. Since any *ex-post* modification to the assignment inevitably changes the preference submission strategy associated with DA, prior to the current paper it was unknown whether or not it was possible to implement a Pareto improvement.

This paper demonstrates that indeed it is possible. Knowing that it is possible to imple-

ment an assignment that dominates the student-optimal fair assignment begs the following question: how should we do so? We have introduced one such algorithm which we feel is natural and intuitive, but the question of how to improve DA is an important topic for future research.

Appendix A Comparison Between Shanghai Mechanism and 2-School DA

Suppose there are three schools, $\{a, b, c\}$, each with a capacity of one; three students, $\{1, 2, 3\}$; and preferences and priorities as follows.

1	2	3	a	b	c
a	b	b	2	1	3
b	a	a	3	2	2
c	c	c	1	3	1

Note that DA assigns 1 to b , 2 to a , and 3 to c . The Shanghai mechanism is defined in Appendix B. Note that 1 and 2 are guaranteed their second choice, so their weakly dominant strategy under Shanghai is to submit their true preferences. Similarly, so long as 3 ranks all three schools, she will be assigned to some school under Shanghai. As c is her least preferred school, she cannot benefit (and may harm) herself by ranking c first or second. Therefore, her only undominated strategies are to submit her true preferences or a, b, c . Under either submission, student 3 is assigned to school c under Shanghai. Therefore, under the Shanghai mechanism there exists a unique equilibrium assignment (in undominated strategies) which coincides with the (inefficient) DA assignment: 1 is assigned to b , 2 is assigned to a , and 3 is assigned to c . On the other hand, under 2-School DA, the following strategies is an equilibrium in undominated strategies: 1 submits a, b ; 2 submits b, a ; and 3 submits b, c . In this equilibrium 1 is assigned to a , 2 is assigned to b , and 3 is assigned to c . This equilibrium assignment Pareto dominates the unique equilibrium of both DA and the Shanghai mechanism.²⁸

²⁸Note that this is not a unique equilibrium for 2-school DA. It is also an equilibrium for 3 to submit a, c in which case the assignment coincides with the DA assignment.

Appendix B Definition of the Mechanisms

Boston Mechanism:

For a given message profile $\theta = (P_i)_{i \in I}$, BM mechanism selects its outcome through the following algorithm:

Step 1: Each student applies to her most preferred school. Each school s accepts the best students according to its priority list, up to q_s , and rejects the rest.

Step $k > 1$: Each student rejected in Step $k - 1$ applies to her k^{th} choice. Each school s accepts the best students among the new applicants, up to the number of remaining seats, and rejects the rest.

School Proposing DA Mechanism:

For a given message profile $\theta = (P_i)_{i \in I}$, school proposing DA mechanism selects its outcome through the following algorithm:

Step 1: Each school s proposes to top q_s students under \succ_s . Each student i accepts the best proposal it gets according to P_i , and rejects the rest.

Step $k > 1$: Each school s proposes to top q_s students under \succ_s who have not rejected it yet. Each student i accepts the best proposal it gets according to P_i , and rejects the rest.

Top Trading Cycles Mechanism:

For a given message profile $\theta = (P_i)_{i \in I}$, TTC mechanism selects its outcome through the following algorithm:

Step 0: Assign a counter to each school and set it equal to the quota of each school. If the counter of a school is zero, then that school is removed.

Step 1: Each student points to his most preferred school among the remaining ones. Each remaining school points to the top-ranked student in its priority order. Due to the finiteness

there is at least one cycle.²⁹ Assign each student in a cycle to the school he points to and remove him. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is removed.

Step $k > 1$: Each student points to his most preferred school among the remaining ones. Each remaining school points to the student with the highest priority among the remaining ones. There is at least one cycle. Assign each student in a cycle to the school he points to and remove him. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is also removed.

Deferred Acceptance-Top Trading Cycles Mechanism

For a given message profile $\theta = (P_i)_{i \in I}$, DA-TTC mechanism selects its outcome through the following algorithm:

Round DA : Run the DA algorithm. Update the priorities by giving highest priorities for each school to the students assigned to it.

Round TTC : Run the TTC algorithm by using preference profile and updated priorities.

Efficiency-Adjusted Deferred Acceptance Mechanism:

In order to define the algorithm selecting the outcome of EADAM, we first present a notion that we use in the definition. If student i is tentatively accepted by school s at some step t and is rejected by s in a later step t' of DA and there exists another student j who is rejected by s in step $t'' \in \{t, t + 1, \dots, t' - 1\}$, then i is called an **interrupter** for s and (i, s) is called an **interrupting pair** of step t' . Under EADAM, each student reports a message $\theta_i = (\alpha_i, P_i)$ where $\alpha_i \in \{0, 1\}$ and $\alpha_i = 1$ if i consents and $\alpha_i = 0$, otherwise. For a given message profile $\theta = (\alpha_i, P_i)_{i \in I}$, EADAM selects its outcome through the following algorithm:

²⁹A cycle is an ordered list of distinct schools and distinct students $(s_1, i_1, s_2, \dots, s_k, i_k)$ where s_1 points to i_1 , i_1 points to s_2 , ..., s_k points to i_k , i_k points to s_1 .

Round 0: Run the DA algorithm.

Round $k > 0$: Find the last step of the DA run in Round $k - 1$ in which a consenting interrupter is rejected from the school for which she is an interrupter. Identify all the interrupting pairs of that step with consenting interrupters. For each identified interrupting pair (i, s) , remove s from the preferences of i without changing the relative order of the other schools. Rerun DA algorithm with the updated preference profile. If there are no more consenting interrupters, then stop.

Shanghai Mechanism

See Chen and Kesten (2016) for a full description of the Shanghai mechanism and its properties. The Shanghai mechanism can be defined as iteratively repeating the 2-school DA. Each student submits a ranking over all schools.

Round 1: Run 2-school DA with each student submitting her top two schools. Finalize the assignments made in this round and adjust the school capacities accordingly.

Round k : Run 2-school DA with each student who has not already been assigned in a previous round submitting her $2k - 1$ and $2k$ favorite schools. Finalize the assignments made in this round and adjust the school capacities accordingly.

The algorithm concludes when either all students have been assigned or else the remaining students have been rejected by every school on their application list.

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