History-Dependent Risk Aversion and the Reinforcement Effect*

Gerelt Tserenjigmid†

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Abstract

This paper studies history-dependent risk aversion and focuses on a well-documented behavior called the *reinforcement effect* (RE), which says that people become less risk-averse after a good history than after a bad history. I show that this seemingly anomalous behavior occurs whenever risk preferences are history-dependent (in a nontrivial way) and satisfy *monotonicity with respect to first-order stochastic dominance*. To study history-dependent risk aversion and the RE formally, I introduce a behaviorally founded model of dynamic choice under risk that generalizes standard discounted expected utility. To illustrate the usefulness of my model, I apply it to the Lucas tree model of asset pricing and draw implications of the RE on asset price dynamics. I find that, compared to history-independent models, the assets are overpriced when the economy is in a good state and are underpriced in a bad state. Moreover, my model generates high, volatile, and predictable asset returns, consistent with empirical evidence.

Keywords: History-Dependent Risk Aversion; Reinforcement Effect; Monotonicity; History-Dependent Model; Behavioral Foundations; Asset Pricing.

JEL Classification Numbers: D03, D11, D80, D90, G12

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†Email: gtserenj@caltech.edu. Division of the Humanities and Social Sciences, California Institute of Technology.

1 Introduction

Empirical evidence suggests that risk preferences evolve over time with personal experiences. This paper studies history-dependent risk aversion and focuses on a well-documented behavior called the *reinforcement effect* (henceforth, RE). The RE says that people become less risk-averse after a good history than after a bad history. I show that this seemingly anomalous behavior occurs whenever a risk preference is history-dependent (in a nontrivial way) and satisfies *monotonicity with respect to first-order stochastic dominance*. Since monotonicity is a reasonable condition, my result provides a theoretical justification for the RE.[1]

To study history-dependent risk aversion and the RE formally, I introduce a behaviorally founded model of dynamic choice under risk. The model generalizes standard discounted expected utility in two ways. First, risk preferences are allowed to reflect past risky choices and their payoffs. For example, if an agent is an expected utility maximizer, then the model is a generalization of discounted expected utility, in which the concavity of the agent's utility function changes with her past risky choices and their payoffs. In Section 2, I informally introduce this example and illustrate the main result. In this example, the RE arises if the utility function after a good history is less concave than the utility function after a bad history.

Second, risk preferences are allowed to violate expected utility and be non-expected utility preferences such as rank-dependent utility preferences (Quiggin 1982 and Tversky and Kahneman 1992) or disappointment aversion theory preferences (Gul 1991). For example, consider a disappointment aversion theory agent, who distorts probabilities by a real number called a *disappointment parameter*. Then the model is a dynamic version of disappointment aversion theory, in which the disappointment parameter is history-dependent. It turns out that, for a fixed utility function, the disappointment parameter dictates the agent's degree of risk aversion. Therefore, the RE arises if the disappointment parameter is smaller after a good history than a bad history. The behavioral foundations of my model and the two special cases above (dynamic versions of expected utility and disappointment aversion theory) are provided.

To illustrate the usefulness of the model, I apply it to the classical Lucas tree model of asset pricing (Lucas 1978) and draw implications of the RE on the dynamics of asset prices. I find that, compared to history-independent models, the assets are overpriced when the

[1] The RE is documented in the lab experiments of Thaler and Johnson (1990), Ackert et al. (2006), Harrison (2007), and Peng et al. (2013), and in the field studies of Massa and Simonov (2005), Kaustia and Knüpfer (2008), Liu et al. (2010), Malmendier and Nagel (2011), Guiso et al. (2013), and Knüpfer et al. (2014). I carefully discuss three well-known examples of the RE in Section 1.1.

economy is in a good state, but they are underpriced when the economy is in a bad state. Moreover, I relate the predictions of my model to empirical facts on asset prices. Specifically, my model generates high, volatile, and predictable asset returns, consistent with empirical evidence.

Let me illustrate the key intuition behind the main result. Suppose there are two periods, today and tomorrow. Today an agent compares two lotteries, a dominant lottery and a dominated lottery. The dominant lottery returns high payoffs with high probability, while the dominated lottery returns high payoffs with low probability. Then *monotonicity* requires that the dominant lottery must be preferred to the dominated one. Tomorrow the agent receives a lottery after choosing one of today's two lotteries. *Dynamic monotonicity*, an extension of monotonicity to the dynamic environment, requires that the dominant lottery is still preferred to the dominated one, independent of tomorrow's lottery.

Suppose the agent's risk preference is history-dependent; that is, the utility of tomorrow's lottery depends on today's payoffs. The RE says that the agent is less risk averse after a good history (high payoff) than after a bad history (low payoff). Since a less risk-averse agent values risky lotteries more than a risk-averse agent does, the RE is equivalent to requiring that tomorrow's lottery is more valuable after a good history than after a bad history.

Now suppose the RE is violated. This implies that tomorrow's lottery is less valuable after a good history than after a bad history. Since the dominant lottery returns high payoffs with high probability, it generates good histories more often than the dominated one does. In other words, tomorrow's lottery is less valuable after the dominant lottery than after the dominated one because good histories make tomorrow's less valuable.

Therefore, when the RE is violated, there is a tradeoff between today and tomorrow: although the dominant lottery is preferred to the dominated one today (the dominant lottery's advantage today), it makes tomorrow's lottery less valuable (disadvantage tomorrow). Therefore, if the dominant lottery's disadvantage tomorrow exceeds its advantage today, then dynamic monotonicity is violated. It turns out that, when tomorrow's lottery is significantly more valuable than today's two lotteries, the dominant lottery's disadvantage tomorrow can be greater than its advantage today.

1.1 Examples of the Reinforcement Effect

The first example of the RE comes from the experimental study of Thaler and Johnson (1990). The authors run the following experiment, which involves two choice scenarios. In the first scenario, the subjects are asked to choose between a risky lottery $(q, \$y_1, 1-q, \$y_2)$

and a sure outcome $\$x$, right after winning $\$z$ from a lottery Z.[2] In the second scenario, the subjects are asked to choose between a risky lottery $(q, \$y_1 + z, 1 - q, \$y_2 + z)$ and a sure outcome $\$x + z$. According to expected utility theory and considering the final wealth of each situation, there is no difference between the two scenarios. However, Thaler and Johnson found that when $z = \$15$ (a good history), 77% of the subjects prefer the risky option in the first scenario, but only 44% of subjects prefer the risky option in the second scenario.[3] By contrast, when $z = -\$4.50$ (a bad history), only 32% of the subjects prefer the risky option in the first scenario, but 57% of subjects prefer the risky option in the second scenario.[4] Therefore, the subjects become less risk-averse after $z = \$15$ (a good history) than after $z = -\$4.50$ (a bad history).

In the second example, using data from the Survey of Consumer Finances from 1960-2007, Malmendier and Nagel (2011) show that individuals' experiences of macroeconomic shocks affect their financial risk taking, consistent with the RE. The authors find that individuals who have experienced low stock market returns throughout their lives report lower willingness to take financial risks, are less likely to participate in the stock market, and invest a lower fraction of their liquid assets in stocks if they participate. These results are robust to controlling for age, year effects, and household characteristics such as wealth, income, and education.

The third example of the RE is from the empirical finance literature. It is well known that some market variables move countercyclically in a way that is consistent with the RE (see Cochrane 2011). For example, consider the stock market Sharpe ratio (sometimes called the "price of risk") – the expected excess return of an asset divided by the standard deviation of return. The Sharpe ratio is an important indicator for risk aversion because the first-order condition of intertemporal utility maximization gives that

$$\text{Sharpe ratio} = \text{degree of risk aversion} \times \text{std.dev}(\Delta c) \times cov(\Delta c; R),$$

where Δc is consumption growth and R is asset return. Empirical evidence (e.g., Tang and Whitelaw 2011) suggests that the Sharpe ratio is countercyclical: when the economy is good, the Sharpe ratio is low and when the economy is bad, the Sharpe ratio is high.

[2] The vector $(q, \$y_1, 1-q, \$y_2)$ is a lottery that gives $\$y_1$ with probability q and $\$y_2$ with probability $1-q$.

[3] Thaler and Johnson phrase their questions in the following way (see p. 652): "You won x, now choose between a gamble A and a sure outcome B." In the followup experiment by Peng et al. (2013), they phrase their questions in the following two ways (p. 154): i) "You won x from from a gamble X, now choose between a gamble A and a sure outcome B" or ii) "You will get allowance x, now choose between gamble A and a sure outcome B." It turns out that, significantly more subjects choose the gamble A in i) compared to that in ii). Therefore, it is important that the subjects know that they won x by chance.

[4] In the case of $z = \$15$, Thaler and Johnson use numbers $x = \$0$, $q = 0.5$, $y_1 = \$4.5$, and $y_2 = -\$4.5$, and in the case of $z = -\$4.50$, they use numbers $x = \$5$, $q = 0.33$, $y_1 = \$15$, and $y_2 = \$0$.

Therefore, if the standard deviation of consumption growth std.dev(Δc) and the covariance between consumption growth and asset return $cov(\Delta c; R)$ do not vary much over time, the countercyclicity of the Sharpe ratio suggests the presence of the RE. That is, when the economy is good, the Sharpe ratio is low, which implies low risk aversion, but when the economy is bad, the Sharpe ratio is high, which implies high risk aversion. Probably for that reason, some of the most successful models of asset pricing (e.g., Campbell and Cochrane 1999 and Barberis et al. 2001) use countercyclical risk aversion, which is consistent with the RE.[5]

The three examples above illustrate that the RE is a robust and economically relevant notion. My explanation of the RE depends on changing preferences for risk, but it is also possible to explain the RE through changes in wealth or beliefs. The first example rules out these two explanations of the RE. First, it unambiguously rules out the wealth effect explanation of the RE (wealthy people are less risk-averse than poor people) since the two scenarios in the experiment generate the same final wealth, but the subjects exhibit the RE. Second, it rules out a belief-based explanation (optimistic people are less risk-averse than pessimistic people) since in the experiment, subjects are asked to compare objective lotteries.[6] The second example illustrates the robustness of the RE to different measures of risk aversion, i.e., willingness to take financial risk, stock market participation, and so on. The third example illustrates that the RE might have market level implications.

1.2 Related Literature

Three well-known classes of models generate history-dependent behavior: the Kreps-Porteus model (Kreps and Porteus 1978 and Selden 1978), the Epstein-Zin model (Epstein and Zin 1989 and Weil 1989), and the habit-formation model (Pollak 1970, Constantinides 1990, and Campbell and Cochrane 1999). In all of these models, an agent's current preference is affected by past outcomes or consumption. The main difference between my model and the above three models is that I allow the agent's current preference to be affected by past outcomes and their distributions. Therefore, my model allows the following history-dependent behavior: an agent becomes more risk averse after winning \$10 from a lottery $(\frac{1}{2}, \$10, \frac{1}{2}, \$20)$ and becomes less risk averse after winning \$10 from a lottery $(\frac{1}{2}, \$10, \frac{1}{2}, \$0)$. Moreover, an additive version of the Kreps-Porteus model is a special case of my model (see Section 3.3).

[5] In fact, empirical finance research directly finds countercyclical risk aversion using an estimator for time-varying risk aversion (e.g., Kim 2014).

[6] Although I consider objective probabilities, in some special cases, the way my model explains the RE is similar to the belief-based explanation. See Section 6.4 for more details.

The closest paper is Dillenberger and Rozen (2015), which models choice over multi-stage compound lotteries and studies the RE. A two-stage compound lottery is a lottery over simple lotteries, and a t-stage compound lottery is a lottery over $(t-1)$-stage compound lotteries. In Dillenberger and Rozen (2015), risk preferences are affected by a realized $(t-1)$-stage compound lottery as well as unrealized $(t-1)$-stage compound lotteries.[7] Since each multi-stage compound lottery corresponds to a distribution over final outcomes in the future, one key difference is that in their model, risk preferences are affected by past distributions over future outcomes, while in my model, risk preferences are affected by past risky choices and their outcomes. In other words, in their model, agents care about "what might have been" in the future, but in my model, agents care about "what might have been" in the past. Interestingly, Dillenberger and Rozen prove that the RE is a result of internal consistency of changes in risk preferences[8], while I prove that the RE is a result of dynamic monotonicity. Thus, we focus on two different channels that may give rise to the RE. Arguably, the first two examples of the RE indicate that the RE is caused by past risky choices rather than future outcomes. Moreover, their model violates monotonicity while my model is built around monotonicity.[9] Dillenberger and Rozen also apply their model to asset pricing and find volatile and history-dependent prices.

The remainder of the paper is organized as follows. First, I outline the main result and the application to asset pricing in Section 2. In Section 3, I introduce the model and state the main result. In Section 4, I then apply my model to the classical Lucas tree model of asset pricing and draw implications of the RE on the dynamics of asset prices. Two different behavioral foundations for my model are provided in Section 5. In Section 6, I introduce three special cases of my model (including dynamic versions of expected utility and disappointment aversion theory) and discuss the RE. The proofs are collected in Appendix A. A more general version of the main result is provided in Appendix B. Behavioral foundations for dynamic versions of expected utility and disappointment aversion are provided in Appendix C.

[7]Note that any compound lottery returns a single outcome at the end and so it does not allow intermediate consumptions.

[8]In their model, internal inconsistency may occur because the evaluations of final outcomes affect the current risk preference, while the current risk preference also affects the evaluations of final outcomes. Dillenberger and Rozen (2015) show that internal consistency also implies a behavior called the *primacy effect*: an agent who experiences a bad history today and good histories forever is more risk averse than an agent who experiences a good history today and bad histories forever. However, the primacy effect seems not consistent with the following findings of Malmendier and Nagel (2011): most recent experiences have a stronger effect on risk preferences than those in early in life.

[9]See p. 461 of Dillenberger and Rozen 2015, where they explain why monotonicity is violated in their model.

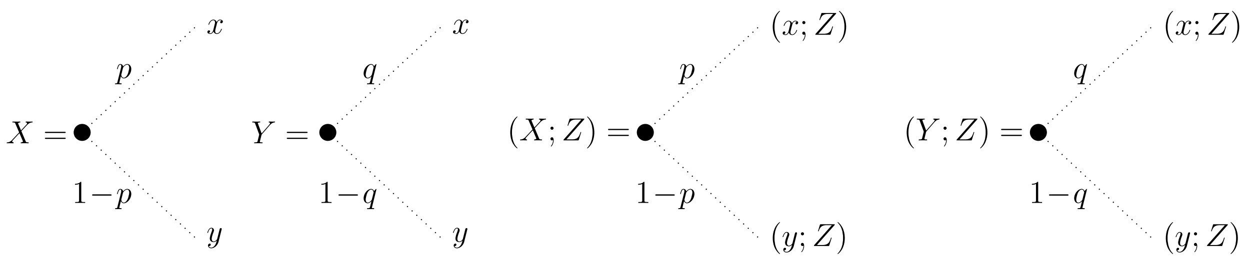

Figure 1: Lotteries X and Y and bundles $(X;Z)$ and $(Y;Z)$

2 Overview of the Main Result and Its Applications

2.1 Main Result

In this subsection, I introduce a special case of my model and illustrate the main result in this simple case. Suppose the agent receives a bundle $(X;Z)$, which gives a lottery X today and a lottery Z tomorrow. Suppose the lottery X gives \$$x$ with probability p and \$$y$ with probability $1-p$, denoted by $X = (p, x, 1-p, y)$, and the lottery Z gives \$$z$ with probability r and nothing with probability $1-r$, denoted by $Z = (r, z, 1-r, 0)$. See Figure 1.

Suppose the agent is an expected utility maximizer, and her risk preference is history-dependent. Suppose that the agent has a CRRA utility function $u(t) = t^\gamma$ where $1-\gamma$ is the agent's degree of risk aversion. Tomorrow the agent's risk attitude changes with today's outcome. In particular, the agent's degree of aversion $1-\gamma$ changes to $1-\gamma(x)$ when the outcome x is realized from the lottery X, and the certainty equivalent of Z is $\mu_x(Z) = \big(r\, z^{\gamma(x)} + (1-r)\, 0\big)^{\frac{1}{\gamma(x)}}$. Therefore, in my model, the utility of the bundle $(X;Z)$ is

$$p\, x^\gamma + (1-p)\, y^\gamma + \beta \left(p \left(\mu_x(Z)\right)^\gamma + (1-p) \left(\mu_y(Z)\right)^\gamma \right),$$

where β is the discount factor. Note that the utility of a sure bundle $(x;z)$ is $x^\gamma + \beta\, z^\gamma$.

The key assumption will be a form of monotonicity of risk preferences. Take a lottery $Y = (q, x, 1-q, y)$ with $x > y$ and $q < p$. *Monotonicity* requires that the lottery X must preferred to the lottery Y because X first-order stochastically dominates Y. I define the following extension of monotonicity to the dynamic environment called *dynamic monotonicity*: the bundle $(X;Z)$ must be preferred to the bundle $(Y;Z)$. Intuitively, since the two bundles provide a common lottery Z tomorrow and X first-order stochastically dominates Y, the first bundle must be preferred to the second. In my model, dynamic monotonicity is equivalent to the following inequality:

$$x^\gamma - y^\gamma \geq \beta \left(\left(\mu_y(Z)\right)^\gamma - \left(\mu_x(Z)\right)^\gamma \right) = \beta \left(r^{\frac{\gamma}{\gamma(y)}} - r^{\frac{\gamma}{\gamma(x)}} \right) z^\gamma. \tag{1}$$

The RE states that the agent is less risk-averse after a good history (when x is realized) than after a bad history (when y is realized); i.e., $\gamma(x) \geq \gamma(y)$.[10] Equivalently, $\mu_x(Z) \geq \mu_y(Z)$. Therefore, if the RE is violated, when z is large enough, then the RHS of (1) exceeds the LHS of (1); i.e., dynamic monotonicity is violated.

As I show later, the above argument is true under an assumption called *nontriviality*. It requires that if the agent's risk preference is history-dependent; that is, $\mu_x(Z) \neq \mu_y(Z)$, then the utilities $u(\mu_x(Z))$ and $u(\mu_y(Z))$ must be significantly different as z increases. In three special cases of my model discussed in Section 6, nontriviality is equivalent to the condition that the agent's Bernoulli utility function u is unbounded. In the above example, nontriviality is satisfied since $u(z) = z^\gamma$ is unbounded.

I prove the result in a general case in Section 3.2. In particular, the certainty equivalents $\mu_x(Z)$ and $\mu_y(Z)$ of Z not only depend on the outcomes x and y, but also depend on lotteries X and Y. Moreover, the agent's risk preference can be a non-expected utility preference such as a rank-dependent utility preference or disappointment aversion theory preference.

2.2 Application to Asset Pricing

I apply my model to the Lucas tree model of asset pricing (Lucas 1978) and show that the RE has important implications for asset prices. In Section 4, I show that I can obtain predictable, high, and volatile asset returns, consistent with empirical data.[11] I use a simple model of an economy that follows a two-state Markov process in which the states, high and low, are persistent. The price of an asset increases when agents become less risk-averse, since less risk-averse agents value the asset more than risk-averse agents. Therefore, the first implication of the RE is as follows: the asset is overpriced in a high state, but it is underpriced in a low state, compared to models with history-independent risk aversion. In the LHS of Figure 2, the price-dividend ratio is plotted against the degree of risk aversion. The blue (dotted) lines illustrate the price-dividend ratio in my history-dependent model (discounted expected utility). Note that in a high state, the price-dividend ratio is above the dotted line; i.e., assets are overpriced, while in a low state, the price-dividend ratio is below the dotted line; i.e., assets are underpriced.

[10] Note that in this definition, the RE does not rule out the standard case $\gamma = \gamma(x) = \gamma(y)$.

[11] See Campbell (2003) for a survey of important stylized facts of aggregate stock prices.

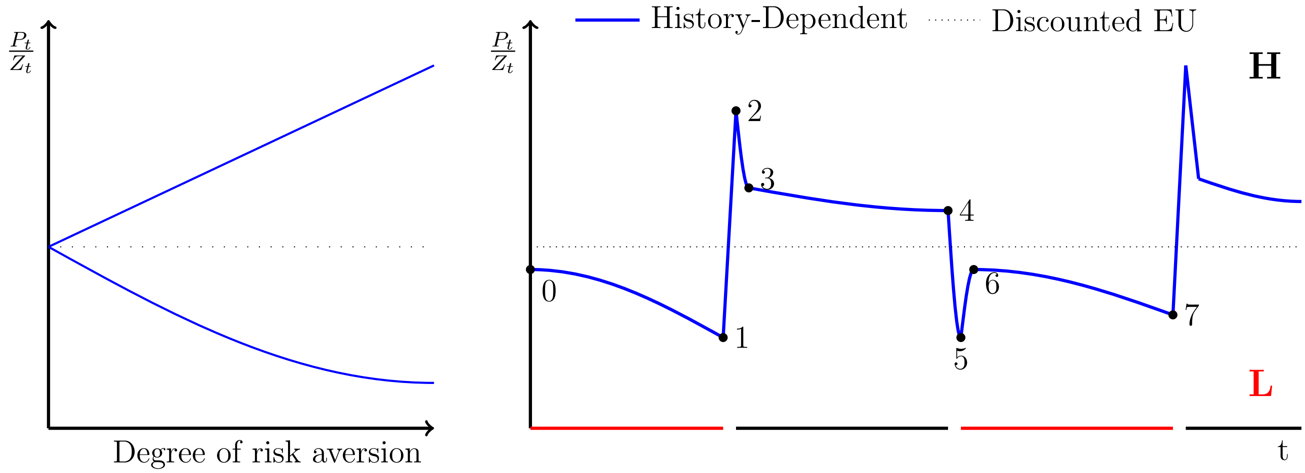

Figure 2: The Price-Dividend Ratio

The second implication of the RE is that there is asymmetry between high and low states. This implication relies on the following property of the model: the degree of mispricing (overpricing and underpricing) increases as agents become more risk-averse. Therefore, as illustrated in the LHS of Figure 2 in high states, the price-dividend ratio increases as agents become more risk-averse because the degree of overpricing increases. However, in low states, the price-dividend ratio decreases as agents become more risk-averse because the degree of underpricing increases.

I now discuss the dynamics of asset prices using the RHS of Figure 2. Since states are persistent, low states continue for a while; after that, high states continue for a while, and so on. Suppose the economy is in a low state first. By the second implication of the RE, in a low state, asset prices decrease as agents become more risk-averse (Point 0 to Point 1). But, when the economy recovers, asset prices overshoot because underpricing turns to overpricing as the state changes (Point 1 to Point 2). Moreover, the overshooting is large because the degrees of risk aversion and overpricing are very high after a long period of low states.

Now the economy is in a high state. By the second implication of the RE, asset prices decrease because the degrees of risk aversion and overpricing decrease (Point 2 to Point 4). Moreover, when the economy declines, asset prices drop because overpricing turns to underpricing as the state changes (Point 4 to Point 5). However, the drop is not as large as the overshooting since the degree of risk aversion decreased after a long period of high states (between Point 3 to Point 4).

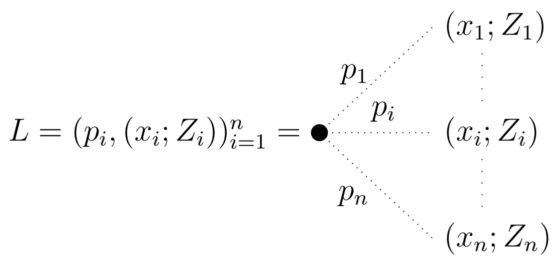

Figure 3: Intertemporal Consumption Lotteries

3 Model

3.1 Basic Setup and Model

I now introduce a model of dynamic choice under risk that generalizes standard discounted expected utility. There are two periods, today and tomorrow.[12] An agent evaluates intertemporal consumption lotteries, which gives a lottery today and another lottery tomorrow depending on the realization of today's lottery. First, I define what a lottery is. Let $\mathbb{R}_+$ be the set of all (monetary) outcomes. For any set $\mathcal{A}$, I define the set of all finite lotteries over $\mathcal{A}$, denoted by $\Delta(\mathcal{A})$. That is,

$$\Delta(\mathcal{A}) \equiv \Big\{A = (p_1, a_1, \dots, p_n, a_n) \in \big([0,1] \times \mathcal{A}\big)^n \text{ with } \sum_{i=1}^{n} p_i = 1 \text{ for some } n\Big\},$$

where a lottery $A = (p_i, a_i)_{i=1}^n$ gives a_i with probability p_i for each i. I call elements of $\Delta(\mathbb{R}_+)$ simple lotteries.

I focus on elements of $\mathscr{L} = \Delta(\mathbb{R}_+ \times \Delta(\mathbb{R}_+))$ called intertemporal consumption lotteries. Figure 3 illustrates an intertemporal consumption lottery $L = (p_i, (x_i; Z_i))_{i=1}^n$, an element of $\mathscr{L} = \Delta(\mathbb{R}_+ \times \Delta(\mathbb{R}_+))$. The lottery L gives an intertemporal consumption bundle $(x_i; Z_i)$ with probability p_i for each i, and the bundle $(x_i; Z_i)$ gives an outcome x_i today and gives a simple lottery Z_i tomorrow. In other words, the agent receives a simple lottery $X = (p_i, x_i)_{i=1}^n$ today and receives another simple lottery Z_i tomorrow after an outcome x_i is realized from the lottery X. I call a pair (x_i, X) a history when $x_i \in \mathit{supp}(X) \equiv \{x_1, \dots, x_n\}$, the support of X. So the agent receives a simple lottery Z_i after the history (x_i, X).

The main focus of this paper is to study how the value of a lottery Z_i is affected by the history (x_i, X). I assume that there exists a utility function $W\colon \mathscr{L} \to \mathbb{R}_+$ for intertemporal consumption lotteries. After specifying W, I analyze and compare risk preferences after

[12]The number of periods is not important. In fact, I use an infinite horizon version of the model for the application to asset pricing in Section 4.

different histories (x, X) and (x', X) with $x > x'$.

The benchmark model is *discounted expected utility* (henceforth, DEU). In DEU, the utility of an intertemporal consumption lottery $(p_i, (x_i; Z_i))_{i=1}^n \in \mathscr{L}$ is

$$W\big((p_i, (x_i; Z_i))_{i=1}^n\big) = \mathbb{E}[u(X)] + \beta\, \mathbb{E}_X\big[\mathbb{E}[u(Z_i)]\big] = \sum_{i=1}^n p_i\big(u(x_i) + \beta\, \mathbb{E}[u(Z_i)]\big) \tag{2}$$

where $u : \mathbb{R}_+ \to \mathbb{R}_+$ is a Bernoulli utility function, $\beta \in (0, 1)$ is a discount factor, $\mathbb{E}$ is the standard expectation operator, and $\mathbb{E}_X$ is the expectation operator with respect to the distribution of X. Note the following four properties of DEU:

1. *Simple Expected Utility:* the agent uses expected utility theory when she evaluates simple lotteries;

2. *History Independence:* tomorrow's risk preference is the same as today's risk preference; i.e, she uses the same Bernoulli utility function u to calculate expected utilities $\mathbb{E}[u(X)]$ and $\mathbb{E}[u(Z_i)]$;

3. *Discounted Utility:* the agent uses discounted utility theory when she aggregates utilities of today and tomorrow; i.e., the utility of $(x; z)$ is $u(x) + \beta\, u(z)$;

4. *Expected Utility Aggregator:* once tomorrow's lotteries are evaluated ($\mathbb{E}[u(Z_i)]$ is calculated for each i), she aggregates them using expected utility theory ($\mathbb{E}_X\big[\mathbb{E}[u(Z_i)]\big] = \sum_{i=1}^n p_i\, \mathbb{E}[u(Z_i)]$).

I generalize DEU by weakening the first two of the above four properties of DEU while retaining the latter two. Indeed, weakening the second property, history independence, is essential for analyzing history-dependent risk preferences. However, I weaken the first property, simple expected utility, not only to demonstrate the generality of the main result, but also to include well-known non-expected utility models such as rank-dependent utility theory (Quiggin 1982 and Tversky and Kahneman 1992) and disappointment aversion theory (Gul 1991). In fact, to demonstrate the usefulness of the model, I apply a dynamic version of disappointment aversion theory to asset pricing in Section 4. The role of the third property, discounted utility, will be discussed in Sections 3.2-3.3.

Therefore, in my model, the agent uses a function $V_0 : \Delta(\mathbb{R}_+) \to \mathbb{R}_+$ to evaluate simple lotteries today, but she uses a history-dependent function $V_{(x,X)} : \Delta(\mathbb{R}_+) \to \mathbb{R}_+$ to evaluate simple lotteries tomorrow after a history (x, X). Once simple lotteries are evaluated, the agent aggregates them using the last two of the above four properties of DEU. Formally, I study the following model.

Definition 1 (History-Dependent Model)**.** A utility W is a *history-dependent model* if there exists a triplet $(V_0, \beta, \{V_{(x,X)}\})$ such that the utility of an intertemporal consumption lottery $(p_i, (x_i; Z_i))_{i=1}^n \in \mathscr{L}$ can be represented as

$$W\big((p_i, (x_i; Z_i))_{i=1}^n\big) = V_0(X) + \beta \sum_{i=1}^n p_i V_{(x_i, X)}(Z_i). \tag{3}$$

Indeed, the history-dependent model (3) reduces to DEU when $V_0(Z) = V_{(x,X)}(Z) = \mathbb{E}[u(Z)]$ for some u. Two different behavioral foundations for the history-dependent model (3) are provided in Section 5. For notational simplicity, when there is no danger of confusion, I also call $\{V_{(x,X)}\}$ a history-dependent model.[13]

In order to discuss a relation between the RE and monotonicity, I impose several assumptions on the history-dependent model (3). I require two continuity properties on $V_{(x,X)}$. First, let me assume that $V_{(x,X)}(Z)$ is continuous in outcomes of Z. I do not assume that $V_{(x,X)}$ is continuous in x because X is a finite lottery. However, I assume the following property of continuity of $V_{(x,X)}$ with respect to X.

Definition 2. (Right-continuity) A history-dependent model $\{V_{(x,X)}\}$ is right-continuous if for any lotteries $\{X^n\}_{n=1}^\infty$ and X^* such that *supp*$(X^n) =$ *supp*(X^*) and X^n first-order stochastically dominates X^* for each n, $X^n \to^w X^*$ implies $\lim_{n\to\infty} V_{(x,X^n)} = V_{(x,X^*)}$.[14]

Roughly speaking, the above assumes that changes in risk preferences caused by a lottery X are not so extreme as long as x is fixed. Indeed, right-continuity is satisfied when $V_{(x,X)}$ is independent of X. Moreover, the following special case of (3), in which $\{V_{(x,X)}\}$ is represented by two step functions, satisfies right-continuity: for each $x \in \mathbb{R}_+$, there are two functions

[13] One might find it conflicting since I assume that risk preferences over compound lotteries are expected utility preferences (expected utility aggregator) while risk preferences over simple lotteries are possibly non-expected utility preferences (V_0). Non-expected utility preferences are motivated by violations of the independence axiom such as the Allais Paradox. However, it turns out that, the compound independence axiom, a counterpart of the independence axiom for compound lotteries, is less likely to be violated compared to the independence axiom (Segal 1990, Luce 1990, and Camerer and Ho 1994). For that reason, I use expected utility aggregator. Moreover, the main result is still true if I relax expected utility aggregator to non-expected utility aggregators such as disappointment aversion aggregator or rank-dependent utility aggregator.

[14] The notation $X^n \to^w X^*$ denotes the weak convergence; that is, $\mathbb{E}f(X^n) \to \mathbb{E}f(X^*)$ for any bounded continuous function f. Moreover, $\lim_{n\to\infty} V_{(x,X^n)} = V_{(x,X^*)}$ means that for any $\delta > 0$, there exists n^* such that for any $n > n^*$, $|V_{(x,X^n)}(Z) - V_{(x,X^*)}(Z)| < \delta$ for any $Z \in \Delta(\mathbb{R}_+)$. I also can assume left-continuity instead of right-continuity. In Appendix B, I prove the main result under a condition weaker than right-continuity.

$\{\mathcal{V}_x^0, \mathcal{V}_x^1\}$ such that

$$(4) \qquad V_{(x,X)} = \begin{cases} \mathcal{V}_x^1 \text{ when } x > \mu_0(X) \\ \mathcal{V}_x^0 \text{ when } x \leq \mu_0(X), \end{cases}$$

where $\mu_0(X) = V_0^{-1}\big(V_0(X)\big)$ is the certainty equivalent of X. In (4), when the outcome x is greater than the certainty equivalent of the lottery X, the agent is content and uses $\mathcal{V}_x^1$, but when the outcome is equal or less than the certainty equivalent, the agent is disappointed and uses $\mathcal{V}_x^0$. In Section 4, I apply a special case of (4) to asset pricing. I can also consider general cases of (4) in which there are more than 2 step functions. Note that such models generalize a model with state-dependent risk aversion, in which states are endogenously determined by histories.

3.2 The Reinforcement Effect and Monotonicity

I now turn to the discussion of the RE and monotonicity. The RE states that the agent becomes less risk-averse after a good history (x, X) than after a bad history (x', X) where $x > x'$. Since I only focus on changes in risk preferences (not on changes in time preferences), I assume that the utility of money does not change over time. In other words, I assume *time consistency*; that is, $V_0(z) = V_{(x,X)}(z)$ for any $z \in \mathbb{R}_+$. Note that time consistency generalizes the third property, discounted utility, of the aforementioned four properties of DEU. To see this, let $u(z) \equiv V_0(z)$ for any $z \in \mathbb{R}_+$. Then in the history-dependent model (3), the utility of a deterministic bundle $(x; z)$ is $u(x) + \beta\, u(z)$.

Under time consistency, comparisons among V_0, $V_{(x,X)}$, and $V_{(x',X)}$ capture changes in risk attitude. Specifically, the RE is equivalent to stating $V_{(x,X)}$ values risky lotteries more than $V_{(x',X)}$ does. Formally,

Definition 3 (Reinforcement Effect)**.** A history-dependent model $\{V_{(x,X)}\}$ exhibits the *reinforcement effect* if for any lottery $X \in \Delta(\mathbb{R}_+)$ and $x, x' \in \mathit{supp}(X)$ with $x > x'$, $V_{(x,X)}(Z) \geq V_{(x',X)}(Z)$ for any $Z \in \Delta(\mathbb{R}_+)$.

The first-order stochastic dominance is a fundamental concept in risky choice. It defines when one lottery is unambiguously better than another lottery. A lottery X first-order stochastically dominates a lottery Y if for any $z \in \mathbb{R}_+$, the probability that the agent receives at least z from X is not smaller than that from Y; i.e.,

$$\sum_{i=1}^{n} p_i\, \mathbb{1}\{x_i \geq z\} \geq \sum_{j=1}^{m} q_j\, \mathbb{1}\{y_j \geq z\}.$$

Monotonicity of risk preferences requires that if X first-order stochastically dominates Y, then X must be preferred to Y. Monotonicity is appealing because it is rarely violated (e.g., see Hey 2001) and its violation may lead to the "Dutch book" argument (e.g., see Machina 1989). Moreover, in expected utility, monotonicity is equivalent to the monotonicity of the agent's Bernoulli utility function. I now extend monotonicity to the dynamic environment.[15]

Definition 4 (Dynamic Monotonicity)**.** For any $X, Y \in \Delta(\mathbb{R}_+)$, if X first-order stochastically dominates Y, then $(X; Z)$ must be preferred to $(Y; Z)$ for any $Z \in \Delta(\mathbb{R}_+)$.

Dynamic monotonicity says that if two intertemporal consumption lotteries $(X; Z)$ and $(Y; Z)$ share a common lottery Z tomorrow, then monotonicity in the static environment must be satisfied. Dynamic monotonicity is a weak version of monotonicity in which intertemporal consumption lotteries give different lotteries after different histories (in line with Segal 1990), but dynamic monotonicity will be enough for my purpose.

Another key assumption is *nontriviality*, which requires that if two functions $V_{(x,X)}$ and $V_{(x',X)}$ are different, then they must be significantly different. Specifically, if two utilities $V_{(x,X)}(Z)$ and $V_{(x',X)}(Z)$ of $Z = (r, z, 1 - r, 0)$ are different, then i) they are still different as z increases (in the sense of the single crossing property) and ii) the absolute difference between these two utilities goes to infinity as $z \to +\infty$. Formally,

Assumption 1 (Nontriviality)**.** Take any lottery $X \in \Delta(\mathbb{R}_+)$ and $x, x' \in supp(X)$. For any $Z = (r_1, z_1, \ldots, r_m, z_m) \in \Delta(\mathbb{R}_+)$ with $z_1 > \ldots > z_m$, if $V_{(x,X)}(Z) \neq V_{(x',X)}(Z)$, then for any $z_1' \in [z_1, +\infty)$,

$$V_{(x,X)}(r_1, z_1', r_2, z_2, \ldots, r_m, z_m) \neq V_{(x',X)}(r_1, z_1', r_2, z_2, \ldots, r_m, z_m) \text{ and}$$

$$\left|V_{(x,X)}(r_1, z_1', r_2, z_2, \ldots, r_m, z_m) - V_{(x',X)}(r_1, z_1', r_2, z_2, \ldots, r_m, z_m)\right| \to +\infty \text{ as } z_1' \to +\infty.$$[16]

I briefly argue that dynamic monotonicity rather than nontriviality is mostly responsible for the RE for two reasons. First, since nontriviality treats $V_{(x,X)}$ and $V_{(x',X)}$ symmetrically, it does not state whether $V_{(x,X)}(Z)$ or $V_{(x',X)}(Z)$ is greater than other. Second, in the three examples of the history-dependent model (3) discussed in Section 6, I show that nontriviality is essentially unrelated to history dependence. Specifically, in these three examples, I show that the first part of nontriviality is automatically true. Moreover, the second part of nontriviality is equivalent to the condition $u(+\infty) = +\infty$, which is unrelated to histories (x, X) and (x', X). I now state the main result.

[15] A necessary and sufficient condition for a general (differentiable) non-expected utility function to satisfy monotonicity is given in Machina (1982).

[16] I choose the highest outcome z_1 to make sure that $(r_1, z_1', r_2, z_2, \ldots, r_m, z_m)$ never become a degenerate lottery.

Theorem 1 (Dynamic Monotonicity Implies the Reinforcement Effect)**.** *If a history-dependent model* $\{V_{(x,X)}\}$ *satisfies right-continuity, dynamic monotonicity, and nontriviality, then it exhibits the reinforcement effect.*

In the introduction, I argued that the RE is well documented in empirical studies. Since dynamic monotonicity is a natural behavior, Theorem 1 provides a justification for why the RE is well observed. The proof of Theorem 1 is in Appendix A. In the proof, in fact, I only use a weak version of dynamic monotonicity, in which X dominates Y in the following obvious manner: $X = \big(p_i, x_i, p_j, x_j, (p_k, x_k)_{k\neq i,j}\big)$ and $Y = \big(p_i - \epsilon, x_i, p_j + \epsilon, x_j, (p_k, x_k)_{k\neq i,j}\big)$ where $x_i > x_j$.

The main idea behind Theorem 1 is that when the RE is violated, bad histories generate higher utilities than good histories do. Hence, since Y generates bad histories more often than X does, the utility of Z is higher after Y than after X. Therefore, if there exists Z such that the advantage of Z of $(Y;Z)$ over Z of $(X;Z)$ exceeds the advantage of X over Y, then dynamic monotonicity is violated. In fact, nontriviality guarantees that such Z exists. Therefore, under nontriviality, dynamic monotonicity implies the RE.

3.3 Time Consistency and Relation to Existing Models

My model (3) satisfies time consistency: $V_0(z) = V_{(x,X)}(z)$ for any $z \in \mathbb{R}_+$. Because of the additive structure of (3), time consistency has the following interpretation: the utility $V_{(x,X)}(Z)$ of Z is history-independent when Z is a deterministic lottery (i.e., $\frac{\partial V_{(x,X)}(z)}{\partial x} = \frac{\partial V_0(z)}{\partial x} = 0$).[17] As I argue in this subsection, time consistency is an important difference between my model and other history-dependent models. In the habit-formation model (e.g., Pollak 1970 and Constantinides 1990), the utility of a deterministic bundle $(x;z)$ is

$$u(x) + \beta\, u(z - \alpha\, x) \text{ for some } \alpha \in (0,1).$$

The habit formation model not only violates time consistency ($\frac{\partial u(z-\alpha x)}{\partial x} \neq 0$), but also deviates from discounted utility.

The Kreps-Porteus model (Kreps and Porteus 1978 and Selden 1978) does not necessarily violate time consistency. In the Kreps-Porteus model, the utility of $(p_i, (x_i; Z_i))_{i=1}^n$ is

$$(5) \qquad \mathbb{E}_X\, V\Big(x_i, u_{x_i}^{-1}\big(\mathbb{E}\, u_{x_i}(Z_i)\big)\Big),$$

where V is a time aggregator (see Koopmans 1960) and $u_{x_i}^{-1}\big(\mathbb{E}\, u_{x_i}(Z_i)\big)$ is the certainty

[17] This separability property is captured by a behavioral axiom called *weak separability between today and tomorrow.* See Section 5 for the behavioral foundations of (3).

equivalent of Z_i after x_i. In fact, my model generalizes the following additive version of the Kreps-Porteus model (5), in which $V(x,z) = u(x) + \beta\, u(z)$: the utility of $(p_i, (x_i; Z_i))_{i=1}^n$ is

$$\mathbb{E}_X \left(u(x_i) + \beta\, u\big(u_{x_i}^{-1}\big(\mathbb{E}\, u_{x_i}(Z_i)\big)\big) \right). \tag{6}$$

Since the additive Kreps-Porteus model (6) is a special case of the history-dependent model (3), I can derive the following corollary of Theorem 1 for (6).

Corollary 1. Suppose dynamic monotonicity and nontriviality are satisfied. If there are utility functions $\{u_x\}_{x\in\mathbb{R}_+}$ that satisfy (6), then for any $x, x' \in \mathbb{R}_+$ with $x > x'$, $u_{x'}$ is more concave than u_x; that is, there exists a concave function $f_{x,x'}$ such that $u_{x'} = f_{x,x'} \circ u_x$.

However, the Kreps-Porteus model (5) does not nest my model (3) because I allow i) u_{x_i} to be dependent on X and ii) risk preferences to violate expected utility.

The Epstein-Zin model (Epstein and Zin 1989 and Weil 1989) extends the Kreps-Porteus model (5) by allowing risk preferences to violate expected utility, but they are history-independent. Hence, for example, the utility of a bundle $(x; Z)$ is $V\big(x, \mu(Z)\big)$ where the certainty equivalent $\mu(Z)$ does not necessarily follow expected utility, but is independent of x. The Epstein-Zin model can generate history-dependent behavior because of the time aggregator V. In other words, if V is additive-separable, then the Epstein-Zin model is history-independent. For that reason, the only intersection between my model (3) and the Epstein-Zin model is DEU. For example, consider the most popular version of the Epstein-Zin model, in which $V(x,z) = (x^\rho + \beta\, z^\rho)^{\frac{\alpha}{\rho}}$ and $\mu(Z) = \big(\mathbb{E}Z^\alpha\big)^{\frac{1}{\alpha}}$ for some α, ρ. That is, the utility of $(p_i, (x_i; Z_i))_{i=1}^n$ is

$$\mathbb{E}_X \big[x_i^\rho + \beta\big(\mathbb{E}[Z_i^\alpha]\big)^{\frac{\rho}{\alpha}}\big]^{\frac{\alpha}{\rho}}. \tag{7}$$

When $\rho = \alpha$, (7) reduces to DEU. However, when $\rho \neq \alpha$, Z_i is not separable from x_i even if Z_i is a deterministic lottery (i.e., $\frac{\partial V(x_i, \mu(Z_i))}{\partial x_i} \neq 0$).[18]

4 Application: The Lucas Tree Model with HDDA Agents

In this section, I study the implications of the RE on the dynamics of asset prices. I consider a special case of my model, a dynamic version of the disappointment aversion theory of Gul

[18] In fact, the Epstein-Zin model (7) violates the aforementioned *weak separability between today and tomorrow* when $\rho \neq \alpha$.

(1991). I apply this model to the classical Lucas tree model of asset pricing.[19] I first describe the economy and preferences.

The Economy: There is one unit of identical agents who live forever. An asset produces a stochastic dividend stream $\{Z_t\}$. There are two states, H and L. At each date $t \in \{1, 2, \ldots\}$, for a given state $s_{t-1} \in \{H, L\}$ and dividend z_{t-1}, Z_t takes the value of z_t^H with probability $\rho(H|s_{t-1})$ and z_t^L with probability $\rho(L|s_{t-1})$ where $z_t^H > z_t^L$. That is,

$$Z_t = (\rho(H|s_{t-1}), z_t^H, \rho(L|s_{t-1}), z_t^L).$$

I assume that states are persistent; i.e., $\rho(H|H) = \rho(L|L) = \rho > \frac{1}{2}$. In each period, after z_t is realized from Z_t, agents trade the consumption good, c_t, and the asset in a competitive spot market at price p_t. For a given state s_t, each agent faces the following budget constraint:

$$c_t^s + p_t\, x_{t+1}^s = (z_t^s + p_t)\, x_t,$$

where x_t^s is the asset demand at date t ($x_0 = 1$).

History-Dependent Disappointment Aversion: The preferences of agents in this Lucas economy are defined by a dynamic version of the disappointment aversion theory of Gul (1991).[20]

The disappointment aversion theory is a one-parameter generalization of expected utility theory. In the disappointment aversion theory, an agent overweights probabilities of small outcomes and underweights probabilities of large outcomes. The degree of such probability distortion is summarized by a single parameter δ_0, the *disappointment parameter*. Namely, the utility of $C = (\rho, c^H, 1-\rho, c^L)$ with $c^H > c^L$ is

$$u(\mu(C|\delta_0)) = \frac{\rho\, u(c^H) + (1-\rho)(1+\delta_0)\, u(c^L)}{\rho + (1-\rho)(1+\delta_0)}, \tag{8}$$

where $\mu(C|\delta_0)$ is the certainty equivalent of C. For a fixed utility function u, the disappointment parameter captures the degree of risk aversion; that is, a higher disappointment parameter implies a higher degree of risk aversion.

In *history-dependent disappointment aversion* (HDDA), the agent's disappointment parameter changes with her experiences. In particular, tomorrow, the agent's initial disappointment parameter δ_0 changes to $\delta(x, X)$ after a history (x, X). Then the utility of C

[19] A dynamic version of expected utility is directly applicable to this environment, but finding a closed-form solution is difficult because the Bernoulli utility function is history-dependent. As I show, finding a closed-form solution is easy in the case of disappointment aversion theory because the Bernoulli utility function is history-independent while the disappointment parameter is history-dependent.

[20] See Section 6 for a more detailed discussion of disappointment aversion theory.

after (x, X) is

$$u(\mu(C|\delta(x,X))) = \frac{\rho\, u(c^H) + (1-\rho)(1+\delta(x,X))\, u(c^L)}{\rho + (1-\rho)(1+\delta(x,X))}. \tag{9}$$

In other words, the agent's distorted probabilities change with her experiences. Since the disappointment parameter captures the degree of risk aversion, the RE is equivalent to the following simple condition:

$$\delta(x, X) \leq \delta(x', X) \text{ when } x > x'.$$

<u>Preferences:</u> In this Lucas economy, each agent has a HDDA preference with an initial disappointment parameter $\delta_0 > 0$. Consider a special case of the representation with step functions (4), in which the disappointment parameter $\{\delta_t\}$ follows the following transition law: for some $\alpha \in (0, 1)$,

$$\delta^s_{t+1} = \begin{cases} \alpha \cdot \delta_t & \text{when } s_t = H \\ \frac{\delta_t}{\alpha} & \text{when } s_t = L. \end{cases}$$

Since I fix the utility function u, δ_t dictates the agents' degree of risk aversion and the transition law captures the RE.

At each date t, for a given (z_{t-1}, s_{t-1}), the disappointment parameter δ_t, and the asset demand x_t, the agents' continuation value of the asset is $V(z_{t-1}, s_{t-1}, \delta_t, x_t)$. So agents solve the following Bellman equation for given $\{Z_t\}, \{p_t\}$, and $\{x_t\}$:

$$\begin{aligned} V(z_{t-1}, s_{t-1}, \delta_t, x_t) \quad &= \quad \max_{C_t}\{u\big(\mu(C_t|\delta_t)\big) + \beta E_{Z_t} V(Z_t, S_t, \delta^s_{t+1}, x^s_{t+1})\} \\ \text{s.t} \quad & \quad c^s_t + p_t\, x^s_{t+1} = (z^s_t + p_t)\, x_t, \end{aligned}$$

where $\mu(C_t|\delta_t)$ is the certainty equivalent of $C_t = (\rho(H|s_{t-1}), c^H_t, \rho(L|s_{t-1}), c^L_t)$ for a given disappointment parameter δ_t. In order to find a closed-form solution, I assume $u(x) = \log(x)$.

To emphasize history dependence and the RE, I compare HDDA with history-independent disappointment aversion (henceforth, HIDA, in which the disappointment parameter is constant) and discounted expected utility (DEU).

4.1 Optimal Consumption Profile

Let me start with the optimal consumption path for given processes $\{Z_t\}$ and $\{p_t\}$. For given s_t, agents solve the following problem:

$$\max_{c_t^s} \pi(s_t)\log(c_t^s) + \beta\,\rho(s_t)\,V(z_t^s, s_t, \delta_{t+1}^s, \frac{(z_t^s + p_t^s)x_t - c_t^s}{p_t^s}),$$

where $\pi(s_t)$ is the distorted probability of $\rho(s_t)$ such that

$$\pi(s_t) = \begin{cases} \frac{\rho(H)}{1+(1-\rho(H))\,\delta_t} & \text{when } s_t = H \\ \frac{\rho(L)(1+\delta_t)}{1+\rho(L)\delta_t} = \frac{(1-\rho(H))(1+\delta_t)}{1+(1-\rho(H))\delta_t} & \text{when } s_t = L, \end{cases}$$

and $V(z_t^s, s_t, \delta_{t+1}^s, \frac{(z_t^s+p_t^s)x_t-c_t^s}{p_t^s})$ is the continuation value of the asset for a given z_t^s and s_t.

Note that when $s_t = H$, the probability $\rho(H)$ is underweighted; that is, $\pi(H) < \rho(H)$, but when $s_t = L$, the probability $\rho(L)$ is overweighted; that is, $\pi(L) > \rho(L)$. Let me denote the size of the probability distortion by

$$\lambda_t^s = \begin{cases} \frac{\rho(H)}{\pi(H)} = 1 + (1-\rho(H))\delta_t & \text{when } s_t = H \\ \frac{\rho(L)}{\pi(L)} = \frac{1+(1-\rho(H))\delta_t}{1+\delta_t} & \text{when } s_t = L. \end{cases}$$

One important feature of HDDA is that agents use distorted probabilities $\pi(s_t)$ when they evaluate the expected utility of today's consumption, but they use objective probabilities $\rho(s_t)$ when they evaluate the expected continuation value of the asset. Therefore, the size of the probability distortion λ_t plays an important role in my analysis. Since δ_t dictates the degree of risk aversion, $\lambda_t^H = 1 + (1-\rho(H))\delta_t$ increases and $\lambda_t^L = \frac{1+(1-\rho(H))\,\delta_t}{1+\delta_t}$ decreases as agents become more risk-averse (as δ_t increases). I can interpret δ_t as the degree of pessimism because as δ_t increases, agents pay more attention to bad states. As δ_t increases, agents become more pessimistic; consequently, they think a high state will occur with very low probability. Moreover, the less likely the high state is, the larger the distortion is; that is, both $\lambda_t^H = 1 + (1-\rho(H))\delta_t$ and $\lambda_t^L = \frac{1+(1-\rho(H))\,\delta_t}{1+\delta_t}$ are decreasing in $\rho(H)$. I now solve the optimal consumption path.

Optimal Consumption: For given $\{Z_t\}$ and $\{p_t\}$, the optimal consumption level at time t is

$$c_t^{*s} = \frac{(p_t^s + z_t^s)x_t}{1 + \frac{\beta}{1-\beta}\cdot\lambda_t^s}.$$

Note that $(p_t^s + z_t^s)x_t$ is the agents' wealth at time t. In DEU, agents consume $\frac{1}{1+\frac{\beta}{1-\beta}}$ fraction of their wealth independently of the current state since $\lambda_t = 1$. However, in HDDA, agents consume $\frac{1}{1+\frac{\beta}{1-\beta}\cdot\lambda_t^s}$ fraction of their wealth. Since $\lambda_t^H > 1 > \lambda_t^L$, I have

$$c_t^H(DEU) = \frac{(p_t^H + z_t^H)x_t}{1+\frac{\beta}{1-\beta}} > c_t^{*H} = \frac{(p_t^H + z_t^H)x_t}{1+\frac{\beta}{1-\beta}\cdot\lambda_t^H} > c_t^{*L} = \frac{(p_t^L + z_t^L)x_t}{1+\frac{\beta}{1-\beta}\cdot\lambda_t^H} > c_t^L(DEU) = \frac{(p_t^L + z_t^L)x_t}{1+\frac{\beta}{1-\beta}}.$$

Therefore, in HDDA, for given $\{p_t\}$, the optimal consumption path is much smoother than that in DEU. The main reason is that the HDDA agents are pessimistic in general and so they will save more in high states.

4.2 Market Clearing and Equilibrium Asset Price

I now solve the equilibrium price $\{p_t\}$. In order for the prices to clear the asset market, at each time t, I must have the conditions:

Market Clearing: $x_t = x^* = 1$ and $C_t = Z_t$.

Now it is simple to find the equilibrium asset prices.

Equilibrium Asset Price: For given process $\{Z_t\}$, the equilibrium asset price at time t is

$$p_t^{*s} = \frac{\beta}{1-\beta}\cdot\lambda_t^s\cdot z_t^s.$$

In DEU, the price-dividend ratio $\frac{p_t}{z_t}$ is constant $\frac{\beta}{1-\beta}$. In HDDA, the price-dividend ratio is $\frac{\beta}{1-\beta}$ times the size of the probability distortion, so

$$p_t^{*H} = \frac{\beta}{1-\beta}\cdot\lambda_t^H\cdot z_t^H > p_t^H(DEU) = \frac{\beta}{1-\beta}\cdot z_t^H > p_t^L(DEU) = \frac{\beta}{1-\beta}\cdot z_t^L > p_t^{*L} = \frac{\beta}{1-\beta}\cdot\lambda_t^L\cdot z_t^L.$$

In HDDA, the asset is mispriced by the size of the probability distortion. In a high state, the asset is overpriced since agents undervalue the expected utility of today's consumption compared to the expected continuation value of the asset. In a low state, the asset is underpriced since agents overvalue the expected utility of today's consumption compared to the expected continuation value of the asset. In Figure 4, the price-dividend ratio $\frac{p_t}{z_t}$ is plotted against δ_t. As I described earlier, in high states the price-dividend ratio in HDDA is greater than the price-dividend ratio in DEU (the dotted line), and in low states the price-dividend ratio in HDDA is smaller than the price-dividend ratio in DEU. In a high state, the asset price can take two forms because $\rho(H)$ is either ρ or $1-\rho$ depending on the previous state s_{t-1}: $p_t^H(L)$ is the price in a high state after a low state and $p_t^H(H)$ is the price in a

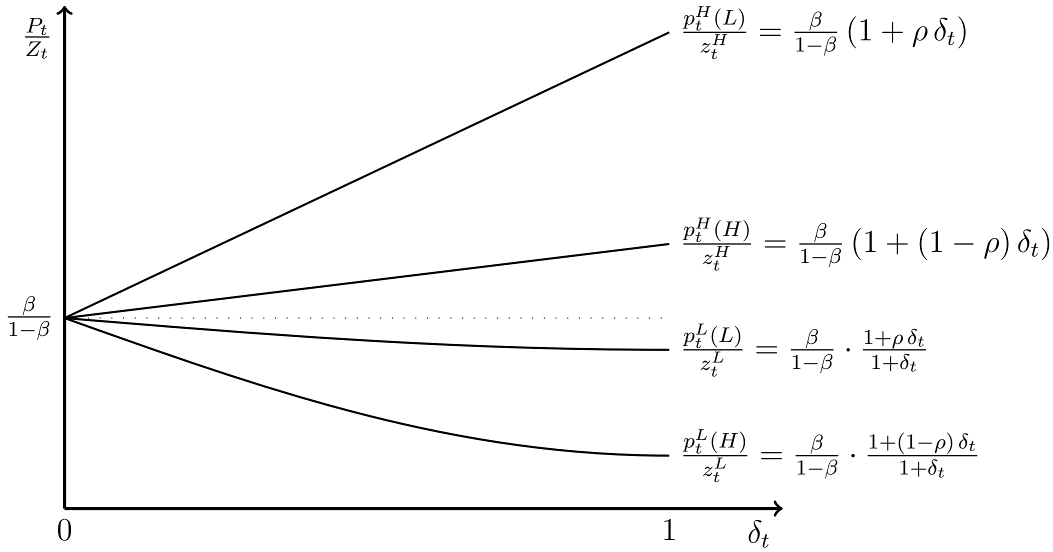

Figure 4: The Price-Dividend Ratio and the Disappointment Parameter

high state after a high state. Similarly, the asset price in a low state can take two forms, $p^L(L)$ and $p^L(H)$. Figure 4 illustrates that the asset prices for high states increase while the asset prices for low states decrease as agents become more risk-averse (as δ_t increases). Now I turn to the dynamics of the price-dividend ratio in HDDA and compare it with HIDA.

4.3 The Dynamics of the Price-Dividend Ratio

Figure 5 illustrates the dynamics of the price-dividend ratio. Since states are persistent, as illustrated in Figure 5, low states continue for a while, then high states continue for a while. Suppose we start with a low state at time τ_0 (Point 0). By the RE, as low states continue to τ_1, agents become more and more risk-averse (δ_t increases). Hence, the price-dividend ratio $\frac{p_t^L(L)}{z_t^L} = \frac{1+\rho\,\delta_t}{\delta_t}$ decreases between τ_0 and τ_1 (Point 0 to Point 1). However, in HIDA illustrated by the orange lines, since the disappointment parameter is constant, the price-dividend ratio is constant between τ_0 and τ_1. Once the economy starts to recover, the asset price overshoots (Point 1 to Point 2). The overshooting in HDDA is larger than in HIDA because agents with HDDA preferences are more risk-averse at τ_1+1 compared to τ_0. At τ_1+2, the asset price will be adjusted down (Point 2 to Point 3) because the transition probability $\rho(H|L)$ switches to $\rho(H|H) > \rho(H|L)$ (in Figure 4, the first line $\frac{p_t^H(L)}{z_t^H}$ switches to the second line $\frac{p_t^H(H)}{z_t^H}$).

Now high states continue for a while, then by the RE, agents become less risk-averse (δ_t decreases). Hence, the price-dividend ratio $\frac{p_t^H(H)}{z_t^L} = 1 + (1-\rho)\,\delta_t$ decreases between τ_1+2 and τ_2 (Point 3 to Point 4). However, in HIDA, the price-dividend ratio is constant between τ_1+2 and τ_2. Once the economy starts to decline, the price-dividend ratio drops below the

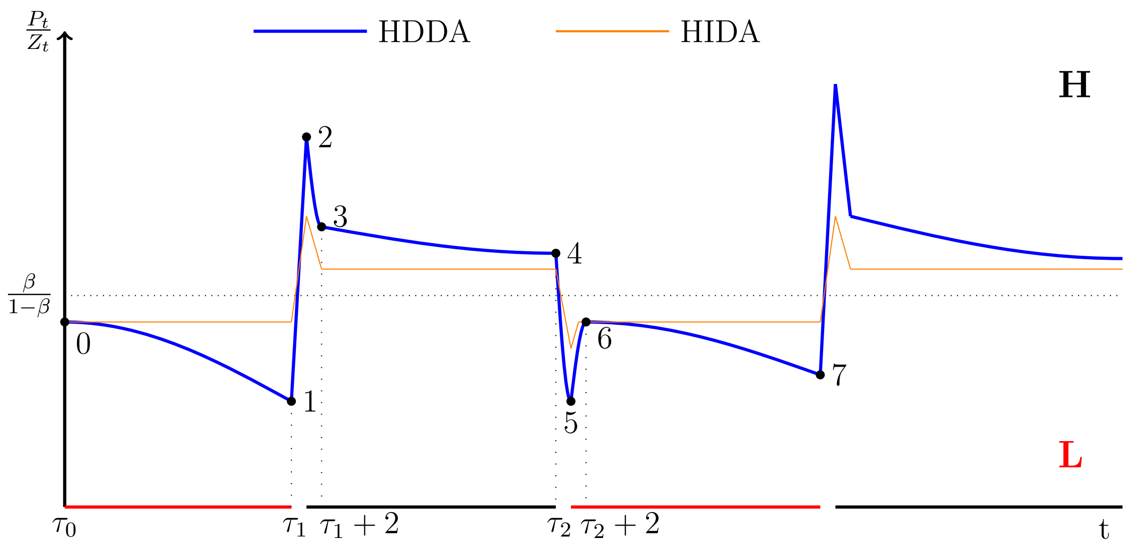

Figure 5: The Dynamics of the Price-Dividend Ratio

dotted line (Point 4 to Point 5). But the drop is not as large as the overshooting at τ_1 because agents are already less risk-averse after a long period of high states. At $\tau_2 + 2$, the price-dividend ratio will be adjusted up (Point 5 to Point 6) because the transition probability $\rho(H|H)$ switches to $\rho(H|L) < \rho(H|H)$ (in Figure 4, the fourth line $\frac{p_t^L(H)}{z_t^L}$ switches to the third line $\frac{p_t^L(L)}{z_t^L}$). So the state is low again.

A sharp prediction that can be spotted easily from Figure 5 is that the price-dividend ratio decreases most of the time to correct the overshooting that happens during state changes.

4.4 Empirical Regularities: High, Volatile, and Predictable Asset Returns

Finally, I relate the predictions of the model to three empirical facts on asset pricing: i) predictability of asset return, ii) high excess return, and iii) time-varying high volatility of asset return (see Campbell 2003 for a survey).

Predictability: The predictability of asset return stands for the predictable dynamics of asset return. In particular, it states that when asset prices are high, subsequent long-horizon real asset returns tend to be low. To check predictability, the following simple regression is usually used:

$$\log(R_{t+1}) = a + b\,\log(\frac{p_t}{z_t}) + \epsilon_{t+1},$$

where $R_{t+1} = \frac{p_{t+1}+z_{t+1}}{p_t}$ is the asset return. The predictability suggests that b must be

Table 1: Simulated Asset Returns

	$\delta_0 = 0.1$	$\delta_0 = 0.15$	$\delta_0 = 0.55$	Empirical value
HDDA				
Mean	5.8%	7.3%		6.98%
Standard deviation	10.2%	16.5%		16.54%
Average DA parameter	0.36	0.55		
HIDA				
Mean	3.8%	3.9%	4.6%	
Standard deviation	2.3%	2.8%	7.2%	
Discounted EU				
Mean	3.8%			
Standard deviation	1.6%			

Empirical values are from Mehra and Prescott (1985) (annualized real returns on S&P500 for the period 1889-1978)

negative since a high $\frac{p_t}{z_t}$ must be followed by a low R_{t+1}. I run simulations with numbers $\beta = 0.98$, $\rho = 0.9$, $\alpha = 0.95$, $\delta_0 = 0.15$ and estimate the same regression. The simulations are illustrated in Figure 6. I find that $\hat{a} = 2.45$ and $\hat{b} = -0.61$ (with standard error 0.11). This regression suggests that a 5% increase in the price-dividend ratio implies a 3% decrease in the next period return.

High and Volatile Returns: Another empirical fact on asset pricing is that asset returns are high and volatile compared to risk-free assets. For example, Mehra and Prescott (1985) found that the historical average of asset return in the U.S. is 6.98% with the standard deviation of 16.54%. The challenge in obtaining a high average return in standard models is that consumption growth is too smooth so unreasonably high risk aversion is required to have a high volatile return. My model can generate a high and volatile return with a reasonable degree of risk aversion.

To emphasize that, I choose the dividend process $\{z_t\}$ (equal to $\{c_t\}$ at the equilibrium) in a way that the first and second moments of the consumption growth match with historical U.S. data. In particular, I choose growth rates $g_h = \frac{z_t^H}{z_{t-1}}$ and $g_L = \frac{z_t^L}{z_{t-1}}$ to satisfy $E(g) = 1.018$ and $\sigma(g) = 0.036$ (Mehra and Prescott 1985). Therefore, I have $g_h = 1.054$ and $g_l = 0.982$. Table 1 shows the average asset return and the standard deviation calculated from the simulated data. For example, for HDDA, I find the average return of 7.3% with a standard deviation of 16.5% when the initial disappointment parameter δ_0 is 0.15. However, for HIDA and DEU, the average returns are only 3.9% and 3.8%, respectively (no significant difference between HIDA and DEU).[21]

[21] I can calculate the "shadow" bond return from the Euler equation and obtain a low average bond return (1.5%) and a high excess return (6%) with reasonable parameters.

In HDDA, when $\delta_0 = 0.15$, the average disappointment parameter is 0.55. Hence, one might think that HIDA can generate high average returns with a high disappointment parameter. So I compare HDDA with $\delta_0 = 0.15$ to HIDA with $\delta_0 = 0.55$. It turns out, in HIDA, the average return is 4.6% with a standard deviation of 7.2%, which are relatively small numbers compared to HDDA. This discussion essentially illustrates the importance of history-dependent δ_t, i.e., the importance of the RE.[22]

The main reason HDDA generates high average returns is that agents undervalue high returns because of the probability distortion, so they demand very high returns. Moreover, when returns are very high, agents happen to be very risk-averse (Point 1 to Point 2 in Figure 5), so they underappreciate high returns.

5 Behavioral Foundations of the History-Dependent Model

In this section, I provide behavioral foundations for the history-dependent model (3) with axioms on the primitive: a preference relation $\succeq$ on the set of intertemporal consumption lotteries $\mathscr{L} = \Delta(\mathbb{R}_+ \times \Delta(\mathbb{R}_+))$. There are two different approaches to characterizing (3). First, since the history-dependent model (3) is a generalization of DEU and a characterization of DEU is standard, I can obtain (3) from DEU by dropping some restrictions of DEU. Second, I can directly characterize (3) by imposing axioms on $\succeq$.

I start with the first approach. For notational simplicity, I write $\big(p_i, (x_i; Z_i); (p_k, (x_k; 0))_{k \neq i}\big)$ as $\big(p_i, (x_i; Z_i); (X_{-i}; 0)\big)$ when $X = (p_k, x_k)_{i=1}^n$. First, recall the four properties of DEU discussed in Section 3.1: i) simple expected utility, ii) history independence, iii) discounted utility, and iv) expected utility aggregator. Then recall that the history-dependent model (3) weakens the first two properties, simple expected utility and history independence. Therefore, I introduce two axioms that capture the above two properties.

First, I state an axiom for history independence, which states that if μ is the certainty equivalent of Z today, then μ is the certainty equivalent of Z after any history (x_i, X). Formally,

[22] The standard deviation in HIDA is higher than the standard deviation in DEU because of the probability distortion. But the probability distortion alone does not help for having a high asset return, so a high disappointment parameter $\delta_0 = 0.55$ does not lead to a high return. For HIDA to obtain the average return 7.3% (which can be obtained by HDDA with $\delta_0 = 0.15$), δ_0 must be more than 1.4, which leads to a relatively high risk aversion and a relatively high probability distortion. Note that $\delta_0 = 0.55$ means that the distorted probability $\frac{0.1\cdot(1+0.55)}{1+0.1\cdot 0.55}$ of 10% is 14.7% while $\delta_0 = 1.4$ means that the distorted probability $\frac{0.1\cdot(1+1.4)}{1+0.1\cdot 1.4}$ of 10% is 21%.

Axiom 1 (Axiom for History Independence). For any $X, Z \in \Delta(\mathbb{R}_+)$ and $\mu \in \mathbb{R}_+$,

$$(Z; 0) \succeq (\mu; 0) \text{ iff } \big(p_i, (x_i; Z), (X_{-i}; 0)\big) \succeq \big(p_i, (x_i; \mu), (X_{-i}; 0)\big) \text{ for any } i.$$

Second, I state an axiom for simple expected utility. Simple expected utility is captured by an axiom on $\succeq$ called *strong independence*, which states that if a lottery X today is equivalent to an outcome z tomorrow and a lottery Y today is equivalent to an outcome t tomorrow; then a mixture $\alpha X + (1-\alpha)Y$ is equivalent to a compound lottery $(\alpha, (0; z), 1 - \alpha, (0; t))$. Formally,

Axiom 2 (Strong Independence). For any $X, Y \in \Delta(\mathbb{R}_+)$, $z, t \in \mathbb{R}_+$, and $\alpha \in (0, 1)$,

$$\text{if } (X; 0) \succeq (0; z) \text{ and } (Y; 0) \succeq (0; t), \text{then } (\alpha X + (1-\alpha)Y; 0) \succeq (\alpha, (0; z), 1 - \alpha, (0; t)).$$

Strong independence is slightly stronger than the independence axiom of expected utility theory. I now can state the first characterization result.

Theorem 2 (Discounted Expected Utility)**.** *A continuous preference relation $\succeq$ on $\mathscr{L}$ is represented by a history-dependent model $\{V_0, \beta, V_{(x,X)}\}$ and satisfies time consistency; that is, $V_0(z) = V_{(x,X)}(z)$ for any $z \in \mathbb{R}_+$, the axiom for history independence, and strong independence if and only if there exists a continuous function $u : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $L = (p_i, (x_i; Z_i))_{i=1}^n, L' = (p'_k, (x'_k; Z'_k))_{k=1}^m \in \mathscr{L}$,*

$$L \succeq L' \textit{ iff } \sum_{i=1}^{n} p_i \left(u(x_i) + \beta\, \mathbb{E}u(Z_i)\right) \geq \sum_{k=1}^{m} p'_k \left(u(x'_k) + \beta\, \mathbb{E}u(Z'_k)\right).$$

Theorem 2 formally shows that the history-dependent model (3) is a result of dropping two properties of DEU, history-independence and simple expected utility.

Now I turn to the second approach: imposing three axioms on $\succeq$. The first axiom is called *regularity*, a collection of four standard postulates.

Axiom 3 (Regularity). A preference relation $\succeq$ on $\mathscr{L}$ satisfies the following four conditions.

1. The preference relation $\succeq$ is complete, transitive, and continuous.

2. (Deterministic Monotonicity) For any $z, z' \in \mathbb{R}_+$ with $z > z'$ and $X \in \Delta(\mathbb{R}_+)$,

$$(z; 0) \succ (z'; 0) \text{ and } \big(p_i, (x_i; z), (X_{-i}; 0)\big) \succ \big(p_i, (x_i; z'), (X_{-i}; 0)\big).$$

3. (Discounted Utility) There exist a utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ and a discount factor $\beta \in (0,1)$ such that for any $(x; z), (x'; z') \in \mathbb{R}^2_+$.

$$(x; z) \succeq (x'; z') \text{ iff } u(x) + \beta\, u(z) \geq u(x') + \beta\, u(z'), \tag{10}$$

4. (Expected Utility Aggregator) There exists $U_2 : \mathbb{R}_+ \to \mathbb{R}$ such that for any $Z = (r_k, z_k)_{k=1}^m, Z' = (r'_k, z'_k)_{k'=1}^{m'} \in \Delta(\mathbb{R}_+)$,

$$(r_k, (0; z_k))_{k=1}^m \succeq (r'_k, (0; z'_k))_{k=1}^{m'} \text{ iff } \mathbb{E}U_2(Z) \geq \mathbb{E}U_2(Z'). \tag{11}$$

The first part of regularity collects completeness, transitivity, and continuity. I also assume a very weak form of monotonicity called *deterministic monotonicity*. The second half of regularity imposes the other two properties of DEU, discounted utility and expected utility aggregator. Specifically, the third part states that the agent uses discounted utility theory when she aggregates utilities of today and tomorrow; i.e., the utility of $(x; z)$ is $u(x) + \beta\, u(z)$ (discounted utility). The fourth part states that compound lotteries are evaluated by expected utility theory; i.e, the utility of a compound lottery $(r_k, (0; z_k))_{k=1}^m$ is $\sum_{k=1}^m r_k U_2(z_k)$ (expected utility aggregator).

The next two axioms are novel axioms. The second axiom (A.4) is called *separability*, which consists of two properties of separability. Separability is essential for studying history-dependent risk aversion because it allows me to define a risk preferences for each history independently of other histories. Separability also implies time consistency.

Axiom 4 (Separability). A preference relation $\succeq$ on $\mathscr{L}$ satisfies the following two properties.

1. (Separability between Parallel Histories) For any $(p_i, (x_i; Z_i))_{i=1}^n \in \mathscr{L}$ and $Y, Y' \in \Delta(\mathbb{R}_+)$,

$$\big(p_i, (x_i; Y), (X_{-i}; 0)\big) \succeq \big(p_i, (x_i; Y'), (X_{-i}; 0)\big)$$
$$\text{iff } \big(p_i, (x_i; Y), (p_k, (x_k; Z_k))_{k\neq i}\big) \succeq \big(p_i, (x_i; Y'), (p_k, (x_k; Z_k))_{k\neq i}\big).$$

2. (Weak Separability between Today and Tomorrow) For any $(p_i, (x_i; z_i))_{i=1}^n \in \Delta(\mathbb{R}_+ \times \mathbb{R}_+)$ and $\mu, z \in \mathbb{R}_+$,
   i) $(X; 0) \succeq (\mu; 0)$ if and only if $(X; z) \succeq (\mu; z)$ and
   ii) $(p_i, (0; z_i))_{i=1}^n \succeq (0; \mu)$ if and only if $(p_i, (x_i; z_i))_{i=1}^n \succeq (X; \mu)$.

Suppose the agent receives either of two simple lotteries Y and Y' after a history (x_i, X). Separability between parallel histories requires that a comparison between the two simple lotteries Y and Y' cannot be affected by what she would receive in histories other than (x_i, X). This axiom is essentially a dynamic version of an axiom called *replacement separability*, introduced in Machina (1989). Weak separability between today and tomorrow essentially requires that the utility of a deterministic outcome z is history-independent (recall time consistency); i.e., the utility of z is not affected by a lottery X and a deterministic outcome μ. Specifically, i) states that if μ is the certainty equivalent of a simple lottery X, then μ is still the certainty equivalent of X even if the agent will receive a deterministic outcome z in the second period. Moreover, ii) states that if μ is the certainty equivalent of a compound lottery $(p_i, (0; z_i))_{i=1}^n$, then $(0; \mu)$ is still the certainty equivalent of $(p_i, (0; z_i))_{i=1}^n$ even if the agent receives a lottery X in the first period.

The third axiom (A.5) is called *additivity*.[23] Let $X = (p, x, 1-p, x')$. Additivity requires that if receiving z after a history (x, X) is equivalent to receiving μ tomorrow and receiving z' after a history (x', X) is equivalent to receiving λ today, then receiving z after (x, X) and z' after (x', X) is equivalent to receiving λ today and μ tomorrow.

Axiom 5 (Additivity). For any $(p_k, (x_k; z_k))_{k=1}^n \in \Delta(\mathbb{R}_+ \times \mathbb{R}_+)$ and $\lambda, \mu \in \mathbb{R}_+$, and for any i, j,

$$\text{if } \big(p_i, (x_i; z_i), (X_{-i}; 0)\big) \sim (X; \mu) \text{ and } \big(p_j, (x_j; z_j), (X_{-i}; 0)\big) \sim (\lambda; 0),$$

$$\text{then } \big(p_i, (x_i; z_i), p_j, (x_j; z_j), (X_{-i,-j}; 0)\big) \sim (\lambda; \mu).$$

Finally, I can state the second characterization theorem.

Theorem 3 (History-Dependent Model). *A preference relation $\succeq$ on $\mathscr{L}$ satisfies regularity, separability, and additivity if and only if there are strictly increasing continuous functions $V_0 : \Delta(\mathbb{R}_+) \to \mathbb{R}_+$ and $V_{(x,X)} : \Delta(\mathbb{R}_+) \to \mathbb{R}_+$ such that for any $L = (p_i, (x_i; Z_i))_{i=1}^n, L' = (p'_k, (x'_k; Z'_k))_{k=1}^m \in \mathscr{L}$,*

$$(12) \qquad L \succeq L' \text{ iff } V_0(X) + \beta \sum_{i=1}^n p_i V_{(x_i, X)}(Z_i) \geq V_0(X') + \beta \sum_{k=1}^m p'_k V_{(x'_k, X')}(Z'_k),$$

and $V_0(z) = V_{(x,X)}(z)$ for each $z \in \mathbb{R}_+$.

I also have a strong uniqueness result.

Proposition 1 (Uniqueness). *Take any preference relation $\succeq$ on $\mathscr{L}$ that satisfies regularity 1-2. If it is represented by triplets $(V_0, \beta, \{V_{(x,X)}\})$ and $(V'_0, \beta', \{V'_{(x,X)}\})$ that satisfy time consistency, $V_0(0) = V'_0(0)$, and $V_0(1) = V'_0(1)$, then the two triplets are identical.*

[23] Additivity is not essential to Theorem 1. I can relax it and Theorem 1 can be modified accordingly.

I conclude this section by illustrating that Theorem 1 can be stated in terms of axioms on $\succeq$ instead of using the history-dependent model $\{V_{(x,X)}\}$. Since dynamic monotonicity is already defined on the primitive $\succeq$, and the history-dependent model (3) is characterized by Theorem 3, it is sufficient to define the RE in terms of conditions on $\succeq$.[24] It turns out, the RE is equivalent to the following condition.

Definition 5 (Reinforcement Effect)**.** For any simple lottery $X \in \Delta(\mathbb{R}_+)$ and $x_i, x_j \in supp(X)$ with $x_i > x_j$, for any $Z \in \Delta(\mathbb{R}_+)$ and $\mu \in \mathbb{R}_+$,

$$\text{if } \big(p_j, (x_j; Z), (X_{-j}; 0)\big) \succeq \big(p_j, (x_j; \mu), (X_{-j}; 0)\big), \text{ then}$$

$$\big(p_i, (x_i; Z), (X_{-i}; 0)\big) \succeq \big(p_i, (x_i; \mu), (X_{-i}; 0)\big).$$

6 Specifying Risk Preferences: Expected Utility, Disappointment Aversion, and Rank-Dependent Utility

So far, I have discussed a relation between risk preferences for different histories, but I have not focused on specific functional forms for V_0 and $V_{(x,X)}$. In this section, I obtain three special cases of the history-dependent model (3) by specifying V_0 and $V_{(x,X)}$. Specifically, I apply the history-dependent model (3) to three well-known models of choice under risk: expected utility, the disappointment aversion theory of Gul (1991), and the rank-dependent utility of Quiggin (1982) (which subsumes the cumulative prospect theory of Tversky and Kahneman 1992). By doing so, I demonstrate the generality of Theorem 1 in the following two ways. First, three special cases demonstrate the generality of the history-dependent model (3). Second, for these three special cases, I show that nontriviality, the main assumption of Theorem 1, is equivalent to the condition $u(+\infty) = +\infty$ where u is a Bernoulli utility function. This illustrates that nontriviality is equivalent to a condition that is unrelated to history dependence; i.e., dynamic monotonicity is almost fully responsible for the RE.

Specifying V_0 and $V_{(x,X)}$ is useful for the following three reasons. First, I can derive a simple condition that is equivalent to the RE in each special case, which allows me to provide further interpretations for the RE. Second, although I consider objective lotteries, I can demonstrate that the way my model explains the RE is similar to a belief-based explanation (Section 6.4). Lastly, these specifications make the history-dependent model (3) more tractable and applicable (demonstrated in Section 4).

[24] I can also state right-continuity and nontriviality in terms of axioms on the primite $\succeq$.

I call the three special cases the *history-dependent expected utility* (HDEU), *history-dependent disappointment aversion* (HDDA), and *history-dependent rank-dependent utility* (HDRDU), respectively. Behavioral foundations for HDEU and HDDA are provided in Appendix C. Let me start with the definition of HDEU.

6.1 History-Dependent Expected Utility (HDEU)

Here I introduce a dynamic version of expected utility theory in which the concavity of an agent's Bernoulli utility function changes with her experiences. More specifically, today, the agent uses expected utility theory with some Bernoulli utility function u; i.e., the utility of a simple lottery X is

$$V_0(X) = \mathbb{E}[u(X)] = \sum_{i=1}^{n} p_i\, u(x_i).$$

Tomorrow, u becomes $u^{\gamma(x,X)}$ after a history (x, X) for some positive real number $\gamma(x, X)$. In other words, the utility of Z after the history (x, X) is

$$V_{(x,X)}(Z) = \left(\mathbb{E}[(u(Z))^{\gamma(x,X)}]\right)^{\frac{1}{\gamma(x,X)}} = \left(\sum_{k=1}^{m} r_k \left(u(z_k)\right)^{\gamma(x,X)}\right)^{\frac{1}{\gamma(x,X)}}.$$

Therefore, in the *history-dependent expected utility* (HDEU), the utility of an intertemporal consumption lottery $(p_i, (x_i; Z_i))_{i=1}^n \in \mathscr{L}$ is

$$W\left((p_i, (x_i; Z_i))_{i=1}^n\right) = \mathbb{E}[u(X)] + \beta \sum_{i=1}^{n} p_i \left(\mathbb{E}[(u(Z_i))^{\gamma(x_i,X)}]\right)^{\frac{1}{\gamma(x_i,X)}}. \tag{13}$$

Since $\gamma(x, X)$ dictates the concavity of the utility function $u^{\gamma(x,X)}$, the RE is equivalent to the condition:

$$\gamma(x, X) \geq \gamma(x', X) \text{ when } x > x'.$$

6.2 History-Dependent Disappointment Aversion (HDDA)

I now introduce a dynamic version of the disappointment aversion theory of Gul (1991). Disappointment aversion theory is a one-parameter generalization of expected utility theory. In disappointment aversion theory, an agent overweights the probabilities of small outcomes and underweights the probabilities of large outcomes. The degree of such probability distortion is summarized by a single parameter δ_0, a *disappointment parameter*. In particular, for a given lottery X, the agent overweights the probabilities of outcomes that are not larger than the certainty equivalent of X by $1 + \delta_0$. Then the certainty equivalent of the lottery

X, denoted by $\mu(X|\delta_0)$, is a unique solution of the following implicit formula where μ is variable:

(14) $$u(\mu) = \frac{\sum_{i=1}^{n} p_i \left(1 + \delta_0 \, \mathbb{1}\{x_i \leq \mu\}\right) u(x_i)}{\sum_{i=1}^{n} p_i \left(1 + \delta_0 \, \mathbb{1}\{x_i \leq \mu\}\right)}.$$ [25]

Note that the disappointment parameter dictates the degree of risk aversion (for a fixed u); that is, a higher δ_0 implies a higher degree of risk aversion.

In *history-dependent disappointment aversion* (HDDA), the agent's disappointment parameters change with her experiences. In particular, tomorrow, the agent's initial disappointment parameter δ_0 changes to $\delta(x, X)$ after a history (x, X). Then the certainty equivalent of a lottery Z after (x, X), denoted by $\mu(Z|\delta(x, X))$, is a unique solution to

$$u\big(\mu\big) = \frac{\sum_{k=1}^{m} r_k \left(1 + \delta(x, X) \, \mathbb{1}\{z_k \leq \mu\}\right) u(z_k)}{\sum_{k=1}^{m} r_k \left(1 + \delta(x, X) \, \mathbb{1}\{z_k \leq \mu\}\right)}.$$

Therefore, in HDDA, the utility of $(p_i, (x_i; Z_i))_{i=1}^{n} \in \mathscr{L}$ is

(15) $$u(\mu(X|\delta_0)) + \beta \sum_{i=1}^{n} p_i \, u\big(\mu(Z_i|\delta(x_i, X))\big).$$

Since the disappointment parameter dictates the degree of risk aversion, the RE is equivalent to the following simple condition:

$$\delta(x, X) \leq \delta(x', X) \text{ when } x > x'.$$

6.3 History-Dependent Rank-Dependent Utility (HDRDU)

I then introduce a dynamic version of rank-dependent utility in which distorted probabilities change with histories. The rank dependent utility of Quiggin (1982) is a modification of the prospect theory of Kahneman and Tversky (1979) that satisfies monotonicity in the static environment.

In rank-dependent utility, the agent distorts probabilities by some function π and the utility of X is

$$V_0(X) = \sum_{i=1}^{n} \big(\pi(P_i) - \pi(P_{i+1})\big) \, u(x_i),$$

where $x_1 > \ldots > x_n$ and $P_i = \sum_{k=i}^{n} p_k$. In the *history-dependent rank-dependent utility*, π

[25] Note that (8) is a special case of this formula.

becomes $\pi^{\gamma(x,\,X)}$ after a history (x, X). In other words, the utility of Z after (x, X) is

$$V_{(x,\,X)}(Z) = \sum_{k=1}^{m} \left[\left(\pi(R_k)\right)^{\gamma(x,\,X)} - \left(\pi(R_{k+1})\right)^{\gamma(x,\,X)}\right] u(z_k),$$

where $z_1 > \ldots > z_m$ and $R_k = \sum_{s=k}^{m} r_s$. Therefore, in HDRDU, the utility of an intertemporal consumption lottery $(p_i, (x_i; Z_i))_{i=1}^{n} \in \mathscr{L}$ is

$$(16)\quad \sum_{i=1}^{n}\left(\pi(P_i)-\pi(P_{i+1})\right)u(x_i) + \beta \sum_{i=1}^{n} p_i \left(\sum_{k=1}^{m_i}\left[\left(\pi(R_{i,k})\right)^{\gamma(x_i,\,X)} - \left(\pi(R_{i,k+1})\right)^{\gamma(x_i,\,X)}\right] u(z_{i,k})\right),$$

where $z_{i,1} > \ldots > z_{i,m_i}$ and $R_{i,k} = \sum_{s=k}^{m_i} r_{i,s}$ for each i.

It turns out that, the concavity of the distortion function dictates the degree of risk-aversion. Therefore, in HDRDU, the RE is equivalent to the following simple condition:

$$\gamma(x,\, X) \geq \gamma(x',\, X) \text{ when } x > x'.$$

6.4 Probability Distortion, Belief Change, and Nontriviality

My model provides the preference-based explanation of the RE; that is, an agent becomes less risk-averse after a good history than after a bad history. However, the belief-based explanation is another obvious possibility: the agent becomes less pessimistic after a good history and acts as if she is less risk-averse because she thinks probabilities of high outcomes are higher than she previously thought. I have two remarks on the belief-based explanation. First, the belief-based explanation is not consistent with the experiment of Thaler and Johnson (1990) because in their experiment, the subjects are asked to compare objective lotteries. Second, the way the above three special cases of the history-dependent model explain the RE is very similar to the belief-based explanation. To illustrate, consider HDDA, in which the utility of $Z = (r, z, 1-r, 0)$ after (x, X) is

$$V_{(x,\,X)}(Z) = \frac{r}{1 + (1-r)\,\delta(x, X)}\, u(z).$$

Note that the distorted probability $\frac{r}{1+(1-r)\,\delta(x,X)}$ can be interpreted as the agent's subjective belief. Therefore, the agent exhibits the RE because she acts as if her belief changes with her experiences. In particular, when $\delta(x, X) \leq \delta(x', X)$, I have

$$\frac{r}{1 + (1-r)\,\delta(x, X)} \geq \frac{r}{1 + (1-r)\,\delta(x', X)}.$$

Therefore, the agent acts as if she becomes less pessimistic after a good history (x, X) than a bad history (x', X). Similar to HDDA, in HDRDU, the utility of $Z = (r, z, 1-r, 0)$ after (x, X) is

$$V_{(x,\,X)}(Z) = \left[1 - (\pi(1-r))^{\gamma(x,\,X)}\right] u(z)$$

and the distorted probability $1 - (\pi(1-r))^{\gamma(x,\,X)}$ can be interpreted as the agent's subjective belief. In HDEU, the utility of $Z = (r, z, 1-r, 0)$ after (x, X) is

$$V_{(x,\,X)}(Z) = r^{\frac{1}{\gamma(x,\,X)}}\, u(z),$$

and $r^{\frac{1}{\gamma(x,\,X)}}$ can be interpreted as the agent's subjective belief.[26]

Now it is easy see that why nontriviality is equivalent to the condition $u(+\infty) = +\infty$.[27] Let $\pi_{(x,\,X)}(r)$ be a distorted version of probability r after a history (x, X), which can be obtained in all three cases. Then nontriviality is equivalent to

$$\begin{aligned}&|V_{(x,\,X)}(r_1, z_1', r_2, z_2, \ldots, r_n, z_n) - V_{(x',\,X)}(r_1, z_1', r_2, z_2, \ldots, r_n, z_n)|\\ &\approx |\pi_{(x,\,X)}(r_1) - \pi_{(x',\,X)}(r_1)|\, u(z_1') \to +\infty \text{ as } z_1' \to +\infty.\end{aligned}$$

7 Appendix A: Proofs

7.1 Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1. Take any lottery X and its two outcomes x_i, x_j with $x_i > x_j$. I shall prove that $V_{(x_i,\,X)}(Z) \geq V_{(x_j,\,X)}(Z)$ for any $Z \in \Delta(\mathbb{R}_+)$. Let me consider the following special case of dynamic monotonicity.

First define the following new lotteries: for any $\epsilon \in (0, p_j)$, $X_\epsilon \equiv (p_i + \epsilon, x_i, p_j - \epsilon, x_j, X_{-i,-j})$. Since X_ϵ first-order stochastically dominates X, I must have:

Weak Dynamic Monotonicity. $W(X_\epsilon; Z) \geq W(X; Z)$ for any $Z \in \Delta(\mathbb{R}_+)$.

In terms of (3), the above is equivalent to the inequality:

$$V_0(X_\epsilon) + \beta\left((p_i + \epsilon)\, V_{(x_i,\,X_\epsilon)}(Z) + (p_j - \epsilon)\, V_{(x_j,\,X_\epsilon)}(Z) + \sum_{s \neq i,j} p_s\, V_{(x_s,\,X_\epsilon)}(Z)\right) \geq$$

[26]In HDEU, the belief-based interpretation is limited to binary lotteries while in HDDA and HDRDU, the belief-based interpretation works for any lottery Z.

[27]In the three special cases, the first part of nontriviality is automatically true. Therefore, I only discuss the second part of nontriviality.

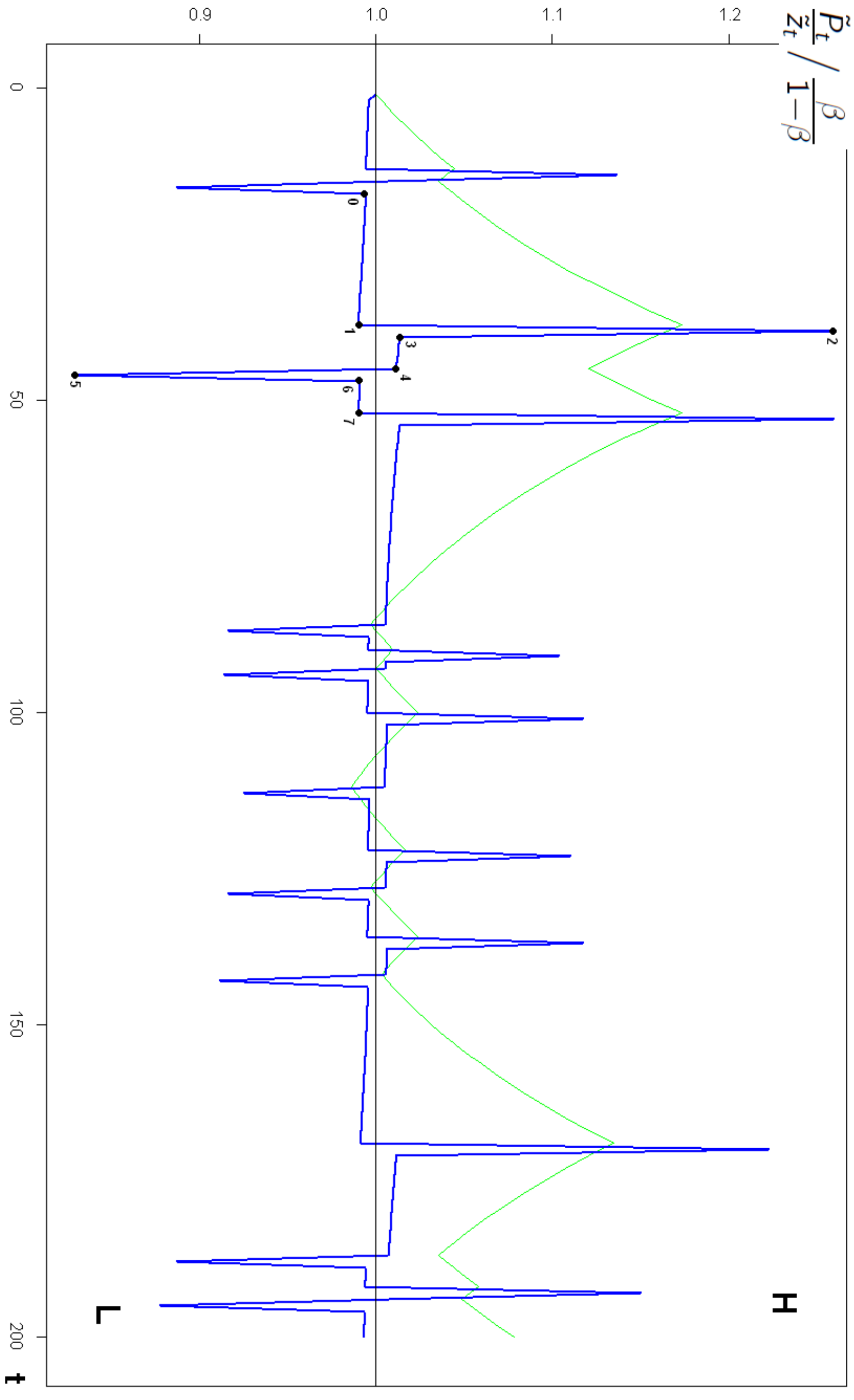

Figure 6: The Price-Dividend Ratio (blue) and the Disappointment Parameter (green)

$$V_0(X) + \beta \left(p_i\, V_{(x_i,\,X)}(Z) + p_j\, V_{(x_j,\,X)}(Z) + \sum_{s \neq i,\,j} p_s\, V_{(x_s,\,X)}(Z)\right);$$

equivalently,

$$\frac{V_0(X_\epsilon) - V_0(X)}{\beta} + \epsilon \left(V_{(x_i,\,X)}(Z) - V_{(x_j,\,X)}(Z)\right) + (p_i + \epsilon) \left(V_{(x_i,\,X_\epsilon)}(Z) - V_{(x_i,\,X)}(Z)\right) +$$

$$(p_j - \epsilon) \left(V_{(x_j,\,X_\epsilon)}(Z) - V_{(x_j,\,X)}(Z)\right) + \sum_{s \neq i,\,j} p_s \left(V_{(x_s,\,X_\epsilon)}(Z) - V_{(x_s,\,X)}(Z)\right) \geq 0.$$

By right-continuity, $\lim_{\epsilon \to 0} V_{(x_s,\,X_\epsilon)} = V_{(x_s,\,X)}$ for each s. That is, for any $\delta > 0$, there exists $\epsilon^* > 0$ such that for any $\epsilon \in (0, \epsilon^*)$,

$$|V_{(x_s,\,X_\epsilon)}(Z) - V_{(x_s,\,X)}(Z)| < \delta \text{ for any } Z \in \Delta(\mathbb{R}_+).$$

Fix any $\delta > 0$. Therefore, for any s, there exists $\epsilon_s(\delta) > 0$ such that for any $\epsilon < \epsilon_s(\delta)$,

$$|V_{(x_s,\,X_\epsilon)}(Z) - V_{(x_s,\,X)}(Z)| < \delta \text{ for any } Z \in \Delta(\mathbb{R}_+).$$

Let $\epsilon^*(\delta) = \min\{\epsilon_1(\delta), \ldots, \epsilon_n(\delta)\}$. Then for any $\epsilon < \epsilon^*(\delta)$ and s,

$$|V_{(x_s,\,X_\epsilon)}(Z) - V_{(x_s,\,X)}(Z)| < \delta \text{ for any } Z \in \Delta(\mathbb{R}_+).$$

Fix any $\epsilon \in (0, \epsilon^*(\delta))$. Then the inequality for weak dynamic monotonicity implies that

$$0 \leq \frac{V_0(X_\epsilon) - V_0(X)}{\beta} + \epsilon \left(V_{(x_i,\,X)}(Z) - V_{(x_j,\,X)}(Z)\right) + (p_i + \epsilon) \left(V_{(x_i,\,X_\epsilon)}(Z) - V_{(x_i,\,X)}(Z)\right) +$$

$$(p_j - \epsilon) \left(V_{(x_j,\,X_\epsilon)}(Z) - V_{(x_j,\,X)}(Z)\right) + \sum_{s \neq i,\,j} p_s \left(V_{(x_s,\,X_\epsilon)}(Z) - V_{(x_s,\,X)}(Z)\right) <$$

$$< \frac{V_0(X_\epsilon) - V_0(X)}{\beta} + \epsilon \left(V_{(x_i,\,X)}(Z) - V_{(x_j,\,X)}(Z)\right) + \delta.$$

Now I shall prove the Theorem 1. By way of contradiction, suppose there exists $Z = (r_1, z_1, \ldots, r_m, z_m) \in \Delta(\mathbb{R}_+)$ with $z_1 > \ldots > z_m$ such that $V_{(x_i,\,X)}(Z) - V_{(x_j,\,X)}(Z) < 0$.

By the second part of nontriviality, $V_{(x_i,\,X)}(Z) \neq V_{(x_j,\,X)}(Z)$ implies that

$$|V_{(x_i,\,X)}(r_1, z_1', r_2, z_2, \ldots, r_m, z_m) - V_{(x_j,\,X)}(r_1, z_1', r_2, z_2, \ldots, r_m, z_m)| \to +\infty \text{ as } z_1' \to +\infty.$$

Therefore, there exists $z_1^* > z_1$ such that

$$|V_{(x_i,X)}(r_1, z_1^*, r_2, z_2, \ldots, r_m, z_m) - V_{(x_j,X)}(r_1, z_1^*, r_2, z_2, \ldots, r_m, z_m)| > |\frac{V_0(X_\epsilon) - V_0(X) + \beta\,\delta}{\epsilon\,\beta}|.$$

Note that if

$$V_{(x_i,X)}(r_1, z_1^*, r_2, z_2, \ldots, r_m, z_m) - V_{(x_j,X)}(r_1, z_1^*, r_2, z_2, \ldots, r_m, z_m) < 0,$$

then I obtain a violation of weak dynamic monotonicity since

$$0 < \frac{V_0(X_\epsilon) - V_0(X)}{\beta} + \epsilon\left(V_{(x_i,X)}(r_1, z_1^*, r_2, z_2, \ldots, r_m, z_m) - V_{(x_j,X)}(r_1, z_1^*, r_2, z_2, \ldots, r_m, z_m)\right) + \delta <$$

$$< \frac{V_0(X_\epsilon) - V_0(X)}{\beta} + \delta - |\frac{V_0(X_\epsilon) - V_0(X) + \beta\,\delta}{\beta}| \leq 0.$$

Now suppose

$$V_{(x_i,X)}(r_1, z_1^*, r_2, z_2, \ldots, r_m, z_m) - V_{(x_j,X)}(r_1, z_1^*, r_2, z_2, \ldots, r_m, z_m) \geq 0.$$

Since $V_{(x_i,X)}(Z) - V_{(x_j,X)}(Z) < 0$ and $V_{(x_i,X)}(Z) - V_{(x_j,X)}(Z)$ is continuous in z_1, by the mean value theorem, there exists $z_1^{**} \in (z_1, z_1^*]$ such that

$$V_{(x_i,X)}(r_1, z_1^{**}, r_2, z_2, \ldots, r_m, z_m) - V_{(x_j,X)}(r_1, z_1^{**}, r_2, z_2, \ldots, r_m, z_m) = 0.$$

Since $z_1^{**} > z_1$ and $V_{(x_i,X)}(Z) \neq V_{(x_j,X)}(Z)$, the above equality violates the first part of nontriviality. So I have obtained a contradiction to $V_{(x_i,X)}(Z) - V_{(x_j,X)}(Z) < 0$.

□

Proof of Corollary 1. For (6), Theorem 1 proves that when $x > x'$,

$$u_x^{-1}\big(\mathbb{E}[u_x(Z)]\big) \geq u_{x'}^{-1}\big(\mathbb{E}[u_{x'}(Z)]\big) \text{ any } Z \in \Delta(\mathbb{R}_+).$$

Now with the Jensen's Inequality, the above inequality implies that $u_{x'}$ is more concave than u_x. To illustrate, let $f_{(x,x')} \equiv u_{x'} \circ u_x^{-1}$ and $t_k = u_x(z_k)$ for each k. Then the above inequality is equivalent to

$$f_{x,x'}\big(\mathbb{E}[T]\big) \geq \mathbb{E}[f_{x,x'}(T)] \text{ any } T \in \Delta(\mathbb{R}_+).$$

By the Jensen's inequality, in order to satisfy the above inequality for any $T \in \Delta(\mathbb{R}_+)$, $f_{x,x'}$ must be concave.

□

7.2 Proofs of Theorems 2-3

Proof of Theorem 2. Suppose a continuous preference relation $\succeq$ on $\mathscr{L}$ is represented by a history-dependent model $\{V_{x,X}\}$ and satisfies time consistency, the axiom for history independence, and strong independence. I prove Theorem 2 with two steps.

Step 1. $V_0(Z) = V_{(x,X)}(Z)$ for any $X, Z \in \Delta(\mathbb{R}_+)$.

First, let $u(z) \equiv V_0(z)$ for any $z \in Z$. Take any $X, Z \in \Delta(\mathbb{R}_+)$ and let $\mu \equiv u^{-1}\big(V_0(Z)\big)$. By (3), $(Z;0) \sim (\mu;0)$. By the axiom for history independence, $(Z;0) \sim (\mu;0)$ implies

$$\big(p_i, (x_i; Z), (X_{-i}; 0)\big) \sim \big(p_i, (x_i; \mu), (X_{-i}; 0)\big);$$

equivalently, by (3), $V_{(x_i,X)}(Z) = V_{(x_i,X)}(\mu)$. By time consistency and the definition of μ, $V_{(x_i,X)}(Z) = V_{(x_i,X)}(\mu) = V_0(\mu) = V_0(Z)$.

Step 2. $V_0(Z) = \mathbb{E}u(Z)$.

Take any $X, Y \in \Delta(\mathbb{R}_+)$. Let $z \equiv u^{-1}(\frac{V_0(X)}{\beta})$ and $t \equiv u^{-1}(\frac{V_0(Y)}{\beta})$. By (3), I have $(X;0) \sim (0;z)$ and $(Y;0) \sim (0;t)$. By strong independence,

$$(\alpha X + (1-\alpha)Y; 0) \sim (\alpha, (0;z), 1-\alpha, (0;t)) \text{ for any } \alpha \in [0,1];$$

equivalently, by (3) and time consistency,

$$V_0(\alpha X + (1-\alpha)Y) = \beta \left(\alpha\, u(z) + (1-\alpha)\, u(t)\right) \text{ for any } \alpha \in [0,1].$$

By the definitions of z and t, I have

$$V_0(\alpha X + (1-\alpha)Y) = \alpha\, V_0(X) + (1-\alpha)\, V_0(Y) \text{ for any } \alpha \in [0,1].$$

Therefore, for any $Z = (r_1, z_1, \ldots, r_m, z_m)$,

$$V_0(Z) = V_0(\sum_{k=1}^{m} r_k\, z_k) = \sum_{k=1}^{m} r_k V_0(z_k) = \mathbb{E}u(Z).$$

□

Proof of Theorem 3. Suppose $\succeq$ on $\mathscr{L}$ satisfies all three axioms. I prove Theorem 3 with the following four steps.

Step 1: First, let me construct V_0 and $V_{(x_i,X)}$.

First, let me define a function $\mu_0 : \Delta(\mathbb{R}_+) \to \mathbb{R}_+$. Take any $X \in \Delta(\mathbb{R}_+)$. Then let $\mu_0(X) \equiv \overline{x}$ whenever $(\overline{x}; 0) \sim (X; 0)$. Indeed, μ_0 is well-defined and $\mu_0(t) = t$ for any $t \in \mathbb{R}_+$. Now let $V_0(X) \equiv u(\mu_0(X))$. Second, let me define a function $\mu_{(x_i, X)}$ for any history (x_i, X). Take any history (x_i, X) and $Z \in \Delta(\mathbb{R}_+)$. Then let $\mu_{(x_i, X)}(Z) \equiv \overline{z}$ whenever

$$\big(p_i, (x_i; Z), (X_{-i}; 0)\big) \sim \big(p_i, (x_i; \overline{z}), (X_{-i}; 0)\big).$$

Indeed, $\mu_{(x_i, X)}$ is well-defined and $\mu_{(x_i, X)}(t) = t$ for any $t \in \mathbb{R}_+$. Let $V_{(x_i, X)}(Z) \equiv u(\mu_{(x_i, X)}(Z))$. By the above construction, I have $V_0(t) = V_{(x_i, X)}(t) = u(t)$ for any $t \in \mathbb{R}_+$.

Step 2: For any $(p_i, (x_i; Y_i))_{i=1}^n \in \mathscr{L}$,

$$(p_i, (x_i; Y_i))_{i=1}^n \sim \big(X; U_2^{-1}\big(\sum_{i=1}^n p_i\, U_2(\mu_{(x_i, X)}(Y_i))\big)\big).$$

Take any $L = (p_i, (x_i; Y_i))_{i=1}^n \in \mathscr{L}$. By Separability Between Parallel Histories,

$$\text{since } \big(p_i, (x_i; Y_i), (X_{-i}; 0)\big) \sim \big(p_i, (x_i; \mu_{(x_i, X)}(Y_i)), (X_{-i}; 0)\big) \text{ for each } i,$$

I have

$$\big(p_i, (x_i; Y_i)\big)_{i=1}^n \sim \big(p_i, (x_i; \mu_{(x_i, X)}(Y_i))_{i=1}^n\big).$$

Let $z_i \equiv \mu_{(x_i, X)}(Y_i)$. Then $\big(p_i, (x_i; Y_i)\big)_{i=1}^n \sim \big(p_i, (x_i; z_i)\big)_{i=1}^n$.

Let me find $\overline{z} \in \mathbb{R}_+$ such that $\big(p_i, (0; z_i)\big)_{i=1}^n \sim (0; \overline{z})$. Then, by Regularity 4, $\overline{z} = U_2^{-1}\big(\sum_{i=1}^n p_i\, U_2(z_i)\big)$. By Weak Separability Between Today and Tomorrow ii),

$$\big(p_i, (x_i; z_i)\big)_{i=1}^n \sim (X; \overline{z}) = \big(X; U_2^{-1}\big(\sum_{i=1}^n p_i\, U_2(z_i)\big)\big) = \big(X; U_2^{-1}\big(\sum_{i=1}^n p_i\, U_2(\mu_{(x_i, X)}(Y_i))\big)\big).$$

Step 3: For any $L = (p_i, (x_i; Y_i))_{i=1}^n, L' = (p'_k, (x'_k; Y'_k))_{k=1}^m \in \mathscr{L}$,

$$L \succeq L' \text{ iff } u\big(\mu_0(X)\big) + \beta\, u\Bigg(U_2^{-1}\big(\sum_{i=1}^n p_i\, U_2(\mu_{(x_i, X)}(Y_i))\big)\Bigg) \geq u\big(\mu_0(X')\big) + \beta\, u\Bigg(U_2^{-1}\big(\sum_{k=1}^m p'_k\, U_2(\mu_{(x'_k, X')}(Y'_k))\big)\Bigg).$$

By Weak Separability Between Today and Tomorrow i),

$$(X; \overline{z}) \sim (\mu_0(X); \overline{z}) \text{ since } (X; 0) \sim (\mu_0(X); 0).$$

By transitivity,

$$(p_i,(x_i;Y_i))_{i=1}^n \sim (X;\overline{z}) \sim (\mu_0(X);\overline{z}) = \big(\mu_0(X);U_2^{-1}\big(\sum_{i=1}^n p_i\,U_2(z_i)\big)\big) = \big(\mu_0(X);U_2^{-1}\big(\sum_{i=1}^n p_i\,U_2(\mu_{(x_i,X)}(Y_i))\big)\big).$$

Then by Regularity 3 and transitivity,

$$(p_i,(x_i;Y_i))_{i=1}^n \succeq (p'_k,(x'_k;Y'_k))_{k=1}^m \text{ iff}$$

$$\Bigg(\mu_0(X);U_2^{-1}\big(\sum_{i=1}^n p_i\,U_2(\mu_{(x_i,X)}(Y_i))\big)\Bigg) \succeq \Bigg(\mu_0(X');U_2^{-1}\big(\sum_{k=1}^m p'_k\,U_2(\mu_{(x'_k,X')}(Y'_k))\big)\Bigg) \text{ iff}$$

$$u\big(\mu_0(X)\big)+\beta\,u\Bigg(U_2^{-1}\big(\sum_{i=1}^n p_i\,U_2(\mu_{(x_i,X)}(Y_i))\big)\Bigg) \geq u\big(\mu_0(X')\big)+\beta\,u\Bigg(U_2^{-1}\big(\sum_{k=1}^m p'_k\,U_2(\mu_{(x'_k,X')}(Y'_k))\big)\Bigg).$$

Step 4: Without loss of generality, suppose $u(0)=U_2(0)=0$ and $u(1)=U_2(1)=1$. Then $U_2=u$.

Take any $(p_i,(x_i;y_i))_{i=1}^n \in \mathscr{L}$ and i. Let me find $\overline{x},\overline{y} \in \mathbb{R}_+$ such that

$$\big(p_i,(x_i;y_i),(X_{-i};0)\big) \sim (X;\overline{y}) \text{ and } \big(p_i,(x_i;0),(p_k,(x_k;y_k))_{k\neq i}\big) \sim (\overline{x};0).$$

By Step 3, since $\mu_{(x_k,X)}(y_k)=y_k$ for each k, I have

$$\overline{y}=U_2^{-1}\big(p_i\,U_2(y_i)\big) \text{ and } u(\overline{x})=u\big(\mu_0(X)\big)+\beta\,u\Bigg(U_2^{-1}\big(\sum_{k\neq i} p_k\,U_2(y_k)\big)\Bigg).$$

By Additivity, I have $(p_i,(x_i;y_i))_{i=1}^n \sim (\overline{x};\overline{y})$; equivalently,

$$u\big(\mu_0(X)\big)+\beta\,u\Bigg(U_2^{-1}\big(\sum_{k=1}^n p_k\,U_2(y_k)\big)\Bigg) = u(\overline{x})+\beta\,u(\overline{y}) =$$

$$= u\big(\mu_0(X)\big)+\beta\,u\Bigg(U_2^{-1}\big(\sum_{k\neq i} p_k\,U_2(y_k)\big)\Bigg)+\beta\,u\big(U_2^{-1}\big(p_i\,U_2(y_i)\big)\big).$$

Let $A \equiv \sum_{k\neq i} p_k\,U_2(y_k)$ and $B \equiv p_i\,U_2(y_i)$. Therefore, I obtain

$$u\big(U_2^{-1}(A+B)\big)=u\big(U_2^{-1}(A)\big)+u\big(U_2^{-1}(B)\big) \text{ for any } A,B \in \mathbb{R}_+.$$

Since the above is a typical Cauchy functional equation for $u \circ U_2^{-1}$, I know that $u \circ U_2^{-1}$ is linear. Therefore, $u=U_2$.

Uniqueness: Since $u = U_2$, I have

$$(r_k, (0; z_k))_{k=1}^{m} \geq (r'_k, (0; z'_k))_{k=1}^{m'} \text{ iff } \mathbb{E}\, u(Z) \geq \mathbb{E}\, u(Z').$$

It is well known that, u that satisfies the above is unique up to the normalization $u(0) = 0$ and $u(1) = 1$. Now it is sufficient to prove that functions μ_0 and $\mu_{(x_i, X)}$ are unique. Recall Step 1. By deterministic monotonicity, μ_0 is unique since for any X, there exists unique $\bar{x}$ such that $(\overline{x}; 0) \sim (X; 0)$. Moreover, by deterministic monotonicity, $\mu_{(x_i, X)}$ is unique since for any Z, there exists unique $\overline{z}$ such that

$$\big(p_i, (x_i; \bar{z}), (X_{-i}; 0)\big) \sim \big(p_i, (x_i; Z), (X_{-i}; 0)\big).$$

□

8 Appendix B: Generalizing Theorem 1

Here I generalize Theorem 1 by weakening right-continuity. The following condition is sufficient: the marginal impact of a lottery X to risk preferences (or to $V_{(x,X)}$) must be smaller than the marginal impact of an outcome x. In other words, the difference between $V_{(x'', X')}$ and $V_{(x'', X)}$ (where lotteries X' and X are close enough), must be smaller than the difference between $V_{(x,X)}$ and $V_{(x',X)}$ (where $x \neq x'$); that is,

$$\text{The Marginal Impact of the Lottery } X = \left|\frac{V_{(x'',X')}(Z) - V_{(x'',X)}(Z)}{\mathbb{E}X' - \mathbb{E}X}\right| <$$

$$< \text{The Marginal Impact of the Outcome } x = \left|\frac{V_{(x,X)}(Z) - V_{(x',X)}(Z)}{x - x'}\right|.$$

when $\mathbb{E}X' - \mathbb{E}X$ is small enough and the largest outcome z_1 of Z is large enough. Formally,

Assumption 2 (Outcome has a Higher Marginal Impact than Lottery)**.** Take any lottery X and its outcomes x, x', x''. Take any Z such that $V_{(x,X)}(Z) \neq V_{(x',X)}(Z)$. There exists $\epsilon^* > 0$ such that for any $\epsilon < \epsilon^*$ and for any lottery X', if X' first-order stochastically dominates X and $\mathbb{E}(X') - \mathbb{E}(X) < \epsilon$, then

$$\lim_{z'_1 \to \infty} \left|\frac{V_{(x'',X')}(Z) - V_{(x'',X)}(Z)}{\mathbb{E}X' - \mathbb{E}X}\right| \Big/ \left|\frac{V_{(x,X)}(Z) - V_{(x',X)}(Z)}{x - x'}\right| < 1.$$

Assumption 2 is weaker than right-continuity. Because when right-continuity is satisfied, the marginal impact of lottery is bounded, but the marginal impact of outcome goes to infinite as $z_1 \to +\infty$; that is, when $\mathbb{E}X' - \mathbb{E}X$ is small enough

$$\text{The Marginal Impact of Lottery} = |\frac{V_{(x'',X')}(Z) - V_{(x'',X)}(Z)}{\mathbb{E}X' - \mathbb{E}X}| < +\infty$$

and

$$\text{The Marginal Impact of Outcome} = |\frac{V_{(x,X)}(Z) - V_{(x',X)}(Z)}{x - x'}| \to +\infty \text{ as } z_1 + \infty.$$

I now state a generalization of Theorem 1.

Theorem 4 (Dynamic Monotonicity Implies the Reinforcement Effect)**.** *If any history-dependent model $\{V_{(x,X)}\}$ satisfies dynamic monotonicity, nontriviality, and Assumption 2, then it satisfies the reinforcement effect.*

Proof of Theorem 4. Take any lottery X and its two outcomes x_i, x_j with $x_i > x_j$. I shall prove that $V_{(x_i,X)}(Z) \geq V_{(x_j,X)}(Z)$ for any $Z \in \Delta(\mathbb{R}_+)$. Let me consider the following special case of dynamic monotonicity.

Let me first define the following new lotteries: for any $\epsilon \in (0, p_j)$, $X_\epsilon \equiv (p_i + \epsilon, x_i, p_j - \epsilon, x_j, X_{-i,-j})$. Since X_ϵ first-order stochastically dominates X, I must have:

Weak Dynamic Monotonicity. $W(X_\epsilon; Z) \geq W(X; Z)$ for any $Z \in \Delta(\mathbb{R}_+)$.

In terms of (3), the above is equivalent to

$$V_0(X_\epsilon) + \beta\left((p_i + \epsilon)\, V_{(x_i,X_\epsilon)}(Z) + (p_j - \epsilon)\, V_{(x_j,X_\epsilon)}(Z) + \sum_{s \neq i,j} p_s\, V_{(x_s,X_\epsilon)}(Z)\right) \geq$$

$$V_0(X) + \beta\left(p_i\, V_{(x_i,X)}(Z) + p_j\, V_{(x_j,X)}(Z) + \sum_{s \neq i,j} p_s\, V_{(x_s,X)}(Z)\right);$$

equivalently,

$$\frac{V_0(X_\epsilon) - V_0(X)}{\beta} + \epsilon\left(V_{(x_i,X)}(Z) - V_{(x_j,X)}(Z)\right) + (p_i + \epsilon)\left(V_{(x_i,X_\epsilon)}(Z) - V_{(x_i,X)}(Z)\right) +$$

$$+ (p_j - \epsilon)\left(V_{(x_j,X_\epsilon)}(Z) - V_{(x_j,X)}(Z)\right) + \sum_{s \neq i,j} p_s\left(V_{(x_s,X_\epsilon)}(Z) - V_{(x_s,X)}(Z)\right) \geq 0.$$

By way of contradiction, suppose there exists Z such that $V_{(x_i,X)}(Z) < V_{(x_j,X)}(Z)$. Fix such Z. By Assumption 2, for any s, there exists $\epsilon^*(Z, s) > 0$ such that for any $\epsilon < \epsilon^*(Z, s)$,

$$\lim_{z_1' \to \infty} |\frac{V_{(x_s,X_\epsilon)}(Z) - V_{(x_s,X)}(Z)}{\epsilon(x_i - x_j)}| \Big/ |\frac{V_{(x_i,X)}(Z) - V_{(x_j,X)}(Z)}{x_i - x_j}| < 1.$$

Let $\epsilon^{**} = \min\{\epsilon(Z, s)\}$. Then there exists $\delta > 0$ such that for any $\epsilon < \epsilon^{**}$ and s,

$$\lim_{z_1' \to \infty} \frac{V_{(x_s, X_\epsilon)}(Z) - V_{(x_s, X)}(Z)}{\epsilon \left| V_{(x_i, X)}(Z) - V_{(x_j, X)}(Z) \right|} < 1 - \delta.$$

Moreover, since by the second part of nontriviality,

$$|V_{(x_i, X)}(r_1, z_1', Z_{-1}) - V_{(x_j, X)}(r_1, z_1', Z_{-1})| \to +\infty \text{ as } z_1 \to +\infty,$$

$$\lim_{z_1 \to \infty} \frac{1}{|V_{(x_i, X)}(Z) - V_{(x_j, X)}(Z)|} \cdot \frac{V_0(X_\epsilon) - V_0(X)}{\beta} = 0.$$

Therefore, when $\epsilon < \epsilon^{**}$, I have

$$0 \leq \lim_{z_1 \to \infty} \frac{1}{\epsilon \left| V_{(x_i, X)}(Z) - V_{(x_j, X)}(Z) \right|} \Bigg(\frac{V_0(X_\epsilon) - V_0(X)}{\beta} + \epsilon \left(V_{(x_i, X)}(Z) - V_{(x_j, X)}(Z) \right)$$

$$+ (p_i + \epsilon) \left(V_{(x_i, X_\epsilon)}(Z) - V_{(x_i, X)}(Z) \right) + (p_j - \epsilon) \left(V_{(x_j, X_\epsilon)}(Z) - V_{(x_j, X)}(Z) \right) + \sum_{s \neq i,j} p_s \left(V_{(x_s, X_\epsilon)}(Z) - V_{(x_s, X)}(Z) \right) \Bigg) \leq$$

$$\leq \lim_{z_1 \to \infty} \frac{V_{(x_i, X)}(Z) - V_{(x_j, X)}(Z)}{|V_{(x_i, X)}(Z) - V_{(x_j, X)}|} + (1 - \delta).$$

Therefore, $V_{(x_i, X)}(Z) - V_{(x_j, X)}(Z)$ must be positive when z_1 is large enough. Since $V_{(x_i, X)}(Z) - V_{(x_j, X)}(Z) < 0$ and $V_{(x_i, X)}(Z) - V_{(x_j, X)}(Z)$ is continuous in z_1, by the mean value theorem, there exists $z_1^* > z_1$ such that

$$V_{(x_i, X)}(r_1, z_1^*, Z_{-1}) - V_{(x_j, X)}(r_1, z_1^*, Z_{-1}) = 0.$$

Since $z_1^* > z_1$ and $V_{(x_i, X)}(Z) \neq V_{(x_j, X)}(Z)$, the above equality violates the first part of nontriviality. So I obtain a contradiction to $V_{(x_i, X)}(Z) - V_{(x_j, X)}(Z) < 0$.

□

9 Appendix C: Behavioral Foundations of HDEU and HDDA

9.1 Characterizing HDEU

I characterize HDEU (13) with additional three axioms.

Axiom 6 (Expected Utility). A preference relation $\succeq$ on $\mathscr{L}$ satisfies the following two conditions.

1. (EU at Period 1) There exists a utility function U_1 such that for any $X, X' \in \Delta(\mathbb{R}_+)$,

$$(X;0) \succeq (X';0) \Leftrightarrow \mathbb{E}U_1(X) \geq \mathbb{E}U_1(X), \tag{17}$$

2. (EU at Period 2) For any history (x_i, X), there exists a utility function $U_{(x_i,X)}$ such that for any $Z, Z' \in \Delta(\mathbb{R}_+)$,

$$\big(p_i,(x_i;Z),(X_{-i};0)\big) \succeq \big(p_i,(x_i;Z'),(X_{-i};0)\big) \text{ iff } \mathbb{E}U_{(x_i,X)}(Z) \geq \mathbb{E}U_{(x_i,X)}(Z'). \tag{18}$$

This axiom assumes that $\succeq$ has an expected utility representation in static case. The first part of A.6 requires that $\succeq$ has an expected utility representation when it compares lotteries in the first period. The second part of A.6 requires that $\succeq$ has an expected utility representation when it compares lotteries in the second period. I allow that utilities for difference periods and different histories can be different.

The second axiom is called *Additivity Today* which exploits additive structure of EU in the first period. It states that if receiving z today with probability r is equivalent to receiving $\overline{y}$ tomorrow and receiving z' today with probability r' is equivalent to receiving $\overline{x}$ today, then receiving $(r, z, r', z', 1-r-r', 0)$ is equivalent to receiving $(\overline{x};\overline{y})$. More formally,

Axiom 7 (Additivity Today). For any $(r, z, r', z', 1-r-r', 0) \in \Delta(\mathbb{R}_+)$ and $\overline{x}, \overline{y} \in \mathbb{R}_+$,

$$\begin{aligned} \text{if } \big((r,z,1-r,0);0\big) &\sim (0;\overline{y}) \text{ and} \\ \big((1-r',0,r',z');0\big) &\sim (\overline{x};0), \text{ then} \\ \big((r,z,r',z',1-r-r',0);0\big) &\sim (\overline{x};\overline{y}). \end{aligned}$$

The third axiom is called Linearity. It states that if receiving an outcome x after a history (x_i, X) is equivalent to receiving a lottery $(p, x, 1-p, 0)$ after a history (y_j, Y), then replacing x with y does not change the equivalence; that is, if receiving an outcome y after the history (x_i, X) is equivalent to receiving a lottery $(p, y, 1-p, 0)$ after the history (y_j, Y). More formally,

Axiom 8 (Linearity). For any $X, Y \in \Delta(\mathbb{R}_+)$ with $(X;0) \sim (Y;0)$, $z, z' > 0$ and $r \in (0,1]$,

$$\text{if } \big(p_i,(x_i;z),(X_{-i};0)\big) \sim \big(q_j,(y_j;(r,z,1-r,0)),(Y_{-j};0)\big),$$

$$\text{then } \big(p_i,(x_i;z'),(X_{-i};0)\big) \sim \big(q_j,(y_j;(r,z',1-r,0)),(Y_{-j};0)\big).$$

Under Axioms 2 and 5, there are functions u, U_1, and $U_{(x_i,X)}$ for each history (x_i, X).

Without loss of generality, I assume $u(0) = U_1(0) = U_{(x_i, X)}(0) = 0$ and $u(1) = U_1(1) = U_{(x_i, X)}(1) = 1$. Now I can state the characterization theorem for HDEU (13).

Theorem 5 (HDEU)**.** *Take any preference $\succeq$ on $\mathscr{L}$ that satisfies (12). If it satisfies A.6 (Expected Utility), Additivity Today, and Linearity, then $u = U_1$ and for any history (x, X), $U_{(x,X)} = u^{\gamma(x,X)}$ for some $\gamma(x, X) > 0$.*

9.2 Characterizing HDDA

I also characterize HDDA (15) with additional two axioms.

Axiom 9 (Axioms of Gul's Disappointment Aversion). A preference relation $\succeq$ on $\mathscr{L}$ satisfies the following two conditions.

1. (Disappointment Aversion at Period 1) There exists a pair (U_0, δ_0) such that for any $X, X' \in \Delta(\mathbb{R}_+)$,

$$(X; 0) \succeq (X'; 0) \Leftrightarrow \mu_0(X|\delta_1) \geq \mu_0(X'|\delta_1), \tag{19}$$

2. (Disappointment Aversion at Period 2) For any history (x_i, X), there exists a pair $(U_{(x_i,X)}, \delta(x_i, X))$ such that for any $Z, Z' \in \Delta(\mathbb{R}_+)$,

(20)
$$\big(p_i, (x_i; Z), (X_{-i}; 0)\big) \succeq (p_i, (x_i; Z'), (X_{-i}; 0)) \text{ iff } \mu_{(x_i,X)}(Z|\delta(x_i, X)) \geq \mu_{(x_i,X)}(Z'|\delta(x_i, X)).$$

This axiom assumes that $\succeq$ has a Gul's Disappointment Aversion representation in the static case. The first part of A.9 requires that $\succeq$ has a Gul's Disappointment Aversion representation when it compares lotteries in the first period. The second part of A.9 requires that $\succeq$ has a Gul's Disappointment Aversion representation when it compares lotteries in the second period. I allow that utilities and disappointment parameters for different periods and different histories can be different.

The second axiom is a weakening of Additivity Today, I call it *Weak Additivity.* First, let me rewrite Additivity Today in the following way: when $r + r' \leq 1$ and $s + s' \leq 1$,

$$\begin{aligned} \text{if } \ (r, z, 1-r, 0) &\succeq (s, t, 1-s, 0) \ \text{ and} \\ (1-r', 0, r', z') &\succeq (1-s', 0, s', t'), \text{ then} \\ (r, z, r', z', 1-r-r', 0) &\succeq (s, t, s', t', 1-s-s', 0). \end{aligned}$$

However, consistent violations of Additivity Today are documented in lab experiments. One well-known violation of Additivity Today is the Common Consequence Effect, a version of the

Allais Paradox. For example, Kahneman and Tversky (1979) found the following instance of the Common Consequence Effect:

$$\begin{aligned}(0.33,\$27,0.67,\$0) &\succ (0.34,\$24,0.66,\$0) \text{ and}\\ (0.34,\$0,0.66,\$24) &= (0.34,\$0,0.66,\$24), \text{ but}\\ (0.33,\$27,0.66,\$24,0.01,\$0) &\prec (1,\$24).\end{aligned}$$

The main reason of the above violation is that the lottery $(0.34, \$0, 0.66, \$24)$ completely complements $(0.34, \$24, 0.66, \$0)$. As a result, the agent has to compare a risky lottery $(0.33, \$27, 0.66, \$24, 0.01, \&0)$ with a deterministic lottery (1, \$24), but her behavior will be different from the case when she compares two risky lotteries $(0.33, \$27, 0.67, \$0)$ and $(0.34, \$24, 0.66, \$0)$. Therefore, I modify Additivity Today in the following way: the additive property should hold when i) there is no mixture between risky and deterministic lotteries (e.g., $r + r' = 1$ and $s + s' = 1$); and ii) lotteries do not complement each other (e.g., z' is small enough compared to z and t' is small enough compared to t). More formally, when $(r, z, 1 - r, 0) \succeq z'$ and $(s, t, 1 - s, 0) \succeq t'$,

$$\begin{aligned}\text{if } (r,z,1-r,0) &\succeq (s,t,1-s,0) \text{ and}\\ (r,0,1-r,z') &\succeq (s,0,1-s,t'), \text{ then}\\ (r,z,1-r,z') &\succeq (s,t,1-s,t').\end{aligned}$$

I require that the new additivity property holds for lotteries in the first period and also for lotteries in the second period. Formally,

Axiom 10 (Weak Additivity). Take any $(r, z, 1 - r, z') \in \Delta(\mathbb{R}_+)$ with $(r, z, 1 - r, 0) \succeq z'$.

1. (Weak Additivity Today) For any $\overline{x}, \overline{y} \in \mathbb{R}_+$,

$$\begin{aligned}\text{if } \big((r,z,1-r,0);0\big) &\sim (0;\overline{y}) \text{ and}\\ \big((r,0,1-r,z');0\big) &\sim (\overline{x};0), \text{ then}\\ \big((r,z,1-r,z');0\big) &\sim (\overline{x};\overline{y}).\end{aligned}$$

2. (Weak Additivity Tomorrow) For any $\overline{x}, \overline{y} \in \mathbb{R}_+$ and history (x_i, X),

$$\begin{aligned}\text{if } \big(p_i,(x_i;(r,z,1-r,0)),(X_{-i};\,0)\big) &\sim (X;\overline{y}) \text{ and}\\ \big(p_i,(x_i;(r,0,1-r,z')),(X_{-i};\,0)\big) &\sim (\overline{x};0), \text{ then}\\ \big(p_i,(x_i;(r,z,1-r,z')),(X_{-i};\,0)\big) &\sim (\overline{x};\overline{y}).\end{aligned}$$

Under Axioms 2 and 8, there are (u, β), (U_0, δ_0), and $(U_{(x_i,X)}, \delta(x_i, X))$ for any history (x_i, X). Without loss of generality, I assume $u(0)=U_1(0)=U_{(x_i,X)}(0)=0$ and $u(1)=U_1(1)=U_H(1)=1$. Now I can state the characterization theorem for HDDA (15).

Theorem 6. *Take any preference $\succeq$ on $\mathscr{L}$ that satisfies (12). If it satisfies Axioms of Gul's Disappointment Aversion, and Weak Additivity, then $u = U_0 = U_{(x,X)}$ for any history (x, X); that is, for any $X, Z \in \Delta(\mathbb{R}_+)$,*

$$\mu_0(X|\delta_0) \textit{ is solution to } u(\mu) = \frac{\sum_{i=1}^n p_i\, u(x_i)(1+\delta_0\ \mathbb{1}\{x_i \le \mu\})}{\sum_{i=1}^n p_i\,(1+\delta_0\ \mathbb{1}\{x_i \le \mu\})} \textit{ and}$$

$$\mu_{(x,X)}(Z|\delta(x,X)) \textit{ is solution to } u(\mu) = \frac{\sum_{k=1}^m r_k\, u(z_k)(1+\delta(x,X)\ \mathbb{1}\{z_k \le \mu\})}{\sum_{i=1}^n r_k\,(1+\delta(x,X)\ \mathbb{1}\{z_k \le \mu\})}.$$

9.2.1 Proofs of Theorems 5-6

Proof of Theorem 5. I prove Theorem 5 with three steps. Recall Step 1 of the proof of Theorem 1. There are functions μ_0 and $\{\mu_{(x,X)}\}$ such that for any $L = (p_i, (x_i; Y_i))_{i=1}^n$, $L' = (p'_k, (x'_k; Y'_k))_{k=1}^m \in \mathscr{L}$,

$$(21)\quad L \succeq L' \text{ iff } u\big(\mu_0(X)\big) + \beta \sum_{i=1}^n p_i\, u\big(\mu_{(x_i,X)}(Y_i)\big) \ge u\big(\mu_0(X')\big) + \beta \sum_{k=1}^m p'_k\, u\big(\mu_{(x'_k,X')}(Y'_k)\big).$$

Step 1. By Axiom 5, for any X and Z,

$$\mu_0(X) = U_1^{-1}\big(\mathbb{E}\, U_1(X)\big) \text{ and } \mu_H(Z) = U_H^{-1}\big(\mathbb{E}\, U_H(Z)\big) \text{ where } H = (x_i, X).$$

Take any X and $\bar{x}$ such that $(X; 0) \sim (\bar{x}; 0)$. By (21), I have $\mu_0(X) = \bar{x}$. By Axiom 5, I then obtain the first equation of Step 1. Moreover, take any $H = (x_i, X)$, Z, and $\bar{z}$ such that $\big(p_i, (x_i; Z), (X_{-i}; 0)\big) \sim \big(p_i, (x_i; \bar{z}), (X_{-i}; 0)\big)$. By (21), I have $\mu_H(Z) = \bar{z}$. By Axiom 5, I then obtain the second equation of Step 1.

Step 2. By Additivity Today, $u = U_1$

Take any $(r, z, (1-r), z') \in \Delta(\mathbb{R}_+)$ and $\overline{x}, \overline{y} \in \mathbb{R}_+$. Suppose

$$\big((r, z, 1-r, 0)); 0\big) \sim (0; \overline{y}) \text{ and } \big((r, 0, (1-r), -z'); 0\big) \sim (\overline{x}; 0);$$

equivalently, by (21),

$$u\big(\mu_0(r, z, 1-r, 0)\big) = \beta\, u(\overline{y}) \text{ and } u\big(\mu_0(r, 0, 1-r, z')\big) = u(\overline{x}).$$

By Additivity Today,

$$\big((r, z, 1-r, -z'); 0\big) \sim (\overline{x}; \overline{y});$$

equivalently, by (21),

$$u\big(\mu_0(r, z, 1-r, z')\big) = u(\overline{x}) + \beta\, u(\overline{y}).$$

Therefore, from the above three equalities, I obtain

$$u\big(\mu_0(r, z, (1-r), 0)\big) + u\big(\mu_0(r, 0, 1-r, z')\big) = u\big(\mu_0(r, z, 1-r, z')\big); \tag{22}$$

equivalently

$$u\big(U_1^{-1}\big(rU_1(z)\big)\big) + u\big(U_1^{-1}\big((1-r)U_1(z')\big)\big) = u\big(U_1^{-1}\big(rU_1(z) + (1-r)U_1(z')\big)\big). \tag{23}$$

Let $A = rU_1(z)$ and $B = (1-r)U_1(z')$. Therefore, I have

$$u\big(U_1^{-1}(A)\big) + u\big(U_1^{-1}(B)\big) = u\big(U_1^{-1}(A+B)\big) \text{ for any } A, B \geq 0.$$

Since the above is a typical Cauchy functional equation, $u \circ U_1^{-1}$ is a linear function. Since $u(1) = U_1(1) = 1$, I have $u = U_1$.

Step 3: For any history H, by Linearity, $U_H = u^{\gamma(H)}$ for some $\gamma(H) > 0$.

Take any $X, Y \in \Delta(\mathbb{R}_+)$ with $(X; 0) \sim (Y; 0)$ and $\big(p_i, (x_i; x), (X_{-i}; 0)\big) \sim \big(q_j, (y_j; (p, x, 1-p, 0)), (Y_{-j}; 0)\big)$ for some $x > 0$ and $p > 0$. By (21),

$$\mathbb{E}\, u(X) = \mathbb{E}\, u(Y) \text{ and}$$

$$\mathbb{E}\, u(X) + \beta\, p_i\, u(x) = \mathbb{E}\, u(Y) + \beta\, q_j\, u\big(U_H^{-1}\big(p\, U_H(x)\big)\big) \text{ where } H = (y_j, Y).$$

From the above two equalities, I obtain

$$p_i\, u(x) = q_j\, u\big(U_H^{-1}\big(p\, U_H(x)\big)\big).$$

By Linearity, I have

$$\big(p_i, (x_i; y), (X_{-i}; 0)\big) \sim \big(q_j, (y_j; (p, y, 1-p, 0)), (Y_{-j}; 0)\big) \text{ for any } y > 0.$$

Therefore,

$$p_i\, u(x) = q_j\, u\big(U_H^{-1}\big(p\, U_H(x)\big)\big) \text{ iff } p_i\, u(y) = q_j\, u\big(U_H^{-1}\big(p\, U_H(y)\big)\big) \text{ for any } y > 0.$$

Let $y = 1$ and $U_H(x) = t$ and $G \equiv u \circ U_H^{-1}$. Then I have

$$G(p)\,G(t) = G\big(p\,t\big) \text{ for any } t > 0 \text{ and } p \in (0,1).$$

Since $G(1) = 1$, $G(p) = \frac{1}{G(\frac{1}{p})}$ when $p \cdot t = 1$. Therefore, I have

$$G(p)\,G(t) = G\big(p\,t\big) \text{ for any } t > 0 \text{ and } p > 0.$$

Since the above is a typical Cauchy functional equation, there exists $\alpha > 0$ such that $G(t) = t^{\alpha}$; that is, $U_H = u^{\frac{1}{\alpha}}$. Since α depends on H, let $\gamma(H) \equiv \frac{1}{\alpha}$; that is, $U_H = u^{\gamma(H)}$.

□

Proof of Theorem 6. I prove Theorem 6 with three steps.

Step 1. Let H be a history. By Axiom 8, for any X and Z,

$$\mu_0(X|\delta_0) \text{ is a unique solution to } U_1(\mu) = \frac{\sum_{i=1}^{n} p_i\, U_0(x_i)(1 + \delta_0\ \mathbb{1}\{x_i \le \mu\})}{\sum_{i=1}^{n} p_i\,(1 + \delta_0\ \mathbb{1}\{x_i \le \mu\})} \text{ and}$$

$$\mu_H(Z|\delta(H)) \text{ is a unique solution to } U_H(\mu) = \frac{\sum_{k=1}^{m} r_k\, U_H(z_k)(1 + \delta(H)\ \mathbb{1}\{z_k \le \mu\})}{\sum_{i=1}^{n} p_i\,(1 + \delta(H)\ \mathbb{1}\{z_k \le \mu\})}.$$

Take any X and $\bar{x}$ such that $(X;0) \sim (\bar{x};0)$. By (21), $\mu_0(X) = \bar{x}$. By Axiom 8, I then obtain the first equation of Step 1. Moreover, take any $H = (x_i, X)$, Z, and $\bar{z}$ such that $\big(p_i, (x_i; Z), (X_{-i}; 0)\big) \sim \big(p_i, (x_i; \bar{z}), (X_{-i}; 0)\big)$. By (21), $\mu_H(Z) = z$. By Axiom 8, I then obtain the second equation of Step 1.

Step 2. By Weak Additivity Today, $u = U_0$

Take any $(r, z, (1-r), z') \in \Delta(\mathbb{R}_+)$ with $(r, z, (1-r), 0) \succeq z'$, and $\overline{x}, \overline{y} \in \mathbb{R}_+$. Suppose

$$\big((r, z, 1-r, 0)); 0\big) \sim (0; \overline{y}) \text{ and } \big((r, 0, (1-r), -z'); 0\big) \sim (\overline{x}; 0);$$

equivalently, by (21),

$$u\big(\mu_1(r, z, 1-r, 0)\big) = \beta\, u(\overline{y}) \text{ and } u\big(\mu_1(r, 0, 1-r, z')\big) = u(\overline{x}).$$

By Weak Additivity Today,

$$\big((r, z, 1-r, -z'); 0\big) \sim (\overline{x}; \overline{y});$$

equivalently, by (21),

$$u\big(\mu_0(r,z,1-r,z'|\delta_0)\big) = u(\overline{x}) + \beta\, u(\overline{y}).$$

Therefore, from the above three equalities, I obtain

$$u\big(\mu_0(r,z,(1-r),0)\big) + u\big(\mu_0(r,0,1-r,z')\big) = u\big(\mu_0(r,z,1-r,z')\big). \tag{24}$$

Since $(r,z,(1-r),0) \succeq z'$,

$$u\big(U_0^{-1}(\frac{rU_0(z)}{r+(1-r)(1+\delta_0)})\big) + u\big(U_0^{-1}(\frac{(1-r)U_0(z')(1+\delta_0)}{r+(1-r)(1+\delta_0)})\big) = \tag{25}$$

$$u\big(U_0^{-1}(\frac{rU_0(z)+(1-r)U_0(z')(1+\delta_0)}{r+(1-r)(1+\delta_0)})\big).$$

Let $A = \frac{rU_0(z)}{r+(1-r)(1+\delta_0)}$ and $B = \frac{(1-r)U_0(z')(1+\delta_0)}{r+(1-r)(1+\delta_0)}$. Therefore, I have

$$u\big(U_0^{-1}(A)\big) + u\big(U_0^{-1}(B)\big) = u\big(U_0^{-1}(A+B)\big) \text{ for any } A, B \geq 0.$$

Since the above is a typical Cauchy functional equation, $u \circ U_0^{-1}$ is a linear function. Moreover, since $u(1) = U_0(1) = 1$, I have $u = U_0$.

Step 3: Let H be a history. By Weak Additivity Tomorrow, $U_H = u$.

Take any $X, (r,z,(1-r),z') \in \Delta(\mathbb{R}_+)$ with $(r,z,(1-r),0) \succeq z'$, and $\overline{x}, \overline{y} \in \mathbb{R}_+$. Let $H = (x_i, X)$. Suppose

$$\big(p_i, (x_i; (r,z,1-r,0)), (X_{-i}; 0)\big) \sim (X; \overline{y}) \text{ and } \big(p_i, (x_i; (r,0,(1-r),-z')), (X_{-i}; 0)\big) \sim (\overline{x}; 0);$$

equivalently, by (21),

$$u\big(\mu_0(X)\big) + \beta\, p_i\, u\big(\mu_H(r,z,1-r,0)\big) = u\big(\mu_0(X)\big) + \beta\, u(\overline{y})$$

and

$$u\big(\mu_1(X)\big) + \beta\, p_i\, u\big(\mu_H(r,0,1-r,z')\big) = u(\overline{x}).$$

By Weak Additivity, I have

$$\big(p_i, (x_i; (r,z,1-r,-z')), (X_{-i}; 0)\big) \sim (\overline{x}; \overline{y});$$

equivalently, by (21),

$$u\big(\mu_0(X)\big) + \beta\, p_i\, u\big(\mu_H(r, z, 1-r, z')\big) = u(\overline{x}) + \beta\, u(\overline{y}).$$

Therefore, from the above three equalities, I obtain

$$u\big(\mu_H(r, z, (1-r), 0)\big) + u\big(\mu_H(r, 0, 1-r, z')\big) = u\big(\mu_H(r, z, 1-r, z')\big). \tag{26}$$

Since $(r, z, (1-r), 0) \succeq z'$, I have

$$u\big(U_H^{-1}\big(\frac{rU_H(z)}{r+(1-r)(1+\delta_H)}\big)\big) + u\big(U_H^{-1}\big(\frac{(1-r)U_{H_i}(z')(1+\delta_H)}{r+(1-r)(1+\delta_H)}\big)\big) = \tag{27}$$

$$u\big(U_H^{-1}\big(\frac{rU_H(z)+(1-r)U_H(z')(1+\delta_H)}{r+(1-r)(1+\delta_H)}\big)\big).$$

Let $A = \frac{rU_H(z)}{r+(1-r)(1+\delta_H)}$ and $B = \frac{(1-r)U_H(z')(1+\delta_H)}{r+(1-r)(1+\delta_H)}$. Therefore, I have

$$u\big(U_H^{-1}(A)\big) + u\big(U_H^{-1}(B)\big) = u\big(U_H^{-1}(A+B)\big) \text{ for any } A, B \geq 0.$$

Since the above is a typical Cauchy functional equation, $u \circ U_H^{-1}$ is a linear function. Since $u(1) = U_H(1) = 1$, I have $u = U_H$.

□

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